Department of Signal Theory and Communications **09/16/08**
UCIIIM

DIGITAL SIGNAL PROCESSING

THEORY
(Time: 65 minutes. Mark: 3/8)

T1.- A one dimensional decision problem with three equally probable hypotheses is characterized by the following likelihoods:

$$p(x \mid H_0) = 2\left(1 - 2\left|x - \tfrac{1}{2}\right|\right), \qquad 0<x<1$$
$$p(x \mid H_1) = 1, \qquad 0<x<1$$
$$p(x \mid H_2) = 2x, \qquad 0<x<1$$

a) Obtain the decider with minimum error probability.

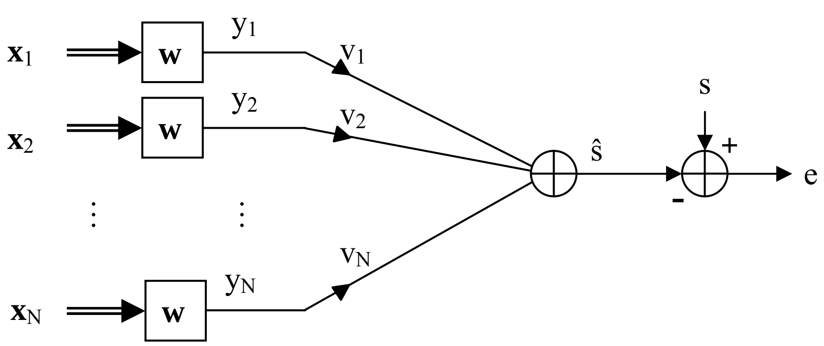
b) Discuss if the previous classifier is equivalent to an alternative one consisting of the two following steps: A first classifier decides, with minimum error probability, between hypotheses H_0 and $H_1 \cup H_2$. Then, in case $H_1 \cup H_2$ is accepted, a second classifier chooses between H_1 and H_2 (again, with minimum error probability).

(25 min; 1p)

T2.- Explain the working principles of parametric, semi-parametric and non-parametric methods for pdf estimation. Illustrate the differences among these methods with an instance of each of them.

(15 min; 1 p)

T3.- The following diagram shows a system for the estimation of a r.v. s from a set of observation vectors, $\mathbf{x}_1, \ldots, \mathbf{x}_N$:



so that $\hat{s} = \sum_{i=1}^{N} v_i \left(\mathbf{w}^T \mathbf{x}_i\right)$.

Provide expressions for the sequential gradient descent rules that allow, using a training set $\{\mathbf{x}_1^{(k)},\ldots, \mathbf{x}_N^{(k)}, s^{(k)}\}$, to search the parameter vectors $\mathbf{v} = [v_1,\ldots,v_N]^T$ and **w** that minimize the mean square error of the estimator.

(25 min; 1 p)

DIGITAL SIGNAL PROCESSING

PROBLEMS

(Time: 165 minutes. Mark: 5/8)

P1.- Consider the estimation of a r.v. s from a r.v. x. The joint pdf of both variables is known, and given by:

$$p(x,s) = \begin{cases} 6x, & 0 \le x \le s \quad 0 \le s \le 1 \\ 0, & \text{otherwise} \end{cases}$$

a) Find the minimum mean square error estimator of s as a function of x, $\hat{s}_{MSE}$.

b) Find the maximum likelihood estimator of s given x, $\hat{s}_{ML}$.

c) Derive analytical expressions for the pdfs of both estimators, $p(\hat{s}_{MSE})$ and $p(\hat{s}_{ML})$. Plot $p(\hat{s}_{MSE})$ and $p(\hat{s}_{ML})$.

d) Calculate the mean and the variance of the errors for both estimators.

(75 min; 2.5 p)

P2.- Consider a one dimensional binary decision problem where it is known that, given the hypothesis H_0, the observed r.v. x follows a symmetric pdf $p(x|H_0)$ with zero mean and variance v. It is also known that $p(x|H_0)$ decreases around its mean. The likelihood for hypothesis H_1 has the same shape of $p(x|H_0)$, but its mean is shifted to $m_1>0$ (i.e., $p(x|H_1) = p(x-m_1|H_0)$). Both hypotheses are equally probable.

a) Design the ML decider.

b) In practice, m_1 is unknown. Therefore, a designer opts for estimating this value using the average (denoted by $\overline{x}_{(1)}$) of K independent observations of x under hypothesis H_1, $\left\{x_{(1)}^{(k)}\right\}_{k=1}^{K}$. From this value, the designer establishes an estimator U of the theoretical threshold, and implements the following classifier:

$$x \underset{D_0}{\overset{D_1}{\gtrless}} U, \qquad \text{if } U>0$$

$$x \underset{D_1}{\overset{D_0}{\gtrless}} U, \qquad \text{if } U<0$$

Justify these decision criteria, and provide an expression for U.

c) Obviously, U is itself a random variable. Calculate the mean and variance of U as a function of the mean and variance of $x|H_1$, m_1 and v.

d) When considering U as a r.v., the application of the previous classifier results in a false alarm in any of the following cases:
– if U>0, when r.v. x is, under hypothesis H_0, above the value of the r.v. U;
– if U<0, when r.v. x is, under hypothesis H_0, smaller than U.
Obtain an expression that would allow computing P_{FA}.

e) When K>>1, and accepting that the Central Limit Theorem can be applied to $p(x|H_0)$, U can be considered Gaussian. Assuming also that $m_1=1$ and

$$p(x\,|\,H_0) = \begin{cases} \frac{1}{2}, & |x|<1 \\ 0, & |x|>1 \end{cases}$$

(note that this pdf is not decreasing around its mean, but the designer ignores this fact), compute P_{FA} for the empirical design (the one based on U).

f) To which value does the error probability of the previous empirical design go, as $K\to\infty$, admitting for symmetry arguments that $P_M \to P_{FA}$? Compare this result with the probability of error of the ML design if m_1 were known, $P_e(T)$.

Remark: The complimentary error function given by the following expression can be used if convenient for the resolution of the problem:

$$\mathrm{erfc}(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left(-\frac{x^2}{2}\right) dx$$

(90 min; 2.5 p)