The HDSL Environment
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Abstract—This paper presents a tutorial on the physical environment in which high bit-rate digital subscriber line (HDSL) transceivers will have to evolve and succeed. Special attention is given to the most damaging impairments that are encountered in subscriber lines, such as propagation loss, linear distortion, crosstalk, bridged taps, and impulse noise. Somewhat less important impairments, such as change of gauge, temperature variation, and thermal noise, are also briefly described. Finally, the paper concludes with a discussion of the capacity of a twisted-pair channel in a crosstalk-dominated environment.

I. INTRODUCTION

The high bit-rate digital subscriber line (HDSL) technology will provide digital access, at the T1 rate of 1.544 Mb/s, over the existing nonloaded loop plant in a repeaterless POTS-like fashion. The HDSL application will be restricted to loops meeting the carrier serving area (CSA) guidelines. The maximum length for this kind of subscriber line is 12 kft, including bridged taps. Research conducted at AT&T Bell Laboratories, Stanford University, and some other institutions has indicated that, in a CSA environment, optimum performance in the presence of crosstalk can be achieved with bandwidth-efficient transceivers that utilize a channel bandwidth going from dc up to about 300 ± 50 kHz.

The 300 kHz bandwidth required for the HDSL application is about four times larger than the 80 kHz bandwidth used for ISDN's basic access, and about five times smaller than the 1.5 MHz bandwidth used for conventional, repeatered T1 service. As a result, the HDSL frequency band has its own idiosyncrasies that set it apart from the frequency bands used for the ISDN and T1 applications. For example, the 300 kHz band completely overlaps the transition region, on the frequency axis, where a loop's loss characteristic departs from the characteristic that is encountered at the lower (<20 kHz) and higher (>200 kHz) frequencies. Also, while the performance of ISDN transceivers is only affected by long bridged taps, HDSL transceivers, on the other hand, will have to contend with average-length (~1 kft) bridged taps, which introduce noticeable channel distortion around 150 kHz.

The purpose of this paper is to present a comprehensive description of the channel characteristics that can influence the performance and/or cost of implementation of HDSL transceivers. An attempt has been made to present, in a unified fashion, various results and data which have been published in the open literature, either in books on digital transmission [1]–[4] or in articles dealing with ISDN's basic access [5]–[12]. Various new experimental results and data are also provided whenever possible. The rest of the material is organized as follows. A brief description of the characteristics of the cables that are used in the U.S. domestic loop plant is given in the next section. Linear impairments, such as propagation loss, amplitude, phase, and delay distortion, are discussed in Section III. The issue of near- and far-end crosstalk modeling is discussed in Section IV, and Section V describes the effects of bridged taps. A brief overview of the characteristics of impulse noise is given in Section VI. Various other channel impairments, such as change of gauge, temperature variations, and thermal noise are discussed in Section VII. Finally, some results on the Shannon capacity of a twisted-pair channel in a NEXT- or FEXT-dominated environment are presented in Section VIII.

II. CABLE CHARACTERISTICS

The description of actual cables that are used in the loop plant can be quite involved, as shown in Table I. A typical cable consists of the concatenation of several cable sections, which are connected in manholes, for example. These sections may or may not have the same gauge or insulation, and their length can typically vary from a couple of hundred feet to about one kilofoot. The upper left entry in the table says that the cable section between the central office (CO) and manhole #428 (MH-428) is a 261-foot-long 26-gauge polyethylene-insulated cable (PIC). Notice in Table I that the gauge used near the CO is smaller than the gauge used near the subscriber location. This is typical of most cables used in the loop plant. Also, bridged gaps, when they exist, tend to be located near the customer location (see Section V). A typical loop is usually terminated by some additional wiring at the CO and subscriber locations. These additional wires can have lengths that vary from a couple of feet to a couple of kilofoot, and they tend to be of a lesser quality than the twisted pairs used in the loop plant.

The twisted pairs within a cable are usually numbered and have different twist lengths. Fig. 1 shows a typical lay-up by pair number of even-count 25-pair PIC cables and units. Commonly used twist lengths and twist frequencies for such a 25-pair unit are given in Table II. Pairs with different twist lengths have different loss and crosstalk characteristics. Typically, shorter twists provide...
better crosstalk immunity (see Section IV) but tend to introduce slightly more propagation loss. Notice from Table II and Fig. 1 that neighboring pairs have different twist lengths.

The last surveys on loop length, for the GTE and regional Bell operating companies (RBOC's) loop plants, were conducted in 1982 and 1983, respectively [8], [9]. (At the time of the survey, the RBOC's loop plant had about 73 million nonloaded loops.) The main results of these two surveys are summarized in the first two rows of Table III. Notice that the average length, $d$, of nonloaded loops varies from 8785 ft for the GTE loop plant to 7535 ft for the RBOC's loop plant. Table III also shows that only 25% and 15% of the loops in the two surveys have lengths larger than 12 kft and that, in both cases, 2% of the nonloaded loops have lengths larger than 18 kft. Some more localized studies on loop length statistics have also been conducted by several RBOC's such as NYNEX [10]. The main results of the 1985 NYNEX survey are given in the last two rows of Table III. It is interesting to note that this survey did not find any loop in the Manhattan area that was longer than 10 kft.

### III. LINEAR IMPAIRMENTS

Propagation loss and linear (amplitude, phase, and delay) distortion are reasonably well understood impairments for the twisted-pair channel. Thus, these two types of impairments will be discussed first. The dependency of these impairments on physical parameters, such as loop length and gauge, mismatch of impedances, frequency, and so forth, is discussed at great length in [1]. Some of the results in [1] are repeated here and specialized to the frequency band of interest, which is in the 0–500 kHz range.

#### A. Propagation Loss

We first discuss the propagation loss experienced in perfectly terminated single-gauge loops. If $R$, $L$, and $C$ are the primary constants of a cable and $\omega = 2\pi f$, where $f$ is the frequency, then

$$\gamma(\omega) = \alpha(\omega) + j\beta(\omega) = \sqrt{(R + j\omega L)(G + j\omega C)}$$

and

$$Z_{0}(\omega) = \frac{R + j\omega L}{\sqrt{G + j\omega C}}$$

(1)

denote the cable's propagation constant and characteristic impedance, respectively. A loop will be said to be perfectly terminated if it is terminated with its characteristic impedance. For a perfectly terminated loop with length $d$, the transfer function $H(d, f)$ is given by

$$H(d, f) = e^{-\gamma d f} = e^{-\alpha d f} e^{-j\beta d f}$$

(2)

1We will also briefly revisit this subject in Section VII when we discuss the effects of temperature variations.
and the attenuation through the cable at a distance $d$ and frequency $f$ is equal to

$$L_{db}(d, f) = -20 \log_{10} |H(d, f)| = \frac{20}{\ln 10} \log f(d)$$

$$= 8.686 \log d(f)$$

where the attenuation is expressed in dB. For convenience, we will interchangeably use the words attenuation and loss to designate the quantity in (3). However, we caution the reader that, according to standard usage, this is only acceptable under the conditions assumed here, which is perfect termination of single-gauge loops. Notice the linear dependency of the loss $L_{db}$ on the length $d$ of the cable. The loss is also an increasing function of frequency as should be apparent from the expression for the propagation constant in (1). Finally, the third important parameter that influences the loss of signal power is the loop's gauge. Table IV provides a listing of primary constants and computed secondary constants for a typical 24-gauge PIC cable. All the constants have been normalized to a cable length of one mile. Computed values for (3), with $d = 1$ mi, can be found in the second column from the right.

