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Digital Communications

Grades in English

Chapter 0

Introduction

Part I - Objectives of the course

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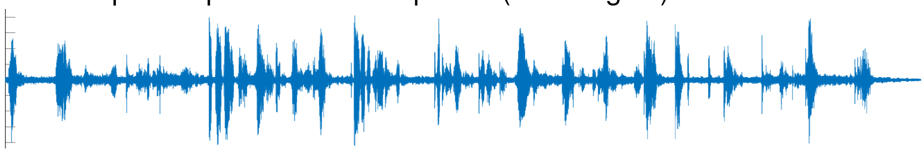
- Definition of communication systems
- Classification: analog and digital communications systems
  - Advantages and drawbacks of digital systems
- Main functional blocks of a digital communication system
- Review of basic Communication Theory
  - Basic principles under some ideal assumptions
- Some realistic constraints to be taken into account
  - Main objectives of this course

Definition: Communication System

- Purpose of a communications system: *transmission*
- Transmission: *process of **sending**, transporting, **information** from one point (source) to another point (destination) through a channel or transmission medium*

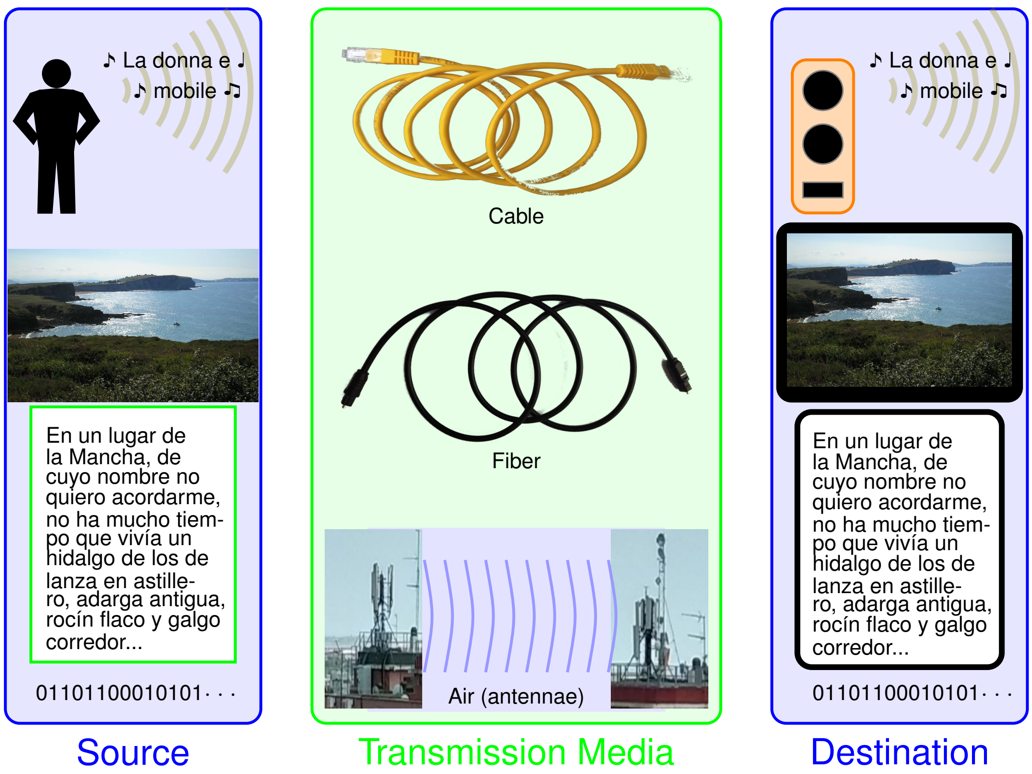
- Taxonomy (format of the information to be transmitted)
  - Analog system: analog signal (continuous time waveform)
    - ★ Information / electrical signal conversion: Transducer
      - Example: output from a microphone (voice signal)



