Sparse Recovery with Non-uniform Chirp Sampling

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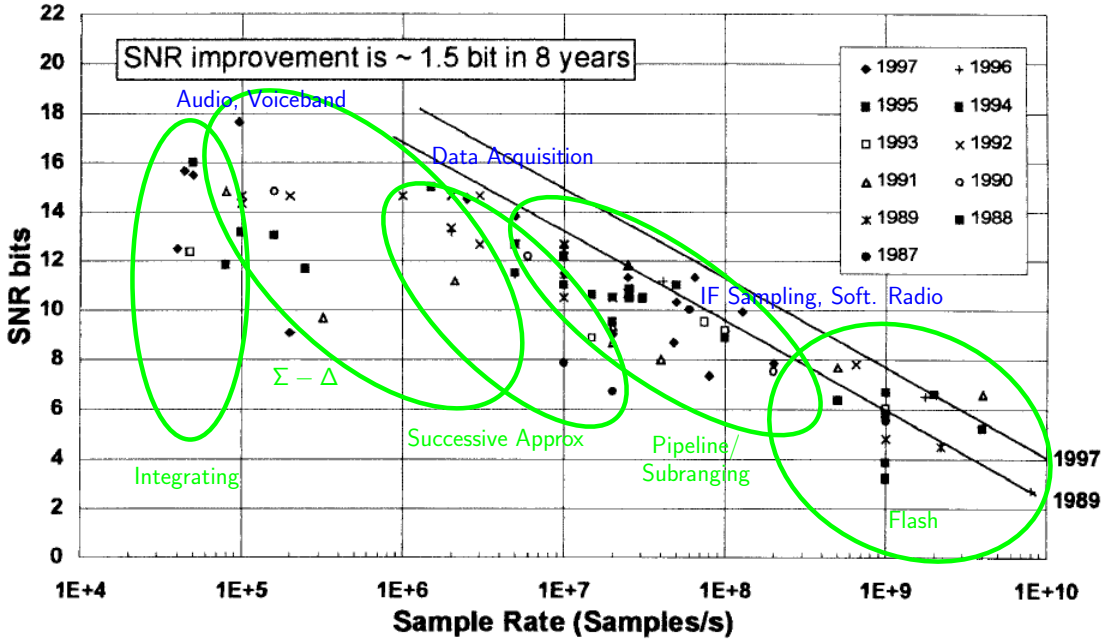
A/D Conversion Metrics and Progress

