Department of Signal Theory and Communications **06/05/09**
UCIIIM

DIGITAL SIGNAL PROCESSING

THEORY
(Time: 55 minutes. Mark: 3/8)

T1.- Consider the two following equally probable hypotheses:

H_0: $x = n$

H_1: $x = n + a \cdot s$

where n and s are two independent Gaussian random variables, with zero means and variances v_n and v_s, respectively, and a is a known constant.

a) Verify that the minimum error probability test can be expressed as

$$c_1 \exp\left(c_2 x^2\right) \gtrless \eta$$

and provide expressions for constants c_1 and c_2, indicating the associated decision criterion.

b) Determine the decision regions over x. Note that such regions can be expressed as functions of constants c_1 and c_2.

(15 min; 1p)

T2.- Explain the working principles of the k-Nearest Neighbors (k-NN) method for pdf estimation, and how it can be applied to the estimation of a posteriori class probabilities (i.e., $P(H_i \mid x)$) in decision problems.

(15 min; 1 p)

T3.- Let s and x be two random variables with joint probability density function

$$p(x,s) = \left(\frac{1}{2} + 2xs\right), \qquad 0 \le x,s \le 1$$

We would like to estimate s given x using a linear estimator $\hat{s} = wx$, following the criterion of minimum expected cost, for a cost function

$$C(s,\hat{s}) = (s - \hat{s})^2$$

a) Obtain an expression for the mean cost as a function of w.

b) Obtain an expression for the steepest descent learning rule that minimizes the mean cost.

c) Assume now that p(x,s) is not known, but we have instead a set of K independent labeled samples, $\{x^{(k)}, s^{(k)}\}$, that can be used to approximate the mean cost through a sample average. Obtain the sequential learning rule that minimizes such average cost.

(25 min; 1 p)

DIGITAL SIGNAL PROCESSING

PROBLEMS

(Time: 120 minutos. Mark: 5/8)

P1.- The joint probability density function of two random variables x and z is given by

$$p(x,z) = x + z, \qquad 0 \le x,z \le 1$$

Consider a decision problem based on the observation of x (but not z) with hypotheses:

H_0: $z < 0'6$

H_1: $z > 0'6$

a) Determine p(z|x).

b) Obtain the MAP decider based on x.

c) Calculate the probability of false alarm for the MAP decider.

d) Obtain the ML decider based on x.

(60 min; 2.5 p)

P2.- A random variable x follows a unilateral exponential pdf with parameter $a > 0$:

$$p(x) = \frac{1}{a}\exp(-x/a), \quad x > 0$$

As it is known, the mean and variance of x are given by a and a^2, respectively.

a) Obtain the maximum likelihood estimator of a, $\hat{a}_{ML}$, from a set of K independent observations of the random variable x, $\left\{x^{(k)}\right\}_{k=1}^{K}$.

b) A new estimator is proposed based on the maximum likelihood one according to:

$$\hat{a} = c \cdot \hat{a}_{ML},$$

where $0 \leq c \leq 1$ is a constant which allows a rescaling of the ML estimator. Obtain the square of the bias, the variance, and the mean square error (MSE) of the new estimator, and plot them together as functions of c.

c) Calculate the value of c that minimizes the MSE, c*, and explain its evolution as the number of observations increases. Compute the MSE of the estimator associated to c*.

d) Obtain now the interval of values of c for which the MSE of $\hat{a}$ is smaller than that of the maximum likelihood estimator, and explain how this interval changes as $K \to \infty$. Discuss your result.

(60 min; 2.5 p)