BLADE RUNNER (Roy Batty): *"I've seen things you people wouldn't believe: attack ships on fire off the shoulder of Orion. I watched C-beams glitter in the dark near the Tannhauser gate. All those moments will be lost in time, like tears in the rain... Time to die."*

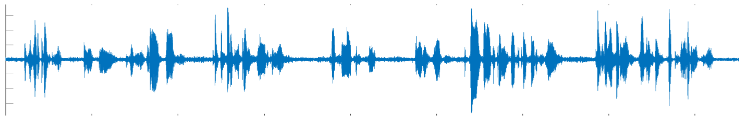
  - Digital system: digital signal 1001110101001010...
    - ★ Posterior conversion to electrical or electromagnetic signal (Modulation)

Physical manifestation of information



Examples of information signals (messages)

- Analog signals: Audio signals (electrical signal at the output of the microphone)



RESERVOIR DOGS (Mr. White): *"If you get a customer, or an employee, who thinks he's Charles Bronson, take the butt of your gun and smash their nose in. Everybody jumps. He falls down screaming, blood squirts out of his nose, nobody says fucking shit after that."*

BLADE RUNNER (Roy Batty): *"I've seen things you people wouldn't believe: attack ships on fire off the shoulder of Orion. I watched C-beams glitter in the dark near the Tannhauser gate. All those moments will be lost in time, like tears in the rain... Time to die."*

- Digital signals

$$B_b[n] = 0100110010001010010010100100100000011101010100101 \cdots$$

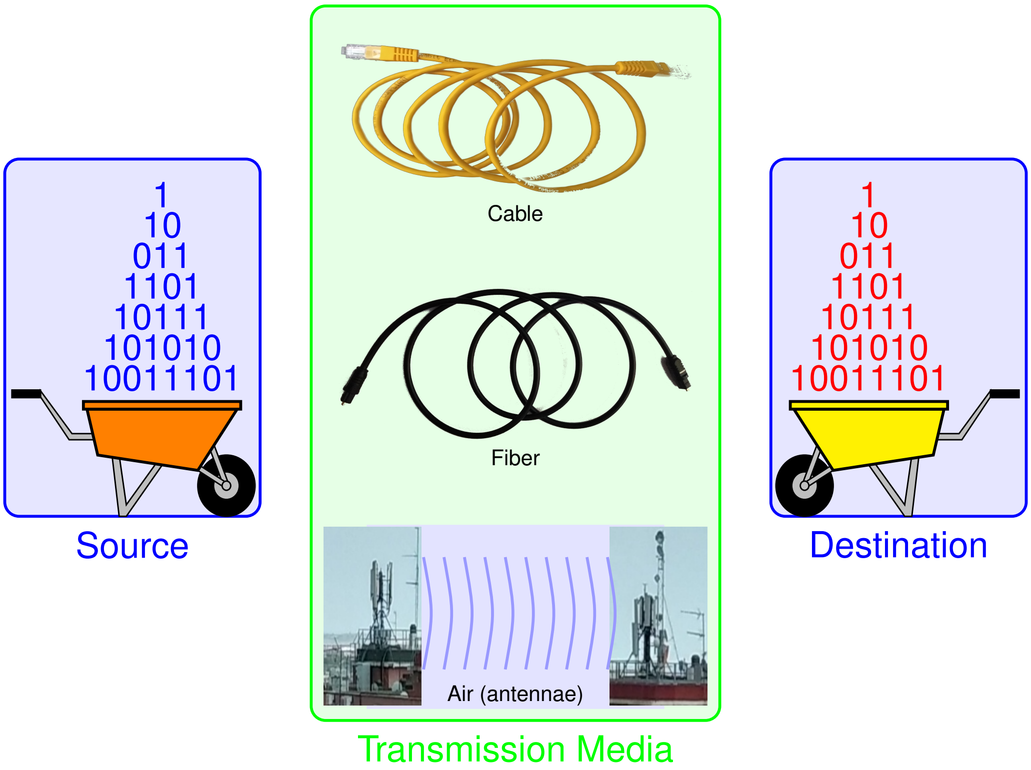
$$B_b[n] = 1101010101001010000101111001001010010101010010100 \cdots$$

