random variable V as $V = \sqrt{X^2 + Y^2}$. Show that

$$f_V(v) = \begin{cases} \frac{v}{\sigma^2} I_0\left(\frac{mv}{\sigma^2}\right) e^{-\frac{v^2+m^2}{2\sigma^2}}, & v > 0 \\ 0, & v \le 0 \end{cases}$$

where

$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{x \cos u}\, du = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x \cos u}\, du$$

is called the *modified Bessel function of the first kind and zero order*. The distribution of V is known as the *Ricean distribution*. Show that, in the special case of $m = 0$, the Ricean distribution simplifies to the Rayleigh distribution.

4.32 A coin whose probability of a head is $\frac{1}{4}$, is flipped 2000 times

1. Using the law of large numbers, find a lower bound to the probability that the total number of heads lies between 480 and 520.
2. Using the central limit theorem, find the probability that the total number of heads lies between 480 and 520.

4.33 Find the covariance matrix of the random vector X in Example 4.2.2.

4.34 Find $m_X(t)$ for the random process $X(t)$ given in Example 4.2.4. Is it independent of t?

4.35 Let the random process $X(t)$ be defined by $X(t) = A + Bt$, where A and B are independent random variables each uniformly distributed on $[-1, 1]$. Find $m_X(t)$ and $R_X(t_1, t_2)$.

4.36 What is the autocorrelation function of the random process given in Example 4.2.5?

4.37 Show that the process given in Example 4.2.4 is a stationary process.

4.38 Is the process given in Example 4.2.5 wide-sense stationary?

4.39 Show that any Mth-order stationary process, $M \ge 2$, is wide-sense stationary.

4.40 Which one of the following functions can be the autocorrelation function of a random process and why?

1. $f(\tau) = \sin(2\pi f_0 \tau)$.
2. $f(\tau) = \tau^2$.
3. $f(\tau) = \begin{cases} 1 - |\tau| & |\tau| \le 1 \\ 1 + |\tau| & |\tau| > 1 \end{cases}$
4. $f(\tau)$ as shown in Figure P-4.40.

4.41 Is the process given in Example 4.2.5 an ergodic process?

4.42 Is the process of Example 4.2.1 power-type or energy-type? Is this process stationary?

4.43 A random process $Z(t)$ takes values 0 and 1. A transition from 0 to 1 or from 1 to 0 occurs randomly, and the probability of having n transitions in a time interval

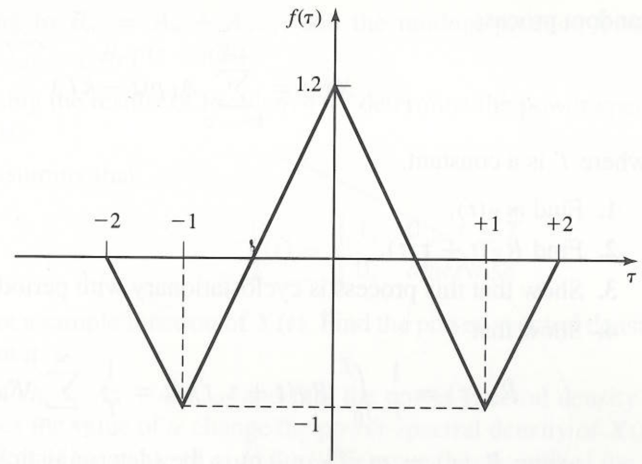

Figure P-4.40

of duration τ, $(\tau > 0)$, is given by

$$p_N(n) = \frac{1}{1+\alpha\tau}\left(\frac{\alpha\tau}{1+\alpha\tau}\right)^n, \quad n = 0, 1, 2, \ldots$$

where $\alpha > 0$ is a constant. We further assume that at $t = 0$, $X(0)$ is equally likely to be 0 or 1.

