Radar Signal Processing

Sensitivity of Golay Pairs to Doppler
Doppler Resilient Golay Pairs
Prouhet-Thue-Morse Pulse Train
Sensitivity of Golay Pairs to Doppler

- Pulse train of Golay pairs \((x_0, x_1), \ldots, (x_{N-2}, x_{N-1})\):

- Correlator output in the presence of Doppler shift:

\[
G(k, \theta) = \sum_{n=0}^{N-1} e^{jn\theta} C_{x_n}(k)
\]

where \(\theta = -\omega T\) is the relative Doppler shift over a PRI, and Doppler shift at the chip rate is ignored.

- We call \(G(k, \theta)\) the “composite ambiguity function”.

- Doppler shift perturbs the perfect auto-correlation property and creates range sidelobes:

\[
\sum_{n=0}^{N-1} e^{jn\theta} C_{x_n}(k) \neq NL\delta_{k,0}
\]

Radar Signal Processing
“Although the autocorrelation sidelobe level is zero, the ambiguity function exhibits relatively high sidelobes for nonzero Doppler.”  

**Range Sidelobes Problem:** A weak target located near a strong target can be masked by the range sidelobes of the ambiguity function centered around the strong target.
**Question:** Is it possible to design a *Doppler resilient sequence* of Golay pairs \((x_0, x_1), \ldots, (x_{N-2}, x_{N-1})\) so that for a reasonable range of Doppler shifts

\[
G(k, \theta) = \sum_{n=0}^{N-1} e^{jn\theta} C_{x_n}(k) \approx NL\delta_{k,0}
\]

- Design the Golay pairs so that the composite ambiguity function has a high-order null along \(\theta = 0\).
Doppler Resilient Golay Pairs

- **Approach:** Select the Golay pairs \((x_0, x_1), \ldots, (x_{N-2}, x_{N-1})\) so that in the Taylor expansion of \(G(k, \theta)\) around \(\theta = 0\) all terms up to a certain order, say \(M\), vanish at all nonzero delays (become impulses).

- **Taylor expansion of \(G(k, \theta)\) around \(\theta = 0\):**

  \[
  G(k, \theta) = \sum_{m=0}^{\infty} D_m(k)(j \theta)^m,
  \]

  \[
  D_m(k) = \sum_{n=0}^{N-1} n^m C_{x_n}(k), \quad \text{for } m = 0, 1, 2, 3, \ldots
  \]

- **Objective:** Design \((x_0, x_1), \ldots, (x_{N-2}, x_{N-1})\) so that \(D_m(k)\), \(m = 1, \ldots, M\) vanish at all nonzero delays.
Transmit 2 Golay pairs \((x_0, x_1)\) and \((x_2, x_3)\) over 4 PRIs.

Making \(D_1(k)\) vanish:

\[
D_1(k) = \underbrace{0C_{x_0}(k) + C_{x_1}(k)}_{1C_{x_1}(k)} + \underbrace{2C_{x_2}(k) + 3C_{x_3}(k)}_{2 \times 2L\delta_{k,0} + 1C_{x_3}(k)}
\]

\[3 \times 2L\delta_{k,0}\]

**Condition:** Golay pairs \((x_0, x_1)\) and \((x_2, x_3)\) must be selected such that \((x_1, x_3)\) is also a Golay pair.

**Example:**

\[
\begin{array}{cccc}
  x_0 & x_1 & x_2 & x_3 \\
  x & y & y & x
\end{array}
\]

where \((x, y)\) is an arbitrary Golay pair.
Transmit 4 Golay pairs \((x_0, x_1), \ldots, (x_6, x_7)\) over 8 PRIs.

Making \(D_1(k)\) vanish:

\[
\begin{align*}
0C_{x_0}(k) &+ 1C_{x_1}(k) &+ 2C_{x_2}(k) &+ 3C_{x_3}(k) &+ 4C_{x_4}(k) &+ 5C_{x_5}(k) &+ 6C_{x_6}(k) &+ 7C_{x_7}(k) \\
2 \times 2L\delta_{k,0} &+ 4 \times 2L\delta_{k,0} &+ 6 \times 2L\delta_{k,0} \\
[(1 - 0) = 1]C_{x_1}(k) &+ [(3 - 2) = 1]C_{x_3}(k) &+ [(5 - 4) = 1]C_{x_5}(k) &+ [(7 - 6) = 1]C_{x_7}(k) \\
3 \times 2L\delta_{k,0} &+ 11 \times 2L\delta_{k,0}
\end{align*}
\]