It is sometimes convenient to define loss equivalence ratios, at some frequency, between cables having different gauges and/or insulations. Table V shows some computed ratios, which allow us to determine the length of a 24-gauge PIC cable that would introduce the same loss as a 19, 22, or 26 AWG cable at frequencies of 100 and 150 kHz. Table V, together with (3) and Table IV, can be used to compute first-order estimates for the propagation loss experienced by an HDSL signal whose center frequency is located at 100 or 150 kHz. For example, we find that the cable in Table I, which is 3067 feet long, introduces the same loss as a 4235-foot-long 24-gauge PIC cable, or 42.7 dB, which was close to the measured value. Table V will also be useful when we discuss the channel capacity in Section VIII.

### B. Amplitude and Phase Distortion

We first consider the lower frequency region for which $\omega L < R$, and we assume that $G$ can be neglected, which is the case for the frequency region under consideration here. The propagation constant in (1) can then be rewritten as

$$\gamma = \alpha + j\beta = \sqrt{\gamma RC} \left[ 1 + \frac{(\omega L)^2}{2R^2} + \frac{(\omega L)^4}{8R^4} + \cdots \right]^{1/2} \exp \left[ \frac{\pi}{4} + \frac{\omega L}{2R} \left( \frac{(\omega L)^3}{6R^3} + \cdots \right) \right]$$

Fig. 2. Measured spectra of transmitted and received carrierless AM/PM signals.
which, for small enough frequencies, can be further simplified to

$$\gamma = \alpha + j\beta = \sqrt{\frac{\omega RC}{2} \left[ 1 - \frac{\omega L}{2R} \right] + j\frac{\omega RC}{2} \left[ 1 + \frac{\omega L}{2R} \right]}.$$  \hspace{1cm} (5)

For frequencies less than 10 kHz, both the real and imaginary parts of (5) are approximately proportional to $\sqrt{f}$. At higher frequencies, around 100-150 kHz, the frequency dependency of the primary constants becomes noticeable, except for $C$, and the propagation constant in (1) can be approximated by

$$\gamma = \alpha + j\beta = \frac{R(\omega)}{2} \sqrt{\frac{C}{L(\omega)}} + j\omega\sqrt{CL(\omega)},$$  \hspace{1cm} (6)

$$f > 150 \text{ kHz}.$$  

As is apparent from Table IV, the imaginary part $\beta$ in (6) is approximately a linear function of frequency. The major variations for the real part $\alpha$ are due to the frequency dependency of $R$, which, for large frequencies, becomes proportional to $\sqrt{f}$ because of the skin effect. Thus, the loss function in (3) has a $\sqrt{f}$ characteristic both at low and high frequencies. In the transition region, the loss function decreases at a somewhat slower rate with frequency and is approximately proportional to $f^{1/4}$, as can be verified from Table IV. The transition region is a function of the gauge, as shown in Fig. 3, which has been redrawn from [1] and gives the attenuation in dB/mi for three different gauges. Fig. 4 shows the attenuation characteristic of a 12-kft 24-gauge loop as well as the attenuation curve that would be obtained for a loop having a $\sqrt{f}$ characteristic over the whole frequency band. Typical frequency transition regions are from 20-200 kHz for a 24-gauge cable, and 30-300 kHz for a 26-gauge cable.

C. Phase and Envelope Delay

The phase and envelope delays of a loop will be expressed in seconds per mile and are defined as

- phase delay $= \tau_\phi(\omega) = \beta(\omega)/\omega$ seconds per mile
- envelope delay $= \tau_e(\omega) = d\beta(\omega)/d\omega$ seconds per mile

where the phase characteristic $\beta$ is the imaginary part of the propagation constant in (1). Related quantities are the phase velocity and group velocity which are the inverses of the phase delay and envelope delay, respectively, and are expressed in miles per second (mi/s). Neglecting $G$ again in (1), we can compute the envelope delay by taking the imaginary part of the derivative of the propagation constant $\gamma(\omega)$, and we get

$$\tau_e(\omega) = \frac{d\beta(\omega)}{d\omega} = \text{Im} \left( \frac{d\gamma(\omega)}{d\omega} \right) = \text{Im} \left( \frac{jRC - 2\omega LC}{2\gamma(\omega)} \right).$$  \hspace{1cm} (7)

For low-enough frequencies, we can take the derivative of the imaginary part of (5)

$$\tau_e(\omega) = \frac{d\beta(\omega)}{d\omega} = \frac{RC}{8\omega} \left[ 1 + \frac{3\omega L}{2R} \right] = \frac{1}{2} \tau_\phi(\omega)$$  \hspace{1cm} (8)

$$f < 10 \text{ kHz}.$$  

Notice that the phase and delay distortions become very large when the frequency goes to zero. For large frequencies, we get from the imaginary part of (6)

$$\tau_e(\omega) = \frac{d\beta(\omega)}{d\omega} = \frac{\beta(\omega)}{\omega} = \sqrt{CL},$$  \hspace{1cm} (9)

$$f > 150 \text{ kHz}.$$  

Thus, for large frequencies, the envelope and phase delays are roughly the same and almost independent of frequency. (Notice from Table IV that, for PIC cables, $C$ is
constant with frequency and that \( L \) only varies by about 15% over the 500 kHz frequency range.) Using the preceding results, it is easily shown that the envelope delay assumes its largest values in the voiceband region. At higher frequencies, it is approximately constant and equal to about 8.7 \( \mu s / mi \). Computed values for the envelope delay introduced by a 24-gauge PIC cable are shown at the top of Fig. 5.

IV. NEAR- AND FAR-END CROSSTALK

The word crosstalk generally refers to interference that enters a communication channel, such as a twisted-pair channel, through some coupling path. Fig. 6 shows two examples of crosstalk generated in a multipair cable. On the left-hand side of the figure, a signal \( v_i(t) \) is generated at the input of pair \( j \). This signal, when propagating through the loop, generates two types of crosstalk in pair \( i \). The crosstalk \( x_i(t) \), which appears on the left, is called near-end crosstalk (NEXT). The crosstalk \( x_i(t) \), which appears on the right, is called far-end crosstalk (FEXT).

From a data communication point of view, NEXT, if it exists, is generally more damaging than FEXT. This is because NEXT, unlike FEXT, does not necessarily propagate through a long loop and thus does not experience the corresponding propagation loss. Crosstalk modeling has its roots in Dr. G. Campbell’s pioneering work at the beginning of this century [14]. Since then, numerous papers have been written on crosstalk, and [15]-[23] are only a small sample of the papers that have been published on this subject in the open literature.