- Standard Performance metric

$$P = 2^{\text{SNR Bits}} f_{sampling}$$

  - Captures bandwidth/resolution trade-off. Eg: $\Delta\Sigma$ modulation

- [Walden (1999)] Slow rate of progress: 1.5 bit increase / 8 Years

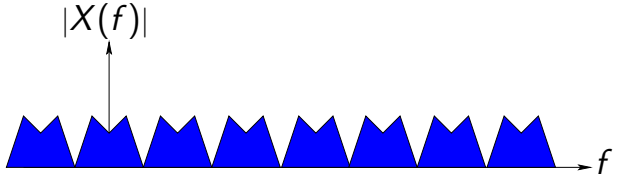


A/D Design Past and Future

- Traditional signal model

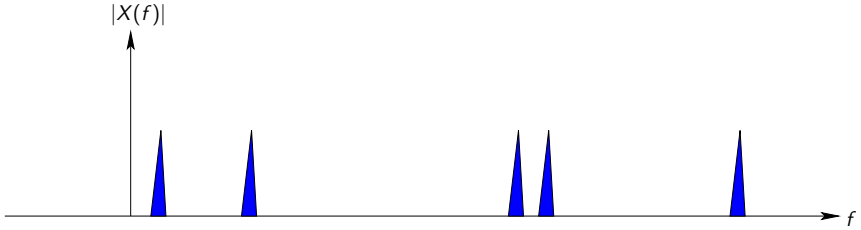
Assumption:
Bandlimited signals

$\longrightarrow$



Implied requirements:
Nyquist converter

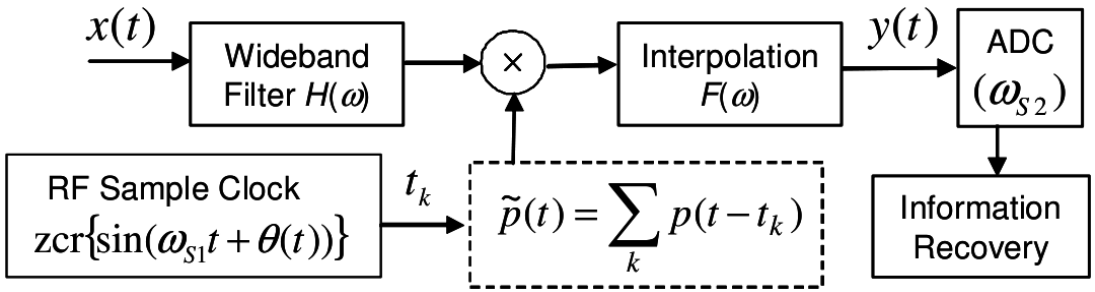
- New signal model



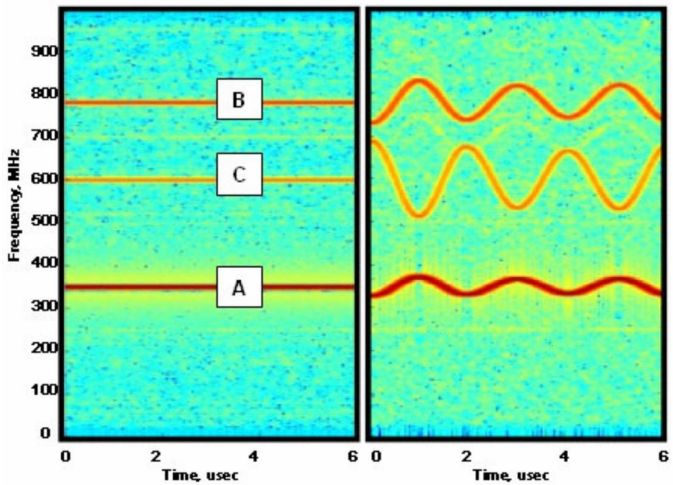
  - Assumption: sparse signals under a transformation
  - Use more general measurements than uniform sampling

Nyquist Folding Analog-to-Information Receiver

G. Fudge et al.



- Sampling at zero-crossing of a phase-modulated signal
- Undersampling aliases "Nyquist zones" together
- Stretching/reflection of phase-modulation resolves "Nyquist zone"
- Recovery visualized by spectrogram



Our Approach: Chirp Sampling

- Intended application: Sparse Fourier signals in a large bandwidth

- Chirp Sample times: $t_n = nT + n^2 T\epsilon$
  - Converts pure tones to linear chirps

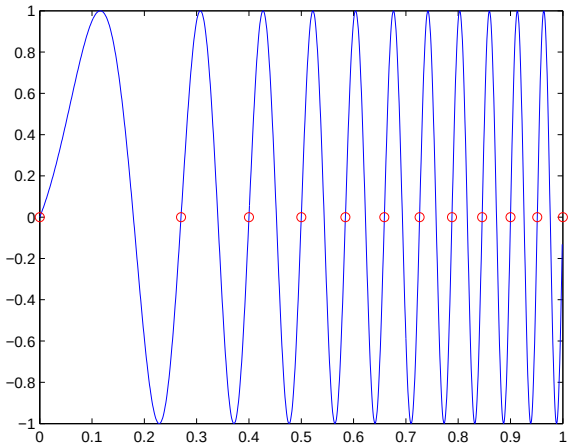
$$e^{j\omega t} \quad \longrightarrow \quad e^{j\omega Tn + j\omega T\epsilon n^2}$$

  - Can pick $T > \frac{1}{\text{bandwidth}}$ (under sampling)

- Motivations:
  - Frequency estimation: improved and achievable Cramér-Rao Bound
  - Recoverability conditions via compressed sensing
  - Simple signal recovery algorithm

Chirps to Chirps

Positive edged zero crossings of continuous chirps $\Rightarrow$ Chirp Sampling



$e^{j2\pi ft + j2\pi rt^2}$ has zero crossings when $rt^2 + ft = n \in \mathbb{Z}$

$$t_n = \frac{f}{2r}\left[-1 + \sqrt{1 + \frac{4rn}{f^2}}\right]$$
$$\approx \frac{1}{f}n - \frac{8r}{f^3}n^2 \quad \text{when} \quad rN \ll f^2$$