**Condition:** Golay pairs must be selected such that \((x_1, x_3)\) and \((x_5, x_7)\) are also Golay pairs.
Making $D_2(k)$ vanish:

\[
\begin{align*}
0^2C_{x_0}(k) + 1^2C_{x_1}(k) + 2^2C_{x_2}(k) + 3^2C_{x_3}(k) + 4^2C_{x_4}(k) + 5^2C_{x_5}(k) + 6^2C_{x_6}(k) + 7^2C_{x_7}(k) & \quad 0^2C_{x_0}(k) + 1^2C_{x_1}(k) + 2^2C_{x_2}(k) + 3^2C_{x_3}(k) + 4^2C_{x_4}(k) + 5^2C_{x_5}(k) + 6^2C_{x_6}(k) + 7^2C_{x_7}(k) \\
4 \times 2L\delta_{k,0} + [(1^2 - 0^2) = 1]C_{x_1}(k) + [(3^2 - 2^2) = 5]C_{x_3}(k) & \quad 16 \times 2L\delta_{k,0} + [(5^2 - 4^2) = 9]C_{x_5}(k) + [(7^2 - 6^2) = 13]C_{x_7}(k) \\
5 \times 2L\delta_{k,0} + [(3^2 - 2^2 - 1^2 + 0^2) = 4]C_{x_3}(k) & \quad 5 \times 2L\delta_{k,0} + [(3^2 - 2^2 - 1^2 + 0^2) = 4]C_{x_3}(k) \\
[0^2 + 1^2 + 2^2 + \ldots + 7^2) = 70] \times 2L\delta_{k,0} & \quad 61 \times 2L\delta_{k,0} + [(7^2 - 6^2 - 5^2 + 4^2) = 4]C_{x_7}(k)
\end{align*}
\]

**Condition:** Golay pairs must be selected such that $(x_1, x_3)$, $(x_5, x_7)$, and $(x_3, x_7)$ are also Golay pairs.

**Example:**

\[
\begin{array}{cccccccc}
x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\
x & y & y & x & y & x & x & y
\end{array}
\]
Is there a Pattern? Yes, it’s the Prouhet-Thue-Morse sequence!

1st order: PTM sequence of length $4 = 2^{1+1}$

$$
\begin{array}{ccccc}
  x_0 & x_1 & x_2 & x_3 \\
  x & y & y & x \\
  0 & 1 & 1 & 0 \\
\end{array}
$$

2nd order: PTM sequence of length $8 = 2^{2+1}$

$$
\begin{array}{cccccccc}
  x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\
  x & y & y & x & y & x & x & y \\
  0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
\end{array}
$$
PTM Pulse Train: Up to Order $M$

- **Prouhet-Thue-Morse Sequence:** The $n$th term in the PTM sequence $p_n$ is the sum of the binary digits of $n$ mod 2:

<table>
<thead>
<tr>
<th>$n$</th>
<th>(0)=0000</th>
<th>(1)=0001</th>
<th>(2)=0010</th>
<th>(3)=0011</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_n$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

- **Theorem:** To zero-force up to $M$ Taylor moments, coordinate the transmission of a Golay pair $(x, y)$ according to the length $N = 2^{M+1}$ PTM sequence, with 0 locations corresponding to $x$ and 1 locations corresponding to $y$.

- The result is the **PTM pulse train**, which is resilient to modest Dopplers.
Length-256 PTM Pulse Train: Zero-Forcing 7 Moments

Parameters: $f_0 = 17$ GHz and $T = 0.5 \mu$sec, $T_c = 100$ nsec.
By transmitting a Golay pair according to the PTM sequence we can clear out the range sidelobes along modest Doppler shifts.
Why PTM Sequence?

- Look at the calculations for zero-forcing the 1st and 2nd order moments.

- Key is partitioning of $S = \{0, 1, \ldots, 7\}$ into disjoint subsets $S_0 = \{0, 3, 5, 6\}$ and $S_1 = \{1, 2, 4, 7\}$ that satisfy

  $$(0^m + 3^m + 5^m + 6^m) - (1^m + 2^m + 4^m + 7^m) = 0, \text{ for } m = 1, 2.$$ 

- **Prouhet’s Problem:** Let $S = \{0, 1, \ldots, N - 1\}$. Given $M$, is it possible to partition $S$ into two disjoint subsets $S_0$ and $S_1$ such that

  $$\sum_{r \in S_0} r^m = \sum_{q \in S_1} q^m$$

  for all $0 \leq m \leq M$?

**Solution:** Possible when $N = 2^{M+1}$. The partitions are identified by the PTM sequence.
References