A. Single Interferer

We first consider the case where crosstalk is generated by a single interferer, as shown in Fig. 6. Let \( V_i(f) \) be the Fourier transform of the disturbing voltage \( v_i(t) \), and assume that pairs \( i \) and \( j \) in Fig. 6 are perfectly terminated. Following [14]-[16], the Fourier transforms \( X_i(d, f) \) and \( X_d(d, f) \) of the NEXT and FEXT voltages \( x_i(t) \) and \( x_d(t) \) can then be written as

\[
X_i(d, f) = j2\pi f V_i(f) \int_0^d C_i(x, f) e^{-2\pi(f / x) dx} \quad (10)
\]

\[
X_d(d, f) = j2\pi f V_i(f) e^{-\pi(f / dx)} \int_0^d C_d(x, f) dx \quad (11)
\]

where \( x \) is the distance from the disturbing source, \( d \) is the length of the cable, and \( C_i(x, f) \) and \( C_d(x, f) \) are the NEXT and FEXT unbalance functions at distance \( x \) and frequency \( f \). For simplicity, we will assume that these unbalance functions are independent of frequency, although this is not strictly true, especially for frequencies below 100 kHz. The expressions in (10) and (11) can be used to compute the Fourier transforms of crosstalk signals corresponding to various input voltages \( v_i(t) \) and various scenarios of unbalance functions between two loops. For example, (10) is used in [29] to predict the spectral characteristics of a variety of impulse events that can be generated in a digital subscriber loop by a cross-coupled switched voice circuit.

We now derive an expression for the average gain characteristics of a NEXT coupling path as a function of frequency. The model’s derivation starts by first considering a given pair combination in an ensemble of similar cables with the same length \( d \). Under various conditions, it is then generally assumed that the covariance of the NEXT unbalance function \( C_i(x) \) satisfies

\[
< C_i(x) C_i(y) > = k_n \delta(x - y) \quad (12)
\]

where \( k_n \) is a random variable, which varies from pair combination to pair combination [15], [16]. The gain of the NEXT crosstalk path is defined as the magnitude of the quantity in (10) when \( V_i(f) \) is equal to one. Thus, the squared gain of the NEXT coupling path can be written as

\[
|X_i(d, f)|^2 = 4\pi^2 f^2 \int_0^d \int_0^d C_i(x) C_i(y) \cdot e^{-2\pi(f / x) dx} e^{-2\pi(f / y) dy} \quad (13)
\]
for a given pair combination. Taking the expectation of (13) and using (12), the expected value \( G_\delta(d, f) \) of the squared gain then becomes

\[
G_\delta(d, f) := \langle |X_\delta(d, f)|^2 \rangle = 4\pi^2 f^2 k_0 \int_0^d e^{-4a_\delta f x} \, dx
\]

\[
= \frac{\pi^2 f^2}{\alpha(f)} \left[ 1 - e^{-4a_\delta f d} \right] = \frac{\pi^2 k_0}{\xi} f^{3/2}
\]

for the pair combination under consideration. The approximation on the right in (14) is valid when the cable length \( d \) is large and for frequency regions where \( \alpha(f) \) is proportional to \( \sqrt{f} \), and \( \xi \) is a constant of proportionality.

A derivation similar to the one used for NEXT can also be used for FEXT, and we find that the average squared far-end crosstalk gain, \( G_\theta(d, f) \), can be written as

\[
G_\theta(d, f) := \langle |X_\theta(d, f)|^2 \rangle = 4\pi^2 k_0 f^2 \int_0^d \frac{e^{-2a_\theta f x}}{x} \, dx
\]

for a given pair combination, and \( k_0 \) is a random variable.

Measured spectra of crosstalk signals generated by a single interferer through the NEXT coupling path are shown in Figs. 7-9. The measurements were made in 1989 on actual twisted pairs deployed in the Ameritech loop plant. In all three cases, the disturbed pair was the same and was passively terminated. The disturbing signal was generated by a passband 1.544 Mb/s carrierless AM/PM transmitter and had the spectrum shown at the top of Fig. 2. Different NEXT interferers were then observed by moving the disturbing signal from pair to pair in the cable. Typical NEXT interferers had the kind of spectra shown in Fig. 7. The spectrum of the worst-case interferer that was observed is shown in Fig. 8. The NEXT coupling gain for this interferer was very close to the \( f^{3/2} \) characteristic given on the right in (14). The most interesting interferer that was observed had the spectrum shown in Fig. 9. The nulls in this spectrum suggest that there were strong points of unbalance between the disturbed and disturbing pairs used in this particular experiment.

B. Multiple Interferers

We now consider crosstalk generated in a given pair by a multiplicity of interfering pairs. We will assume that all the interfering pairs have input voltages with the same power spectral density \( S(f) \). Expressions of the type given in (14) and (15) still apply to the average NEXT and FEXT coupling gains between the disturbed pair and each individual disturbing pair. Also, because of the addition of a multiplicity of interfering signals, the amplitude distribution of crosstalk tends to become Gaussian. Thus, for long-enough loops, when \( d \) is large in (14), we can write the power spectral density \( S_\nu(f) \) for NEXT as

\[
S_\nu(f) = [S(f)] \bar{G}_\nu(d, f),
\]

\[
\bar{G}_\nu(d, f) = \chi f^2 \alpha(f)^{-1} \approx \chi f^{1/2}
\]

where \( \chi \) is a random variable, which is a function of the disturbed pair under consideration. The expression in (16) has proven to be a very good model for the kind of multi-interferer NEXT that is observed in practice [18]. Table VI shows typical NEXT power sum loss values for various pairs in a 24-gauge 50-pair cable consisting of two 25-pair units of the type shown in Fig. 1. The entries in this table were obtained from measurements made on a 1509-foot-long cable, and these data were then extrapolated to a cable length of 10 kft. (From a NEXT perspective, this is almost equivalent to an infinite length.) Notice that there is a difference of about 7 dB in power sum loss between the worst pair (#35) and the best pair (#42).
Both the twist length of a pair and its location within the cable influence the values given in Table VI. However, twist length tends to be somewhat more important and pairs with shorter twists usually experience less crosstalk, as can be verified from Table II. (Pairs #35 and #42 are the same as pairs #10 and #17, respectively.)

An expression similar to (16) holds for the power spectral density of FEXT, and we get

\[ S_d(f) = \frac{S(f)G_d(d, f)}{G_s(d, f)} = \Psi f^2 \frac{d}{e^{-2 \pi f} d} \]

(17)

where \(\Psi\) is a random variable, which is a function of the disturbed pair under consideration.

A commonly used model for NEXT consists of generating \(S(f)\), on the left in (16), by passing white Gaussian noise through a filter that has the same attenuation characteristics as the shaping filter of the transceiver’s transmitter. The resulting signal is then passed through another filter that has the 15 dB per decade frequency characteristic given on the right in (16), where \(\alpha\) is assumed to be fixed. The constant \(\alpha\) is then determined from measured or simulated NEXT power-sum statistics. Two possible scenarios that have often been used to determine \(\alpha\) are the following:

- loss of 57 dB at 80 kHz, which leads to a value \(\alpha = 2.8 \times 10^{-9}\)
- loss of 60 dB at 80 kHz, which leads to a value \(\alpha = 1.4 \times 10^{-9}\)

where frequencies are assumed to be expressed in kilohertz.

A somewhat more refined model for NEXT has been proposed by Bellcore for standards-related studies [17]. This model is shown in Fig. 10 and was obtained by computer simulations for a 22-gauge 18-kft-long cable with 50 pairs. The three curves in Fig. 10 give the 1% worst-case NEXT loss for 1, 10, and 49 disturbers, respectively. Above 20 kHz, the loss curve for 49 disturbers decreases by 14 dB per decade and can be expressed as \(G_{s,db}(f) = -14 \log_{10} f + 83.6\), which corresponds to a loss of about 57 dB at 80 kHz.