1. Find $m_Z(t)$.
2. Find $R_Z(t + \tau, t)$. Is $Z(t)$ stationary? Is it cyclostationary?
3. Determine the power of $Z(t)$.

4.44 The random process $X(t)$ is defined by

$$X(t) = X \cos 2\pi f_0 t + Y \sin 2\pi f_0 t$$

where X and Y are two zero-mean independent Gaussian random variables each with variance σ^2.

1. Find $m_X(t)$.
2. Find $R_X(t + \tau, t)$. Is $X(t)$ stationary? Is it cyclostationary?
3. Find the power-spectral density of $X(t)$.
4. Answer the above questions for the case where $\sigma_X^2 = \sigma_Y^2$

4.45 Let $\{A_k\}_{k=-\infty}^{\infty}$ be a sequence of random variables with $E[A_k] = m$ and $E[A_k A_j] = R_A(k - j)$. We further assume that $R_A(k - j) = R_A(j - k)$. Let $p(t)$ be any deterministic signal whose Fourier transform is $P(f)$, and define the random process

$$X(t) = \sum_{k=-\infty}^{+\infty} A_k p(t - kT)$$

where T is a constant.

1. Find $m_X(t)$.
2. Find $R_X(t + \tau, t)$.
3. Show that this process is cyclostationary with period T.
4. Show that

$$\bar{R}_X(\tau) = \frac{1}{T}\int_0^T R_X(t + \tau, t)\, dt = \frac{1}{T}\sum_{n=-\infty}^{\infty} R_A(n) R_p(\tau - nT)$$

where $R_p(\tau) = p(\tau) \star p(-\tau)$ is the (deterministic) autocorrelation function of $p(t)$.

5. Show that the power-spectral density of $X(t)$ is given by

$$\mathcal{S}_X(f) = \frac{|P(f)|^2}{T}\left[R_A(0) + 2\sum_{k=1}^{\infty} R_A(k)\cos 2\pi k f T\right]$$

4.46 Using the result of Problem 4.45 find the power-spectral density of the random process

$$X(t) = \sum_{n=-\infty}^{\infty} A_n p(t - nT)$$

in the following cases

1. A_n's are independent random variables each taking values ± 1 with equal probability and

$$p(t) = \begin{cases} 1, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

2. A_n's take values 0 and 1 with equal probability, all the other conditions as in part 1.
3. Solve parts 1 and 2 for the case where

$$p(t) = \begin{cases} 1, & 0 \le t \le 3T \\ 0, & \text{otherwise} \end{cases}$$

4. In each of the above cases find the bandwidth that contains 95% of the total power of the process.

4.47 Let A_n's denote a sequence of independent binary valued random variables, each taking values ± 1 with equal probability. Random variables B_n are defined according to $B_n = A_n + A_{n-1}$, and the random process $X(t)$ is defined as $X(t) = \sum_{n=-\infty}^{\infty} B_n p(t - nT)$.

1. Using the results of Problem 4.45 determine the power-spectral density of $X(t)$.
2. Assuming that

$$p(t) = \begin{cases} 1, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

plot a sample function of $X(t)$. Find the power-spectral density of $X(t)$ and plot it.

3. Let $B_n = A_n + \alpha A_{n-1}$ and find the power-spectral density of $X(t)$. How does the value of α change the power-spectral density of $X(t)$. What is the effect of the value of α on the 95%-power bandwidth of the process?

4.48 Let $X(t)$ be a cyclostationary process with period T. From Corollary 4.3.3, we have seen that, in order to find the power-spectral density of $X(t)$, we first determine $\bar{R}(\tau) = \frac{1}{T}\int_0^T R_X(t + \tau, t)\, dt$ and then find $\mathscr{F}[\bar{R}(\tau)]$. We now obtain this result by another approach.