Use of low bandwidth chirps leads to chirp sampling in Nyquist Folding Receiver

Rife and Boorstyn: Frequency Estimation Bounds from Uniform Sampling

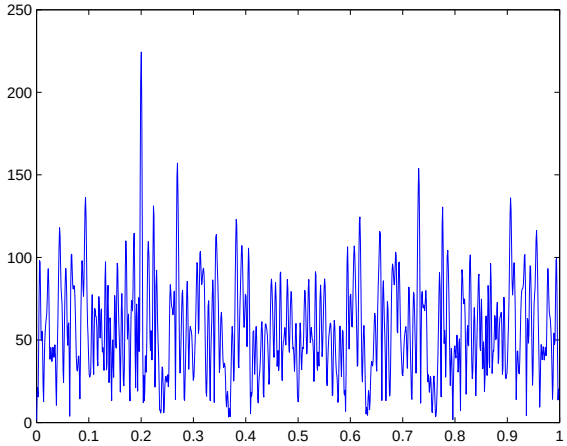
- Derived Cramér-Rao lower bound for estimating single unknown frequency in noise

$$\mathrm{var}\{\hat{\omega}\} \geq \frac{1}{\mathrm{SNR}} \frac{12}{T^2 M(M^2-1)}$$

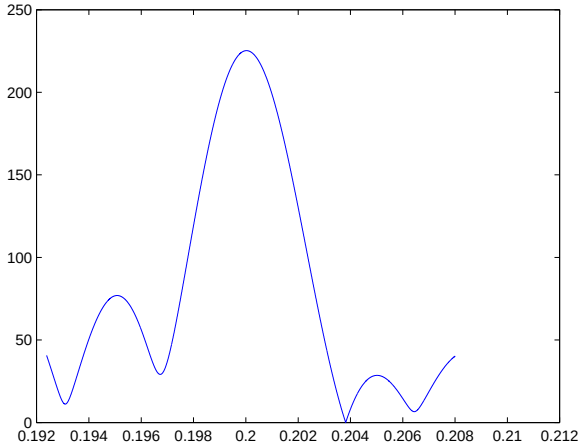
  for samples $nT, n = 0, \ldots, M-1$

- Lower Bound Achieving Algorithm

Initial Coarse FFT Estimate



Local Refinement



Estimation Bounds from Arbitrary Sample Times

For M signal samples in noise given by

$$Z_n = be^{j\omega t_n} + W_n \qquad W_n \sim \mathcal{N}_C(0, 2\sigma^2)$$

Cramér-Rao lower bound on $\hat{\omega}$ with unknown b

$$\text{var}\{\hat{\omega}\} \geq \frac{\sigma^2}{|b|^2(Q - P^2/M)}$$

where

$$P = \sum_{n=0}^{M-1} t_n \quad , \quad Q = \sum_{n=0}^{M-1} t_n^2$$

Uniform sampling: $t_n = nT$

$$\text{var}\{\hat{\omega}\} \geq \frac{\sigma^2}{|b|^2} \frac{12}{T^2 M(M^2 - 1)}$$