We now give a formal definition for the signal-to-noise ratio (SNR), which will be used when we discuss the capacity of the twisted-pair channel. Referring to Fig. 6, the SNR, in the presence of NEXT, is defined as the ratio between the power spectral density of the received signal, on the left on pair \(i\), and the power spectral density of the additive NEXT interference. We will assume that the input signals for the disturbed and disturbing pairs all have the same power spectral density \(S(f)\). The SNR, for a long-enough cable length \(d\), is then obtained by first multiplying the squared magnitude of (2) by \(S(f)\) and then dividing this product by \(s_i(q)\) in (16), and we get

\[ \text{SNR}_a(d, f) = \frac{S(f)|H(d, f)|^2}{S(f)G_d(d, f)} = \frac{e^{-2\text{db}f}}{\Psi f^2} \]

where the expression on the right holds when \(\alpha(f)\) is proportional to \(\sqrt{f}\). Notice that this expression is an exponentially decreasing function of the cable length \(d\) and is independent of \(S(f)\). Fig. 11 shows contours of equal SNR in the \((d, f)\) plane, which were computed using the second expression from the right in (18). The computations were done for a 24-gauge PIC cable and a value \(\alpha = 1.4 \times 10^{-9}\). Of special interest is the curve corresponding to \(\text{SNR} = 0\ dB\), which gives the useful channel bandwidth for a given cable length [25]. As an example, the useful bandwidth for a 12-kft loop is equal to about 400 kHz.

Using (2) and (17), we get the following expression for the SNR in the presence of FEXT:

\[ \text{SNR}_a(d, f) = \frac{S(f)|H(d, f)|^2}{S(f)G_d(d, f)} = \frac{1}{\Psi f^2 d} \]

(19)

where \(\Psi\) is a random variable. This expression is inversely proportional to the cable length \(d\) and is also independent of \(S(f)\). Furthermore, it is also independent of the loop’s transfer function. Notice that the expression in (19) decreases by 20 dB per decade with frequency and by 10 dB per decade with the cable length. This kind of

2This is the so-called “stationary” model for NEXT. It is called stationary because the second-order statistics of the noise process are independent of time. Another model that has been used in data communication applications consists of simulating NEXT by adding several similar data signals with uncorrelated symbols. When the symbol clocks of all the data signals are synchronized, the second-order statistics of such a noise process become periodic with time. This is the so-called “cyclostationary” model for NEXT, which will not be considered any further in this paper.

3When NEXT is generated by signals that are similar to the signal transmitted on the disturbed pair, it is sometimes convenient to refer to it as “self-NEXT.” In order to distinguish it from NEXT generated by other types of signals, in this paper, we only consider NEXT generated by similar signals, so that there is no need to make a distinction. The same comment applies to FEXT.
behavior has been verified experimentally for frequencies larger than about 20 kHz [19]. Stationary FEXT is usually modeled the same way as stationary NEXT. That is, white Gaussian noise is used as the input to a shaping filter cascaded with another filter having a transfer function whose squared magnitude is given on the right in (17), where $\Psi$ is fixed. A possible scenario that is often used for determining $\Psi$ is the following:

- SNR of 37 dB at 1 MHz for a 2-kft cable, which leads to a value $\Psi \approx 10^{-10}$

where frequencies are assumed to be expressed in kilohertz and loop lengths in kilofeet.

V. BRIDGED TAPS

Bridged taps are open-circuited twisted pairs, which are connected in shunt with working twisted pairs. They are intended to provide plant flexibility for future additions and changes in service demands. It has been estimated that 80% of the nonloaded working loops in the GTE loop plant have bridged taps, and that these bridged taps have an average length $d_{bt}$ of about 1.3 kft, as shown in Table III. Bridged taps tend to be located near the subscriber premises, as suggested by the scatter plots in Fig. 12, which has been reproduced from Fig. 9 in [26]. The points in the figure represent computed image impedances at 160 kHz for various loops from the 1983 RBOC survey. The image impedance is a quantity that is closely related to a loop’s characteristic impedance. A scattered points for the CO locations are clustered around the mean value of $112 - j24.4 \Omega$, which suggests that most of the loops entering a CO are 26-gauge loops and that few bridged taps are present near these locations. On the other hand, the much more widely scattered points for the subscriber locations indicate the presence of bridged taps and/or the usage of a greater variety of gauges near these locations.

Fig. 13 shows two of the damaging effects introduced by a bridged tap. When a signal is transmitted from the left, some of its energy fans out from the main signal path and is reflected back to the bridging location by the open-circuited twisted pair. This reflected signal, which is a delayed and distorted version of the main signal, creates two types of interferences. First, part of it is added to the main signal and will appear as a noisy component to a receiver located on the right of Fig. 13. Second, as shown in the figure, part of the reflected signal will propagate back and will appear as an echo to the transceiver located on the left of the figure. A third damaging effect of bridged taps is a net loss in power for the main signal. This loss of power can partly, but not completely, be explained by the fact that some of the main signal’s energy is dissipated by the spurious signals propagating in the bridged taps. Another mechanism which contributes to the loss of power is related to the “nulls” that bridged taps introduce in the transfer function of a loop. A heuristic explanation of the creation and location of these nulls follows.
Going back to Fig. 13 we see that, for certain frequencies, a sine wave reflected by the bridged tap can arrive at the bridging location with a phase difference of 180° with respect to the corresponding sine wave in the main signal. When this occurs, the two frequency components subtract in amplitude, and the result is a noticeable dip in the overall transfer function of the loop around this frequency. In analogy with fading channels, we will refer to these dips as "nulls," even though the transfer function will generally not be zero at these frequencies. For loops with only one bridged tap, nulls occur at frequencies \( f \), for which the bridged tap’s length \( d \) is equal to an odd number of one quarter of the wavelength. Let \( v(f) = 1/\tau(f) \) and \( \lambda(f) = v(f)/f \) be the phase velocity and wavelength at frequency \( f \), respectively. The condition for the first null can then be written as

\[
 f_0 = \frac{v(f_0)}{\lambda(f_0)} = \frac{v(f_0)}{4d_0} = \frac{1}{4d_0 \gamma_0(f_0)}.
\]

Other nulls occur at frequencies which are equal to \((2k + 1)f_0\), where \( k = 1, 2, 3, \ldots \). For working loops with several bridged taps, the location of the nulls can be heuristically determined by superposition.

The expression in (20) is a highly nonlinear function of \( f_0 \) and is generally useless for an arbitrary bridged tap. However, it can be approximately solved for bridged taps with lengths smaller than a couple of kilofoot. In this case, the nulls occur in a frequency region where the phase delay is approximately constant and equal to about 8.7 \( \mu \)s/mi. (See Section III-C.) Using this value in (20) yields the first engineering rule given below. We also provide two other rules, which can be useful in practice. Unless specified otherwise, the rules assume one single open-circuited bridged tap and perfect termination of a single-gauge working loop.

