Department of Signal Theory and Communications **09/19/07**
UCIIIM

DIGITAL SIGNAL PROCESSING

THEORY
(Time: 70 minutes. Mark: 3/8)

T1.- We want to estimate a random vector **s** from a vector of observations **x** which is obtained as:

$$\mathbf{x} = \mathbf{H}\,\mathbf{s} + \mathbf{r}$$

where **H** is a known matrix, **r** is a random noise vector with distribution $G(\mathbf{0}, v_r\mathbf{I})$, and **s** is the random vector to be estimated, which follows a distribution $G(E\{\mathbf{s}\}, \mathbf{V}_s)$.

Knowing also that **s** and **r** are independent random vectors:

a) Obtain the ML estimator of **s**, $\hat{\mathbf{s}}_{ml}$.
b) Is $\hat{\mathbf{s}}_{ml}$ an unbiased estimator? Justify your answer.
c) As it is known, the MSE estimator of **s** is given by:

$$\hat{\mathbf{s}}_{mse} = (\mathbf{H}^T\mathbf{H} + v_r\,\mathbf{V}_s^{-1})^{-1}\,\mathbf{H}^T\,\mathbf{x}$$

Calculate the bias of $\hat{\mathbf{s}}_{mse}$, and explain under which conditions the bias tends to 0.

(25 min; 1 p)

T2.- The following expressions:

a) $\|\mathbf{x}-\mathbf{y}\|^2$

b) $\|\mathbf{x}\|\|\mathbf{y}\|$

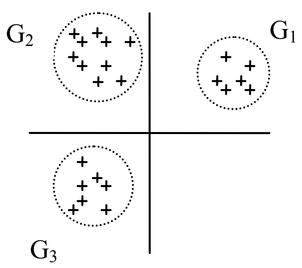
c) $(\|\mathbf{x}\|-\|\mathbf{y}\|)^2$

d) $\|\mathbf{x}\|^2+\|\mathbf{y}\|^2$

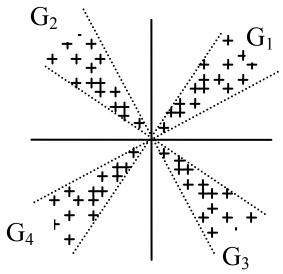
e) $\dfrac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}$

are considered to be used as similarity criteria in a clustering algorithm. Explain which of them you would select to obtain the clusters depicted in the figures below.

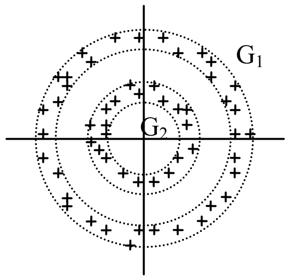
1)



2)



3)



(20 min; 1p)

T3.- We have a set of K samples, $\left\{x^{(k)}\right\}_{k=1}^{K}$, which have been independently drawn from a random variable x whose pdf is given by:

$$p_X(x)=\frac{1}{bx^2}\exp\left(-\frac{1}{bx}\right)u(x)$$

with b>0.

a) Express $\hat{b}_{ml}$ as a function of the available samples.

b) Verify that random variable y = 1/x follows an unilateral exponential pdf, $p_Y(y)$, and obtain the mean value of such a distribution.

c) In the light of previous results, is $\hat{b}_{ml}$ an unbiased estimator?

(25 min; 1 p)

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PROBLEMS

(Time: 135 minutes. Mark: 5/8)

P1.- In a binary classification problem formulated under the ML criterion, the likelihood of the observations are:

$$p(x|H_0)=\begin{cases} 2(1-x)\ ,\ 0<x<1 \\ 0\ ,\ \text{otherwise} \end{cases}$$

$$p(x|H_1)=\begin{cases} 1/a\ \ ,\ \ 0<x<a \\ 0\ ,\ \text{otherwise} \end{cases}$$

where a>1 is a deterministic parameter.

a) Design the optimum decider assuming the value of a is known.
b) Assume now the value of a is unknown. Following a minimax strategy, decisions are taken by comparison with a threshold x_u^* that is fixed to minimize the maximum expected cost (corresponding to the ML criterion), i.e.,

$$x_u^*=\arg\left\{\min_{x_u}\left\{\max_{a}\ \bar{C}_{ML}(x_u,a)\right\}\right\}$$

where x_u is a generic threshold for the decision

$$x \underset{D_0}{\overset{D_1}{\gtrless}} x_u$$

Obtain the value of threshold x_u^*.

c) Compute the increment of the expected cost (ML) when applying the minimax strategy, in comparison to the cost that would be obtained if parameter a were known.

(60 min; 2.5 p)

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P2.- Consider the following family of cost functions:

$$C_N(s,\hat{s}) = \frac{1}{N+1}\hat{s}^{N+1} + \frac{1}{N(N+1)}s^{N+1} - \frac{1}{N}s\hat{s}^N$$

where N is an odd and non-negative integer.

a) Analyze the validity of the invariance principle for such a family of cost functions, and for an arbitrary joint distribution of s and x (where x is the observation).

b) Assuming that

$$p(s,x) = \frac{1}{\lambda x}\exp\left(-\frac{s}{x} - \frac{x}{\lambda}\right)u(s)u(x),\ \ \lambda > 0$$

Obtain the estimator which incurs in a minimum expected cost.

c) Obtain the minimum mean cost.

d) Determine now the coefficient w that minimizes the mean cost of an estimator

$$\hat{s} = wx^m$$

where m is a positive integer.

e) Noting the dependence of w on the value of N, state if this result is in contradiction with the invariance principle. Justify your answer.

Hint: $\int_0^\infty x^N \exp(-x)\, dx = N!$

(75 min; 2.5 p)