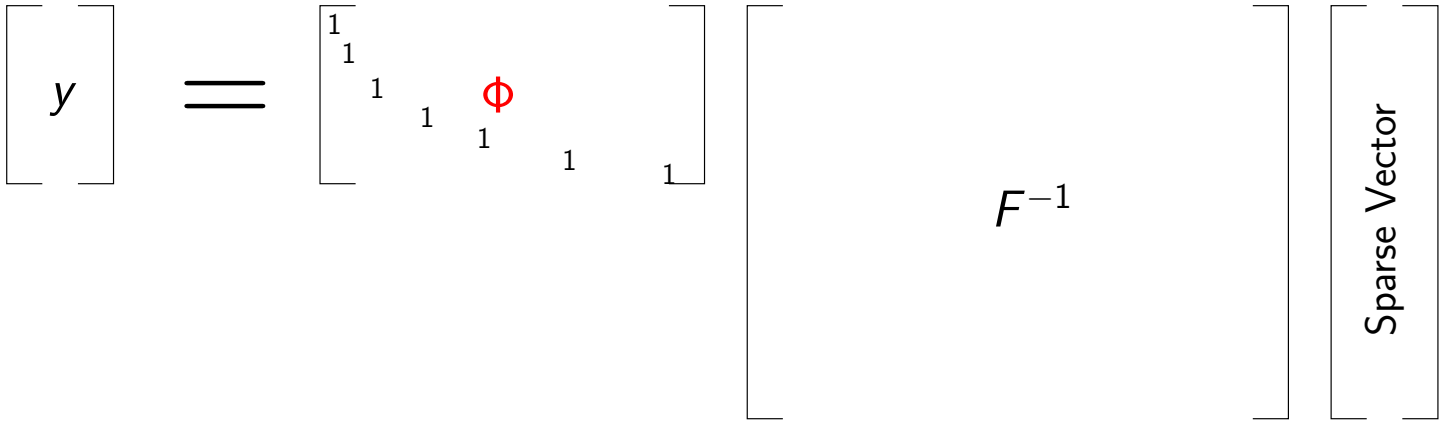
Chirp sampling: $t_n = nT + n^2 T\epsilon$

$$\text{var}\{\hat{\omega}\} \geq \frac{\sigma^2}{|b|^2} \frac{1}{T^2 O(\epsilon^2 M^5)}$$

CRLB is Achievable with good SNR

Leveraging Compressed Sensing Work

Can model chirp sampling of sinsoids using sampling matrix Φ and sparsifying basis F^{-1}



- The matrix $\Theta = \Phi F^{-1}$ has discrete chirp signals as columns
- Similar to a re-ordering of compressed sensing "Chirp Codes"

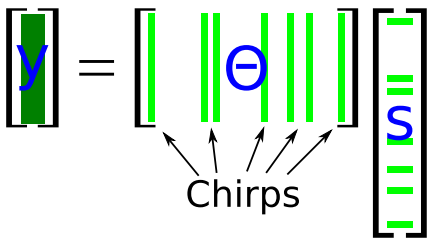
Recovery Guarantees of Chirp Codes

Compressed Sensing: Restricted Isometry Properties of Matrices Θ

$$(1-\delta_M) \leq \text{Eig}\{\Theta_\Gamma^H \Theta_\Gamma\} \leq (1+\delta_M) \quad \forall \quad \text{column sets } |\Gamma| = M$$

Example use: $\delta_{2M} < 1 \Rightarrow$ all M-sparse signals s are uniquely determined by $y = \Theta s$.

Chirps Codes: $\Theta \in \mathbb{C}^{p \times p^2}$

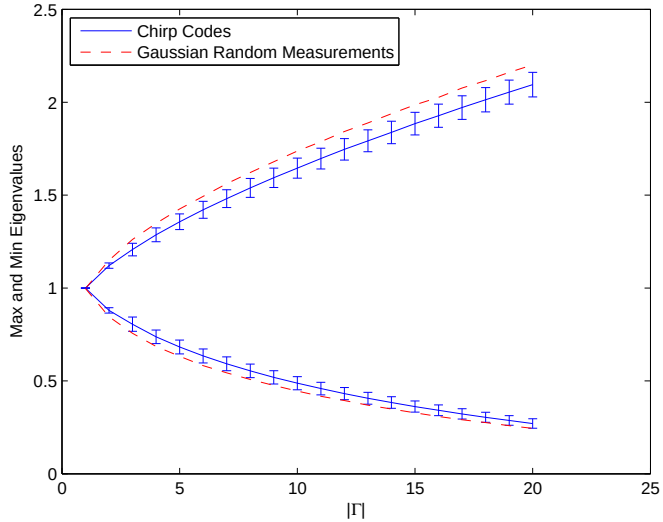


Deterministic Guarantee

All M sparse signals are uniquely determined when

$$M < \frac{\sqrt{P}+1}{2}$$

Stochastically as good as Gaussian Measurements



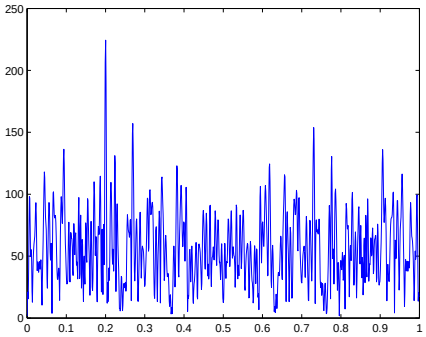
Recovery from Chirp Sampling

- Can leverage efficiency of FFT after simple transform, converting chirps to tones