- Typical bridged taps introduce "nulls" in the loop’s transfer function which are approximately located at odd multiples of

\[
 f_0 = \frac{150}{d_{0,\text{kt}}} = \frac{45}{d_{0,\text{km}}} \text{ kHz}
\]

where \( d_{0,\text{kt}} \) and \( d_{0,\text{km}} \) represent the length of the bridged tap expressed in kilofoot and kilometers, respectively, and \( f_0 \) is expressed in kilohertz.

- The loop’s transfer function is independent of the bridged tap’s location when the loop is perfectly terminated. However, tap location becomes a factor when the terminations are mismatched or when the working portion of the loop has mixed gauges.

- A very long bridged tap with the same characteristics as the working portion of the loop introduces a flat attenuation in the loop’s transfer function that is equal to 2/3, or about 3.5 dB.

Fig. 14 shows measured amplitude and envelope delay curves for an 24-gauge 11-kft-long loop with a 22-gauge 1-kft-long bridged tap. Both curves have a dip around 150 kHz, as predicted by (21). Notice from (21) and Table III that bridged taps with averaged length introduce nulls, which are located slightly below 150 kHz. The last two rules given previously can be proven from the expression of the transfer function of the loop configuration shown in Fig. 13 and Fig. 16(e) in the Appendix. This loop has one bridged tap, and the working portion consists of two sections, with possibly different gauges, which are connected at the bridging location. The transfer function \( H_{li}(f) \) of such an arrangement is derived in the Appendix using the \( ABCD \) matrix representation of a two-port network. (Reference [27] gives another possible approach.)

Let \( d_i \) and \( \gamma_i \) denote the length and propagation constant of the \( i \)th section, respectively. We then have

\[
 H_{li}(f) = H_{oi}(f) = \frac{V_i}{V_i} = \frac{Z_{0i}}{Z_{0i} + Z_{o2} + (Z_{o1})Z_{o1}} e^{-\gamma_1 d_1} e^{-\gamma_2 d_2}
\]

where

\[
 Z_{0i} = Z_{o1} \frac{\cosh(\gamma_1 d_1)}{\sinh(\gamma_1 d_1)}
\]

is the input impedance of the bridged tap, \( Z_{0i} \) is the characteristic impedance of the \( i \)th section, \( i = 1, 2, 3 \), and index 3 refers to the bridged tap. Notice that \( H_{li}(f) \) depends only on the sum \( d_1 + d_2 \) when \( \gamma_1 = \gamma_2 \). That is, the transfer function is independent of the location of the bridged tap when the two working sections in Fig. 13 have the same gauge (second rule). The expressions in (22) and (23) can also be used to prove the third rule. We leave this as an exercise for the reader.
VI. IMPULSE NOISE

In unshielded twisted pairs, impulsive noise can be generated by a variety of man-made equipment and environmental disturbances such as signaling circuits, transmission and switching gear, electrostatic discharges, lightning surges, and so forth [1]. Recent surveys on impulse noise in the loop plant have indicated that, statistically, impulse noise seems to have some reasonably well-defined characteristics. Some of the findings that come out of these surveys are discussed in [24] and can be summarized as follows. Typical impulse noise

- occurs about 1-5 times per minute,
- has peak amplitudes in the 5-20 mV range,
- has most of its energy concentrated below 40 kHz,
- and has a time duration that is in the 30-150 μs range.

There are, of course, many exceptions to these general rules. An extensive study conducted in the Deutsche Bundespost’s loop plant (see [18]) found that the probability density function of the amplitude \( u \) of impulse noise can be approximated by \( p(u) = u_0^2 / u^3 \) over an amplitude range of about 5-40 mV. Thus, the probability that the magnitude of the amplitude \( u \) exceeds some value \( u_0 \) can then be written as

\[
P(|u| > u_0) = 2 \int_{u_0}^{\infty} \left( \frac{u^2}{u_0^3} \right) du = \left( \frac{u_0}{u} \right)^2, \quad u_0 > u_0.
\]

(24)

The hyperbolic distribution in (24) is a special case of a well-known empirical model for impulse noise [1, eq. (31-1)]. Fig. 15 was redrawn from [18, Fig. 6] and shows the measured amplitude probability distribution as well as the curve that would be obtained if the distribution were normal. The distribution curve shown in Fig. 15 follows the hyperbolic distribution in (24) up to about 40 mV. It is suggested in [18] that, above 40 mV, the amplitude probability distribution is better modeled by the function \( u_1 / u_0 \), where \( u_1 = 0.625 \) mV.

The most common and damaging type of impulse noise seems to occur when a disturbed loop shares a common cable sheath with switched voice-frequency pairs. Sharp voltage changes can occur on the analog pairs because of FEXT coupling paths, create spurious, impulsive-like voltages whose amplitude can be quite significant [28]. For example, the disturbing voltage \( v(t) \) in Fig. 6 may be generated by the closure, opening, reopening, or bounces of relays in a CO. Experiments have shown that relay activity always produces relay bounces, and that re-opening events usually happen 5-50 μs after initial closure and may last from 2-20 μs [24], [28].

Impulse noise, which is generated by relay activities, can be modeled mathematically and analyzed to some extent as will be shown in a forthcoming paper [29]. For illustration purposes, assume that the two loops in Fig. 6 are lossless \( R = G = 0 \) and semiinfinite \( d = \infty \) transmission lines, and that the input voltage \( v(t) \) to the NEXT coupling channel is a perfect step function. It is then easily verified from (10) that the impulse generated through the NEXT coupling channel is given by

\[
x(t) = pC_d(p), \quad p = \frac{1}{2\sqrt{LC}}
\]

and is simply proportional to a time-scaled version of the unbalance function \( C_d(x) \). An unbalance at distance \( x \) will determine the amplitude of the impulse \( x(t) \) at time \( t = x^2/\sqrt{LC} \). This simple, idealized example clearly demonstrates the effect of the unbalance function on the time fluctuations of the amplitude of impulses generated through a NEXT coupling channel. For the lossy lines that are encountered in practice, the impulse in (25) would also have an additional damping factor. For example, it is shown in [29] that the average time fluctuation of impulse noise generated through a NEXT coupling channel can be expressed as

\[
\langle |x(t)| \rangle \approx Ke^{-1/2at}I_0^2 at,
\]

(26)

where \( I_0(\cdot) \) is the modified Bessel function of order zero, and (26) is only valid when \( t \) is not too large. Fig. 16 shows computed values of (26), as well as a curve obtained with actual impulses measured in the loop plant.

VII. OTHER IMPAIRMENTS

In addition to the channel impairments described previously, there are some other known impairments for digital transmission systems using unshielded twisted pairs. These impairments do not appear to be overly threatening, and therefore they will not be discussed at great length here.
WERNER: THE HDSL ENVIRONMENT

\[ H(d, f) = \frac{V_f}{V_i} = \frac{Z_o}{Z_o + \alpha} \frac{(1 + \rho_s)e^{-\nu d}}{1 - \rho_s e^{-2\nu d}} \]  

(27)

where

\[ \rho_s = \frac{Z_i - Z_o}{Z_i + Z_o} \]  

(28)

are the reflection coefficients at both ends of the circuit.

