Department of Signal Theory and Communications **09/09/09**
UCIIIM

DIGITAL SIGNAL PROCESSING

THEORY
(Time: 75 minutes. Mark: 3/8)

T1.- K independent observations $\{x^{(k)}, k=1,\dots,K\}$ of a random variable x are drawn from a uniform distribution p(x) in the interval [0,1). Using these observations, the following pdf estimators are built:

- A histogram h(x), based in partitioning the interval [0, 1) into N equally-sized intervals with limits [i/N, (i+1)/N), for i = 0, 1, …, N-1.
- Parzen estimator q(x), using rectangular windows with width 1/N.

a) Obtain the probability with which h(1/2) = 0.

b) Obtain the probability with which q(1/2) = 0.

c) Argue whether the previous results would be different in any of the two cases for any $x \neq 1/2$.

(Hint: For your answer you may consider, e.g., the maximum or minimum values that can be taken by the random variable x)

d) Let $H_{K,N}$ be the probability obtained in section a), and consider the following alternatives:

A1. N = K/10
A2. $N = K^{1/2}$

Determine in which of the these cases it is satisfied that $\lim_{K\to\infty} H_{N,K} = 0$.

Hint: $\lim_{z\to\infty}\left(1-\frac{1}{z}\right)^z = \frac{1}{e}$ $(e \approx 2.72)$

(30 min; 1p)

T2.- Indicate, justifying <u>very briefly</u> your answer, whether the following statements are true or false:

a) Global search methods always achieve strictly better solutions than those obtained with local search, in terms of minimizing the cost function.

b) The NLMS algorithm provides a solution to the zig-zag search problem that is suffered by conventional LMS.

c) Conjugate gradient algorithms select search directions that point to the minimum of the cost function.

d) The fulfillment of the conjugate condition in conjugate gradient algorithms allows the selection, at each iteration, of the optimum value for the step size.

e) We wish to solve an estimation problem applying learning rules based on stochastic gradient methods. Knowing that the a posteriori distribution of the variable to be estimated is Gaussian, invariance principles guarantee that it is equivalent to use gradient rules based on quadratic or absolute cost.

(15 min; 1 p)

T3.- Consider two unidimensional random variables s and x. Random variable x follows the following probability density function:

$$p(x) = \begin{cases} 1/2 & |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

It is known that the minimum mean square error estimator of s given x is given by

$$\hat{s}_{mse}(x) = -\frac{1}{2}\operatorname{sign}(x) = \begin{cases} \dfrac{-1}{2} & x \geq 0 \\ \dfrac{1}{2} & x < 0 \end{cases}$$

It is also known that the mean square error of $\hat{s}_{mse}(x)$ is 1/12. However, for practical reasons, a different estimator is to be used:

$$\hat{s}_1 = -x$$

a) Obtain the bias of estimator $\hat{s}_1$.

b) Calculate the following mathematical expectations: $E\{s \cdot x\}$ and $E\{s^2\}$

c) Compute the mean square error incurred by estimator $\hat{s}_1$.

(30 min; 1 p)

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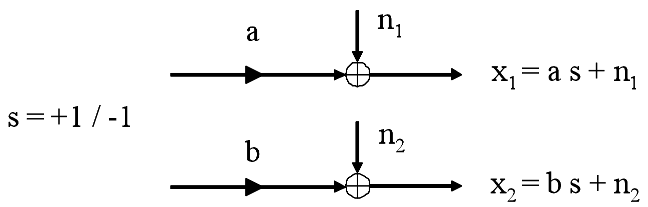
PROBLEMS

(Time: 135 minutes. Mark: 5/8)

P1.- Consider a communication system in which symbols "+1" or "-1" are simultaneously transmitted through two noisy channels, as illustrated in the figure:



$$\mathbf{n}=\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \sim G\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix},\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right]$$

a and b being two unknown constants which characterize the channels. It is also known that the probabilities of transmitting both symbols are the same.

a) If we wish to design a decider to discriminate which was the transmitted symbol, using only one of the two available observations, x_1 or x_2, indicate which of the two variables you would use. Justify your answer as a function of the values of the constants. Provide also the analytical expression for the corresponding ML decider.

b) Design the binary classifier with minimum probability of error, based on the joint observation of x_1 and x_2. Express your result as a function of a, b and ρ, and simplify the expression as much as possible.

c) For $\rho = 0$, calculate the probability of error of the decider obtained in section b). Use the complementary error function to simplify your result:

$$\operatorname{erfc}(u)=\int_{-\infty}^{u}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{t^2}{2}\right)dt$$

d) To adjust constants a, b and ρ, a training sequence is initially transmitted, so that for the design we have access to a set $\left\{x_1^{(k)},x_2^{(k)}\right\}_{k=1}^{K_1}$ of observations corresponding to K_1 independent transmissions of symbol "+1", plus another set $\left\{x_1^{(k)},x_2^{(k)}\right\}_{k=1}^{K_{-1}}$ of K_{-1} observations corresponding to independent transmissions of symbol "-1". Obtain the ML estimators of a, b and ρ, using the available observations.

e) During its normal operation, the decider receives new observations, and classify them according to the criterion obtained in b), using ML estimations for the unknown constants. Explain how you would use these new observations to readjust the values of a, b and ρ. Discuss in which circumstances the proposed procedure would allow a reduction of the probability of error of the classifier, and in which circumstances it would result in performance degradation.

(75 min; 2.5 p)

P2.- We wish to design a low-complexity linear regression model for a random variable s. It is known that s depends on three other random variables x_1, x_2 and x_3, which will act as the observations in the regression model. The following table contains four independent realizations of the random process.

x_1	x_2	x_3	s
3	-1	0	-1
-2	0	1	-2
0	-1	2	0
-1	2	-3	3

The objective of this problem is to evaluate two different strategies to build the aforementioned low-complexity regression model:

- The first strategy consists in designing an exact regression model of minimum quadratic error, using just two of the available observations.
- The second strategy consists in an approximation of the minimum mean square error linear estimator using all three variables. For the approximation, we will assume that the covariance matrix of the observations is diagonal.

a) Determine the two variables to be included in the regression model of the first strategy. Variable selection is carried out in two steps: first, we choose the observation variable whose sample covariance (i.e., estimated from the data) with s has the largest absolute value; the second selected variable will be that having the smallest absolute value of the sample covariance with the observation selected in the first step.

b) Obtain the linear regression model for s which minimizes the mean square error, using just the two variables that were selected in the previous step.

c) Obtain now the approximate linear estimator specified for the second strategy. In order to do so, use the available data to first estimate the diagonal elements of the observations covariance matrix, as well as the covariances between the observations and variable s.

d) Which of the two proposed designs incurs in a smaller average quadratic error across the available data?

(60 min; 2.5 p)