1. Let Θ be a random variable, independent of $X(t)$ and uniformly distributed on $[0, T]$. Show that $Y(t) = X(t + \Theta)$ is stationary and its autocorrelation function is given as

$$R_Y(\tau) = \frac{1}{T}\int_0^T R_X(t + \tau, t)\, dt$$

2. Show that $Y(t)$ and $X(t)$ have equal power-spectral densities.
3. Conclude that

$$\mathcal{S}_X(f) = \mathscr{F}\left[\frac{1}{T}\int_0^T R_X(t + \tau, t)\, dt\right]$$

4.49 The RMS bandwidth of a process is defined as

$$W_{\text{RMS}} = \frac{\int_0^\infty f^2 \mathcal{S}_X(f)\, df}{\int_0^\infty \mathcal{S}_X(f)\, df}$$

Show that for a stationary process, we have

$$W_{\text{RMS}} = -\frac{1}{4\pi^2 R_X(0)} \frac{d^2}{d\tau^2} R_X(\tau)\Big|_{\tau=0}$$

4.50 Show that for jointly stationary processes $X(t)$ and $Y(t)$, we have $R_{XY}(\tau) = R_{YX}(-\tau)$. From this, conclude that $\mathcal{S}_{XY}(f) = \mathcal{S}^*_{YX}(f)$.

4.51 A zero-mean white Gaussian noise with power-spectral density of $\frac{N_0}{2}$ passes through an ideal lowpass filter with bandwidth B.

1. Find the autocorrelation of the output process $Y(t)$.
2. Assuming $\tau = \frac{1}{2B}$, find the joint PDF of the random variables $Y(t)$ and $Y(t+\tau)$. Are these random variables independent?

4.52 Find the output autocorrelation function for a delay line with delay Δ when the input is a stationary process with autocorrelation $R_X(\tau)$. Interpret the result.

4.53 We have proved that when the input to an LTI system is stationary, the output is also stationary. Is the converse of this theorem also true? That is, if we know that the output process is stationary, can we conclude that the input process is necessarily stationary?

4.54 It was shown in this chapter that if a stationary random process $X(t)$ with autocorrelation function $R_X(\tau)$ is applied to an LTI system with impulse response $h(t)$, the output $Y(t)$ is also stationary with autocorrelation function $R_Y(\tau) = R_X(\tau) \star h(\tau) \star h(-\tau)$. In this problem, we show that a similar relation holds for cyclostationary processes.

1. Let $X(t)$ be a cyclostationary process applied to an LTI system with impulse response $h(t)$. Show that the output process is also cyclostationary.
2. Show that

$$\bar{R}_Y(t, t+\tau) = \bar{R}_X(t, t+\tau) \star h(\tau) \star h(-\tau)$$

3. Conclude that the relation

$$\mathcal{S}_Y(f) = \mathcal{S}_X(f)|H(f)|^2$$

is true for both stationary and cyclostationary processes.

4.55 Generalize the result of the Example 4.3.8 to show that if $X(t)$ is stationary,

1. $X(t)$ and $\frac{d}{dt}X(t)$ are uncorrelated processes.
2. The power spectrum of $Z(t) = X(t) + \frac{d}{dt}X(t)$ is the sum of the power spectra of $X(f)$ and $\frac{d}{dt}X(t)$.

Express the power spectrum of the sum in terms of the power spectrum of $X(t)$.

4.56 $X(t)$ is a stationary process with power-spectral density $\mathcal{S}_X(f)$. This process passes through the system shown in Figure P-4.56.

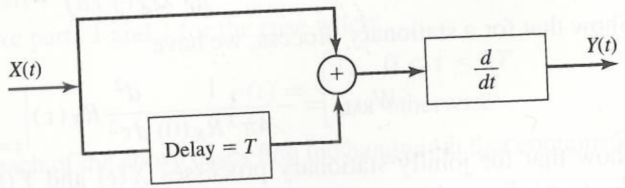

Figure P-4.56

1. Is $Y(t)$ stationary? Why?
2. What is the power-spectral density of $Y(t)$?
3. What frequency components can*not* be present in the output process and why?