For a perfectly terminated loop, we have \( Z_i = Z_o \), so that (27) reduces to (2) except for a factor of one half which is due to the fact that the transfer function in (2) applies to voltages between points B and C in Fig. 18(d). In order to see the effect of one mismatched impedance, we will assume that \( Z_i = Z_o \), so that \( \rho_s = 0 \) in (27). The loss, in dB, introduced by the loop at frequency \( f \) can then be written as

\[ L(d, f) = L_{dB}(d, f) + 20 \log_{10} \left| 1 + \rho_s(f) \right| \]  

(29)

where \( L_{dB}(d, f) \) is the loss in the absence of impedance mismatch. The second term in this expression provides a measure of the amount of energy reflected by the mismatched load impedance.

**Change of temperature** affects the values of the primary constants, especially the resistance \( R \) and, to a lesser degree, the inductance \( L \). For example, for a 24-gauge PIC cable at 150 kHz, \( R \) increases from 295 \( \Omega \)/mi at 0°F at 367 \( \Omega \)/mi at 120°F, or a 25% change. Over the same range of temperature, \( L \) varies from 0.9107 mH/mi to 0.9273 mH/mi, or a 2% change. The other constants, \( C \) and \( G \), are not affected by this temperature variation. From a transmission point of view, the major nuisance created by a change of temperature is its effect on the attenuation characteristics of the loop. In our illustrative example, if the loss of the loop is 30 dB at 0°F it would increase to 36.8 dB at 120°F. This variation in loss is also a function of frequency, as shown by the curves on the bottom of Fig. 5. At high frequencies, a change of temperature affects the delay distortion less than the amplitude distortion. (See (9), for example.) In the previous scenario, at 150 kHz, the envelope delay would change by about 0.08 \( \mu \)s over the 120°F temperature range. However, delay changes can be more pronounced at low frequencies, as shown on the top of Fig. 5.

**Thermal (white) noise** is generated by the Brownian motion of electrons in copper. For a complex impedance \( Z(f) = R(f) + jX(f) \), the open-circuit rms noise voltage in a small frequency band \( df \) is given by

\[ \epsilon_r = \sqrt{4kT_R R(f) df} \]  

(30)

where \( k = 1.3805 \times 10^{-23} \text{ joule/K} \) is Boltzmann's constant, and \( T_R \) is the absolute temperature expressed in kelvins [1, p. 51]. We can use (30) to find an upper bound for the thermal noise generated in the 24-gauge cable that is characterized in Table IV. First, we note that 70°F corresponds to about 294 K. The value that has to be used for \( R(f) \) in (30) is the real part of the loop's input impedance \( Z_{in} \), which is given by

\[ Z_{in} = Z_o \]  

It should be pointed out that the corresponding expression given in eq. (2-17) of Ref. [11] is wrong.
\[ Z_{in} = \frac{Z_I \cosh (\gamma d) + Z_0 \sinh (\gamma d)}{(Z_I/\Zo) \sinh (\gamma d) + \cosh (\gamma d)}. \]  

It is easily verified that \( Z_{in} = Z_i \) for an open-circuited \( (Z_i = \infty) \) and long-enough loop. Notice from Table IV that, for such a loop, the voltage in (30) varies by a factor of about two when going from 1 to 50 kHz. Thus, thermal noise is not white in this frequency region. On the other hand, the noise voltage varies by less than 10% between 50-300 kHz. Therefore, thermal noise is essentially white at these higher frequencies.

We now consider a passband 7 96 IEEE JOURNAL ON SELECTED AREAS IN COMMUNICATIONS. VOL. 9, NO. It is easily verified that \( Z_{in} \) for such a loop, the voltage in (30) varies by a factor of about two when going from 1 to 50 kHz. Thus, thermal noise is not white in this frequency region. On the other hand, the noise voltage varies by less than 10% between 50-300 kHz. Therefore, thermal noise is essentially white at these higher frequencies. We now consider a passband system that uses the frequency band between 10-310 kHz.

Going back to Table III, we see that the maximum value for the real part of \( Z_{in} \) in this frequency band is 182 Ω. Using this value in (30) and integrating over a 300 kHz bandwidth, we get \( e_n = 0.9 \mu V \). This is a small amount and, therefore, thermal noise is not expected to be a major factor for the HDSL application.

Other types of noise, which do not appear to have a clear explanation, also seem to exist in the loop plant. The bottom traces in Figs. 8 and 9 show the power spectral density of background noise that was measured in a central office. When the measurements were made, there were no signals present on any of the loops in the cable. The rms voltage of this noise was smaller than 1 mV, but much larger than the rms voltage that was previously computed for thermal noise. Notice that the power spectral density of this background noise is an increasing function of frequency.

**VIII. CAPACITY OF A TWISTED-PAIR CHANNEL**

The capacity of a twisted-pair channel in a NEXT-dominated environment has been discussed at some length in several publications [30]-[32]. Our treatment of this subject here is somewhat different from previous work and we also consider the case of a FEXT-dominated environment. Unless specified otherwise, we will only consider perfectly terminated single-gauge loop and assume that the crosstalk is stationary and Gaussian. One of the results obtained in this section may be worth mentioning up front: given a 300 kHz bandwidth, the loss in “reach” due to the requirement of an additional 6 dB of margin for NEXT is equal to 1.7 ft for a 24-gauge cable and about 1.2 ft for a 26-gauge cable.

**A. NEXT-Dominated Environment**

In this section, we will assume an \( f^{3/2} \) characteristic for the NEXT coupling gain. Following [25] and using the two expressions on the right in (18) for the SNR of a NEXT-dominated channel, the capacity of a loop with length \( d \) can be written as

\[
C_d(d) = \frac{1}{\ln 2} \int_0^{\infty} \log_2 \left[ 1 + \text{SNR}_d(d, f) \right] df
\]

\[
= \frac{1}{\ln 2} \int_0^{\infty} \log_2 \left[ 1 + e^{-2dW} \right] df
\]

\[
= \frac{1}{\ln 2} \int_0^{\infty} \log_2 \left[ 1 + e^{-2dW} \right] df.
\]  

In general, for an arbitrary loop length \( d \), the integrals in (32) do not have a closed-form solution even for the approximation on the right. However, under the assumptions of the noise model used in (32), it is possible to derive an expression for the capacity of a loop with length zero, which provides an upper bound for the capacity of short loops. Setting \( d = 0 \) in (32), we get

\[
C_d(0) = \frac{2\pi}{\sqrt{2} \ln 2} x^{-2/3} \int_0^{\infty} x^{-5/2} \ln (1 + x) dx.
\]  

The integral in (33) has a closed-form solution [35]:

\[
C_d(0) = \frac{2\pi}{\sqrt{2} \ln 2} x^{-2/3}.
\]  

As an example, if \( x = 2.8 \times 10^{-9} \), we get \( C_d(0) = 2.6 \times 10^8 \text{ kb/s} = 2.6 \text{ Gb/s} \).