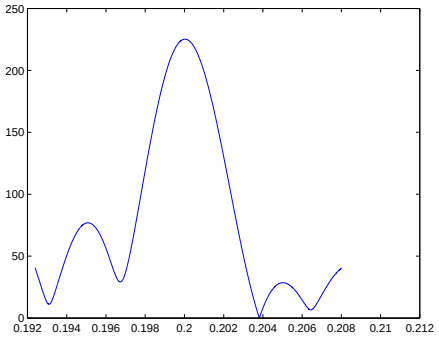
$$
\begin{aligned}
f[n] &= \bar{y}[n]y[n+D] \\
&= |b_1| e^{j\omega_1 D^2 T\epsilon + j\omega_1 DT} e^{j2\omega_1 Dn} + |b_2| e^{j\omega_2 D^2 T\epsilon + j\omega_2 DT} e^{j2\omega_2 DT\epsilon n} \\
&\quad + \cdots + \text{cross terms}
\end{aligned}
$$

- FFT upon $f[n]$ gives initial estimates of ω_i from which we can narrow the search

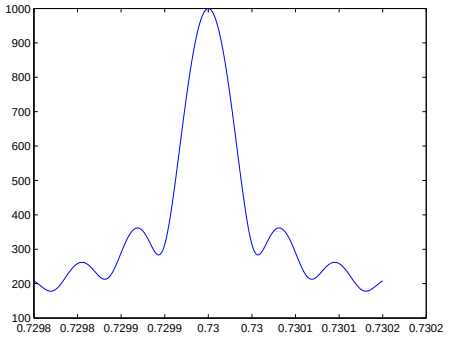
Initial FFT on $f[n]$



Rife Boorstyn refinement on ω_i from $f[n]$



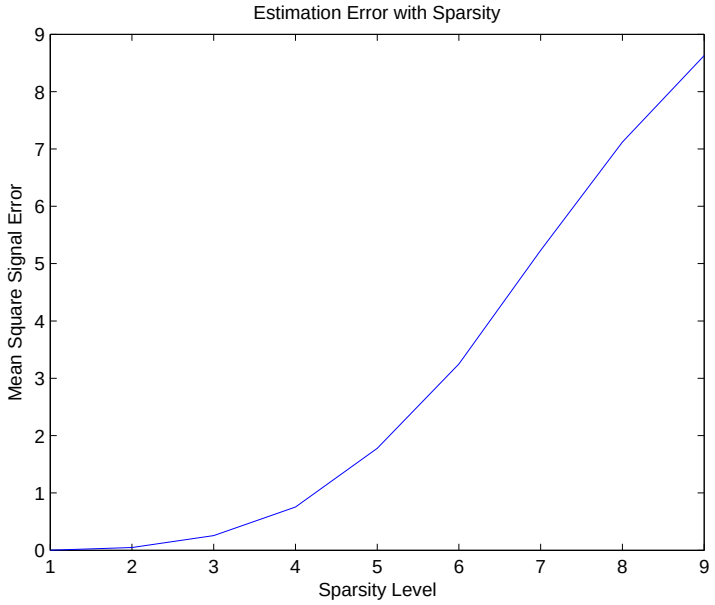
Final refinement on original samples $y[n]$



Simulation Results

	Parameter	Value
M	Samples	256
T	Sample Rate	1/200 s
ϵ	Relative Chirp rate	1/10
$f_{\max}$	Observed BW	1000 Hz

5× undersampling (complex tones)



Summary

- Goal: High fidelity recovery of large bandwidth signals
- Progress on Nyquist converter A/D is slow
- What we know
  - Benefits of continuous time chirp sampling for single frequency estimation
  - Sparse signal recovery conditions from dictionaries of discrete chirps (compressed sensing & chirp codes)
  - Algorithm for recovery from chirp samples
- Work in progress
  - Modelling chirp sampling in a compressed sensing framework
  - Cross-over theory for recovery conditions of continuous time chirps