4.57 The stationary random process $X(t)$ has a power-spectral density denoted by $\mathcal{S}_X(f)$.

1. What is the power-spectral density of $Y(t) = X(t) - X(t-T)$?
2. What is the power-spectral density of $Z(t) = X'(t) - X(t)$?
3. What is the power-spectral density of $W(t) = Y(t) + Z(t)$?

4.58 Show that for two jointly stationary processes $X(t)$ and $Y(t)$, we have

$$|R_{XY}(\tau)| \leq R_X(0)R_Y(0) \leq \tfrac{1}{2}[R_X(0) + R_Y(0)]$$

4.59 The stationary process $X(t)$ is passed through an LTI system and the output process is denoted by $Y(t)$. Find the output autocorrelation function and the crosscorrelation function between the input and the output processes in each of the following cases.

1. A delay system with delay Δ.
2. A system with $h(t) = \frac{1}{t}$.
3. A system with $h(t) = e^{-\alpha t}u(t)$ where $\alpha > 0$.
4. A system described by the differential equation

$$\frac{d}{dt}Y(t) + Y(t) = \frac{d}{dt}X(t) - X(t)$$

5. A finite time averager defined by the input–output relation

$$y(t) = \frac{1}{2T}\int_{t-T}^{t+T} x(\tau)\, d\tau$$

where T is a constant.

4.60 Give an example of two processes $X(t)$ and $Y(t)$ for which $R_{XY}(t+\tau, t)$ is a function of τ but $X(t)$ and $Y(t)$ are not stationary.

4.61 For each of the following processes, find the power-spectral density

1. $X(t) = A\cos(2\pi f_0 t + \Theta)$, where A is a constant and Θ is a random variable uniformly distributed on $[0, \frac{\pi}{4}]$.
2. $X(t) = X + Y$, where X and Y are independent, X is uniform on $[-1, 1]$ and Y is uniform on $[0, 1]$.

4.62 $X(t)$ is a stationary random process with autocorrelation function $R_X(\tau) = e^{-\alpha|\tau|}$, $\alpha > 0$. This process is applied to an LTI system with $h(t) = e^{-\beta t}u(t)$, where $\beta > 0$. Find the power-spectral density of the output process $Y(t)$. Treat the cases $\alpha \neq \beta$ and $\alpha = \beta$ separately.

Example 4.2.2

Let ω_i denote the outcome of a random experiment consisting of independent drawings from a Gaussian random variable distributed according to $\mathcal{N}(0, 1)$. Let the discrete-time random process $\{X_n\}_{n=0}^{\infty}$ be defined by: $X_0 = 0$ and $X_n = X_{n-1} + \omega_n$ for all $n \geq 1$. It follows from the basic properties of the Gaussian random variables that for all $i \geq 1$, $j \geq 1$, and $i < j$, $\{X_n\}_i^j$ is a $j - i + 1$ dimensional Gaussian vector. For this example the second view; i.e., interpreting the random process as a collection of random variables, is more appropriate.

4.2.1 Description of Random Processes

Based on the adopted viewpoint, there are two types of descriptions possible for random processes. If the random process is viewed as a collection of signals, the *analytic description* may be appropriate. In this description, analytic expressions are given for each sample in terms of one or more random variables; i.e., the random process is given as $X(t) = f(t; \boldsymbol{\theta})$ where $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$ is, in general, a random vector with a given joint PDF. This is a very informative description of a random process because it completely describes the analytic form of various realizations of the process. For real-life processes, it is hardly possible to give such a complete description. If an analytic description is not possible, a *statistical description* may be appropriate. Such a description is based on the second viewpoint of random processes, regarding them as a collection of random variables indexed by some index set.