Even though the integrals in (32) do not have a closed-form solution, it is possible to derive a closed-form expression that is a good approximation to the loop’s capacity. First, we will limit the channel bandwidth to some frequency \( W \). We then approximate the loop’s transfer function with another transfer function, which has an \( \sqrt{f} \) characteristic and provides the same loss at frequency \( W/2 \), so that, from (3)

\[
\alpha(W/2) = \frac{\sqrt{W/2}}{10^{\frac{L_{dB}(W/2)}{20}}}
\]

where \( L_{dB}(W/2) \) is the loop’s loss in dB/mi at frequency \( W/2 \), and distances are assumed to be expressed in kilometers. We now consider a loop with a \( \sqrt{f} \) frequency characteristic and length \( d \) such that the SNR at frequency \( W \) is equal to one, that is

\[
\chi W^{3/2} = Ke^{-2dW}
\]

where \( K \) is a constant whose usefulness will be explained a little later. The two expressions in (35) and (36) uniquely define \( d \) as

\[
d_k = \frac{-52.8}{L_{dB}(W/2)} \left[ \log_{10} (\chi W^{3/2}) - \log_{10} K \right].
\]  

We can now compute a lower bound for the capacity of a loop having this length. Neglecting the one in the brackets on the right in (32), we get, after some algebra,

\[
C_d(d, W) = \frac{1}{\ln 2} \int_0^{\infty} \log_2 \left[ 1 + e^{-2dW} \right] df
\]

\[
= \frac{W}{3 \ln 2} \left[ -\ln (\chi W^{3/2}) + 4.5 + \ln K \right].
\]  

The two upper curves in Fig. 17 give computed values for the approximate capacity in (38) as a function of the loop length \( d_k \). The computations were made for perfectly terminated single-gauge 24- and 26-AWG PIC cables (i.e., \( K = 1 \)), and the NEXT loss was assumed to be equal to 57 dB at 80 kHz. The two lower curves give the
useful bandwidth $W$ corresponding to each loop length. Notice that the computed value for a 24-gauge 12-kft PIC cable is about 2.7 Mb/s, which is slightly less than the value that would be obtained by numerical integration of (32). Other loop configurations can be accommodated by choosing the appropriate value for $K$. For example, in the case of the third rule in Section V, which applies to a loop with a long-enough bridged tap, we get a value $K = 4/9$.

Alternatively, $K$ can also be used to account for various “margin” requirements. For example, an additional margin of 6 dB against the NEXT model corresponds to a value $K = 1/4$.

Notice from (38) that the capacity is uniquely defined by the value of $W$. Thus, using (37) we get the following relationship between the relative lengths of two different types of loops that provide the same capacity

$$\frac{d_2}{d_1} = \frac{L_{1,\text{db}}(W/2)}{L_{2,\text{db}}(W/2)}, \quad C_s(d_1, W) = C_s(d_2, W). \quad (39)$$

Another useful result is the loss in reach for a given capacity when $K$ in (37) is different from one. Assuming that $d_1$ and $d_2$ correspond to $K = 1$ and $K \neq 1$, respectively, we get

$$d_1 - d_2 = \frac{-52.8 \times \log_{10} K}{\sqrt{2} L_{1,\text{db}}(W/2)}, \quad C_s(d_1, W) = C_s(d_2, W). \quad (40)$$

It can be shown that the expression in (40) also gives the loss in reach for an HDSL transceiver using an ideal decision feedback equalizer (DFE), or a near-ideal DFE of the type described in [33], if the error performance has to be kept constant. The proof is a direct consequence of the generalization of a result by Price, in [34], to the case of a NEXT-dominated environment [13].

As an example of the usefulness of the expressions in (39) and (40), consider a 300 kHz bandwidth (i.e., $W/2 = 150$ kHz). From Fig. 17, we see that this is the useful bandwidth for a 13.2-kft 24-gauge cable, for which the approximate capacity is about 2.26 Mb/s. Using Table V and (39), we find that the 22-gauge PIC cable that provides the same capacity has a length 13.2/0.746 = 17.7 kft. Consider now an additional margin requirement of 6 dB, which corresponds to $K = 1/4$. Using the previous parameters and (40), we find that the loss in reach is about 1.645 kft for a 24-gauge cable. The corresponding loss is about 1.168 kft for a 26-gauge cable. These would also be the losses experienced by a transceiver using an ideal DFE.

B. FEXT-Dominated Environment

FEXT is not expected to be a major impairment for typical deployment of HDSL. However, there have been some speculation that it may become an issue for HDSL-related applications such as the asymmetric digital subscriber line (ADSL), which will provide simplex operation at 1.5 Mb/s from the CO location to the subscriber location. In this case, there is no NEXT when only ADSL signals are present in a cable, and FEXT becomes the dominant crosstalk.

We will again constrain the bandwidth to some value $W$ and define the loop length $d$ by assuming the value of the SNR in (19) is equal to one, so that

$$\tilde{d} = \frac{1}{\sqrt{W^2}} \quad \text{or} \quad W = \frac{1}{(\tilde{d} \Psi)^2} \quad (41)$$

The capacity for such a loop, in a FEXT-dominated environment, can then be written as

$$C_s(d, W) = \int_{0}^{W} \log_{2} [1 + \text{SNR}_d(f)] df = \frac{1}{\ln 2} \int_{0}^{W} \ln [1 + W^2 f^{-2}] \quad (42)$$

which can be lower and upper bounded in the following way

$$\frac{2W}{\ln 2} < C_s(d, W) < \frac{2W}{\ln 2} + W. \quad (43)$$

The lower bound in (43) is much cruder than the corresponding expression that was obtained for NEXT in (38), but this bound will suffice for our purposes here. As an example, assume that $d = 12$ kft in (41) so that $W = 29$ MHz. The lower bound for the capacity in (43) then becomes: $C(d, W) > 83$ Mb/s. This is much larger than the capacity of about 3 Mb/s that is obtained for a 12-kft loop in a NEXT-dominated environment (see Fig. 17). Unfortunately, it is not possible in practice to take advan-
tage of the huge capacity promised by (43). Assume a 12-kft 24-gauge PIC cable and a transmitted signal using a 29 MHz bandwidth. The loss experienced at the mid-frequency by such a signal when propagating through the cable would be about 284 dB! Such a huge loss would bring most of the received power spectral density well below the power level of the background noise shown in Figs. 8 and 9, for example. It can be concluded from the preceding discussion that the assumption of a FEXT-dominated environment is probably not realistic for HDSL-related applications such as the ADSL. However, FEXT may become a factor for applications using very short loop lengths and bit rates much larger than 1.5 Mb/s.

APPENDIX

TRANSFER FUNCTIONS OF VARIOUS LOOP CONFIGURATIONS

The steady-state behavior of the two-port network shown in Fig. 18(a) can be characterized by the following set of linear equations:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$  \hspace{1cm} (A-1)

where $V_1$ and $I_1$ are the voltage and current at the input port, $V_2$ and $I_2$ are the corresponding voltage and current at the output port, and $A$, $B$, $C$, and $D$ are complex functions of frequency, which characterize the electrical properties of the two-port network [1]. Two simple examples of two-port networks with their $ABCD$ matrices are shown in Fig. 18(b) and (c). Because voltages and currents have to be continuous in a physical system, it is possible to represent a cascade of two-port networks by the corresponding cascade of $ABCD$ matrices. Two examples of circuits, which are amenable to such a chain-matrix representation, are shown in Fig. 18(d) and (e). We will first derive the coefficients of the $ABCD$ matrix for the single-gauge loop shown in Fig. 18(d). The loop's input is driven by a sinusoidal voltage source $V$, with impedance $Z_v$, and its output is terminated by a load with impedance $Z_o$.