Definition 4.2.1. *A complete statistical description* of a random process $X(t)$ is known if for any integer n and any choice of $(t_1, t_2, \ldots, t_n) \in \mathbb{R}^n$ the joint PDF of $(X(t_1), X(t_2), \ldots, X(t_n))$ is given.

If the complete statistical description of the process is given, for any n the joint density function of $(X(t_1), X(t_2), \ldots, X(t_n))$ is given by $f_{X(t_1),X(t_2),\ldots,X(t_n)}(x_1, x_2, \ldots, x_n)$.

Definition 4.2.2. A process $X(t)$ is described by its Mth *order statistics* if for all $n \leq M$ and all $(t_1, t_2, \ldots, t_n) \in \mathbb{R}^n$ the joint PDF of $(X(t_1), X(t_2), \ldots, X(t_n))$ is given.

A very important special case, in the study of communication systems, is the case of $M = 2$, in which second-order statistics are known. This simply means that, at each time instant t, we have the density function of $X(t)$, and for all choices of (t_1, t_2) the joint density function of $(X(t_1), X(t_2))$ is given.

Example 4.2.3

A random process is defined by $X(t) = A\cos(2\pi f_0 t + \Theta)$ where Θ is a random variable uniformly distributed on $[0, 2\pi)$. In this case, we have an analytic description of the random process. Note that by having the analytic description, we can find the complete statistical description. Figure 4.12 shows some samples of this process.

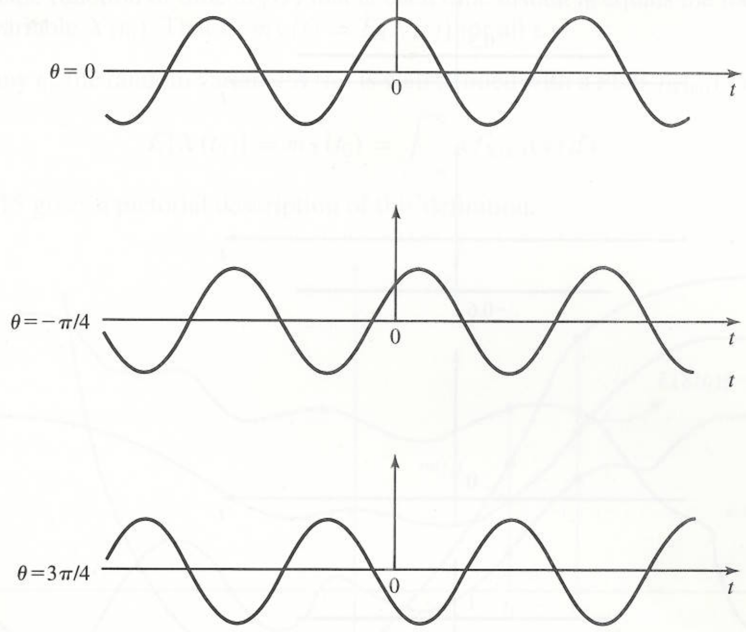

Figure 4.12 Samples of the random process given in Example 4.2.3.

Example 4.2.4

The process $X(t)$ is defined by $X(t) = X$, where X is a random variable uniformly distributed on $[-1, 1]$. In this case again, an analytic description of the random process is given. For this random process, each sample is a constant signal. Samples of this process are shown in Figure 4.13.

Example 4.2.5

The process $X(t)$, $t > 0$, is defined by the property that for any n and any $(t_1, t_2, \ldots, t_n) \in \mathbb{R}^{+n}$, the joint density function of $\{X(t_i)\}_{i=1}^n$ is a jointly Gaussian vector with mean 0 and covariance matrix described by

$$C_{i,j} = \mathrm{COV}(X(t_i), X(t_j)) = \sigma^2 \min(t_i, t_j)$$

This is a complete statistical description of the random process $X(t)$. A sample of this process is shown in Figure 4.14.

Note that in the last example, although a complete statistical description of the process is given, little insight can be obtained in the shape of each realization of the process.