Fig. 19 shows a two-port network representation of an element $dx$ of a loop as a function of the cable's primary constants $R$, $L$, $C$, and $G$. Assuming a sinusoidal input voltage, it is easily verified that the voltage $V$ and current $I$ have to satisfy the following differential equations:

$$\frac{d^2 V}{dx^2} = \gamma^2 V \quad \text{and} \quad \frac{d^2 I}{dx^2} = \gamma^2 I$$  \hspace{1cm} (A-2)

where $\gamma$ is the propagation constant defined in (1). This set of equations can be solved using one of several well-known techniques, and we get the following expressions for the steady-state voltage $V(x, f)$ and current $I(x, f)$ at distance $x$ and frequency $f$:

$\rho_s$ and $\rho_t$ are the reflection coefficients at both ends of the loop, and $Z_0$ is the cable's characteristic impedance, which was previously defined in (1). Notice that $\rho_s$ and $\rho_t$ are equal to zero when the loop is perfectly terminated, i.e., when $Z_i = Z_v = Z_o$, and that, in this case, $V(x, f) = Z_0 I(x, f)$.

SPECIALIZING (A-3) AND (A-4) TO $x = d$ AND $x = 0$, WE GET THE FOLLOWING EXPRESSIONS FOR THE VOLTAGES AND CURRENTS AT THE TWO ENDPOINTS OF THE LOOP:

$$V(d, f) = \frac{Z_0 V(f) e^{-\gamma d} - \rho_t e^{-\gamma (2d - x)} Z_o I(f)}{Z_v + Z_o}$$  \hspace{1cm} (A-3)

$$I(d, f) = \frac{V(f) e^{-\gamma d} - \rho_t e^{-\gamma (2d - x)} Z_o I(f)}{Z_v + Z_o}$$  \hspace{1cm} (A-4)

where $\rho_s = \frac{Z_v - Z_0}{Z_v + Z_o}$ and $\rho_t = \frac{Z_v - Z_0}{Z_v + Z_o}$  \hspace{1cm} (A-5)

are the reflection coefficients at both ends of the loop, and $Z_0$ is the cable's characteristic impedance, which was previously defined in (1). Notice that $\rho_s$ and $\rho_t$ are equal to zero when the loop is perfectly terminated, i.e., when $Z_i = Z_v = Z_o$, and that, in this case, $V(x, f) = Z_0 I(x, f)$.
The expressions in (A-6)-(A-9) can now be used to compute the coefficients of the loop's ABCD matrix. For example, notice from (A-1) that 

\[ A = \frac{V}{V_t} \]

is equal to the ratio of the input and output voltages when the output current is equal to zero. This condition is obtained when the output port is open, i.e., when \( Z_o = 0 \). From (A-5) we then have, from (A-6) and (A-8),

\[ A = B = \frac{Z_o}{Z} \cosh(\gamma d) \]

\[ C = \frac{Z_o}{Z} \sinh(\gamma d) \sinh(\gamma d) \]

The other coefficients of the loop's ABCD matrix are computed in a similar manner, and we get

\[ A = D = \cosh(\gamma d) \]

\[ B = Z_o \sinh(\gamma d) \]

The input impedance of the loop can be computed from (A-1) or (A-8) and (A-9), and is equal to

\[ Z_{in} = \frac{V(0, f)}{I(0, f)} = \frac{Z_o + Z}{Z_o + Z_o} \frac{1 + \rho_0 e^{-2\gamma d}}{1 - \rho_0 e^{-2\gamma d}} \]

We now have all the tools to compute the transfer functions of more complicated loop configurations. Consider, for example, the loop in Fig. 18(e), which consists of two working sections with lengths \( d_1 \) and \( d_2 \), and possibly different characteristic impedances \( Z_{o1} \) and \( Z_{o2} \). A bridged tap with length \( d_3 \) and characteristic impedance \( Z_{o3} \) is connected in shunt with the two working sections as shown in the figure. In this configuration, the ABCD matrix of the bridged tap is equal to the matrix of the shunt impedance in Fig. 18(c) if we replace \( Z \) with the bridged tap's input impedance \( Z_{in} \), which can be computed from (A-12). We then get the following chain matrix for the loop in Fig. 18(e):

\[
\begin{bmatrix}
V_i \\
I_i \\
\end{bmatrix} = \begin{bmatrix}
1 & Z_{o1} & A_1 & B_1 & 0 \\
0 & 1 & C_1 & D_1 & Z_{o1}^{-1} \\
\end{bmatrix} \begin{bmatrix}
1 & 0 & A_2 & B_2 & 1 \\
0 & 1 & C_1 & D_1 & Z_{o2}^{-1} \\
\end{bmatrix} \begin{bmatrix}
V_j \\
I_j \\
\end{bmatrix} = \begin{bmatrix}
V_j \\
I_j \\
\end{bmatrix}
\]

Using the relationships between the coefficients of the ABCD matrix in (A-11) and \( I_i = V_j Z_{o1}^{-1} \), we get the following general expression for the loop's inverse transfer function:

\[ H^{-1}(f) = \frac{V_i}{V_j} = \frac{1 + Z_{o1}^{-1} A_1 A_2 + Z_{o1}^{-1} (A_2 B_1 + B_2 A_1)}{1 + \frac{Z_{o1} B_1}{Z_{o1}^2} A_1^2 + \frac{Z_{o1} B_2}{Z_{o1}^2} A_1 A_2 + \frac{Z_{o2} B_1}{Z_{o2}^2} A_1 + \frac{B_2}{Z_{o2}}} \]

Before specializing (A-14) to more specific cases, let us point out that the transfer functions for single-gauge loops, given in (2) and (24) of the main text, directly follow from (A-6) and need not be derived from the more general expression in (A-14). We now derive the transfer function in (22), which corresponds to a loop with bridged tap and change of gauge. We will assume perfectly matched terminations, i.e., \( Z_o = Z_{o1} \) and \( Z_o = Z_{o2} \). Using the values for the A and B coefficients in (A-11) and the following formula:

\[
\cosh(a) \cos(b) + \sinh(a) \sin(b) = \cosh(a + b)
\]

the inverse transfer function in (A-14) reduces to

\[ H^{-1}(f) = \left[ 1 + \frac{Z_{o1}}{Z_{o2}} + \frac{Z_{o1}}{Z_{o2}} \right] e^{\gamma_1 d \cos(\gamma_3 d)} \]

from which the transfer function in (22) immediately follows. For an open-circuited bridged tap \( Z_o = \infty \), the input impedance in (A-12) can be written as

\[ Z_{in} = Z_{o3} \frac{1 + e^{-2\gamma_3 d}}{1 - e^{-2\gamma_3 d}} = Z_{o3} \cosh(\gamma_3 d) / \sinh(\gamma_3 d) \]

Notice that the magnitude of the ratio in (A-17) passes through minima when the imaginary part \( \gamma_3 d \) of \( \gamma_3 d \) is equal to \((2k + 1) \pi/2\). We have seen previously that, for high-enough frequencies, all the cables tend to have the same characteristic impedance. In this case, minima in the magnitude of (A-17) also correspond to minima in the transfer function in (22). Thus, the locations of the nulls introduced by a bridged tap in the loop's transfer function can be found by solving \( \beta_3(f_0, k) d_3 = (2k + 1) \pi/2 \), with \( k = 0, 1, 2, \cdots \). Using (6) and typical values for \( L \) and \( C \), we find values for \( f_0, k \) which are very close to the ones predicted by (21) in the main text.

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