Analog and digital communication systems

- Analog communication system
  - Designed to send information contained in a continuous waveform

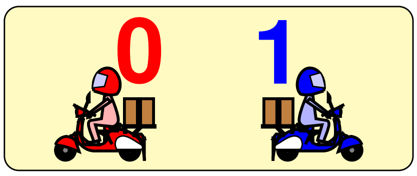
- Digital communication system
  - Designed to send information contained a sequence of symbols pertaining to a finite alphabet (M possible values for each symbol)
    - ★ Example: Bits ($M = 2$): $\{0, 1\}$
      - Information: 011000110111001101010111001001101 0...
  - Transmission at a given rate (symbol rate) R_s symbols/s
    - ★ A symbol is transmitted every $T = \frac{1}{R_s}$ seconds
  - Symbols have to be converted in electrical signals for transmission (digital modulation)
    - ★ Each symbol is mapped onto a waveform
    - ★ Simplest case: waveforms of $T = \frac{1}{R_s}$ seconds
- Digital systems are predominant over analog systems

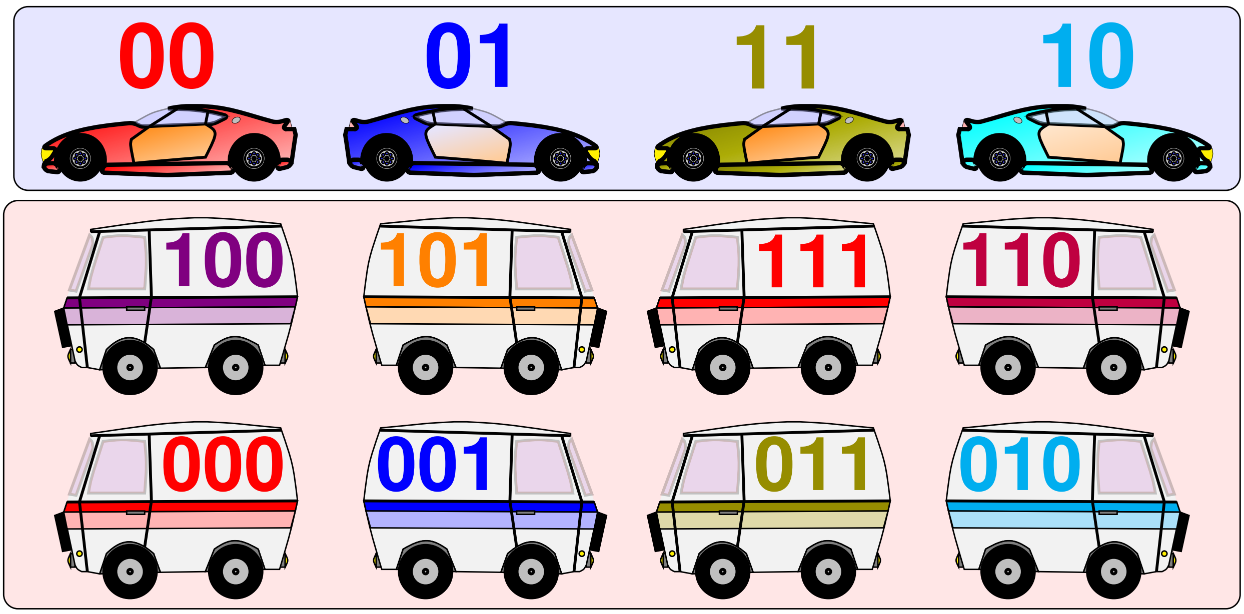
Digital modulation - Bits to signal conversion



Symbols : bit transport vehicles

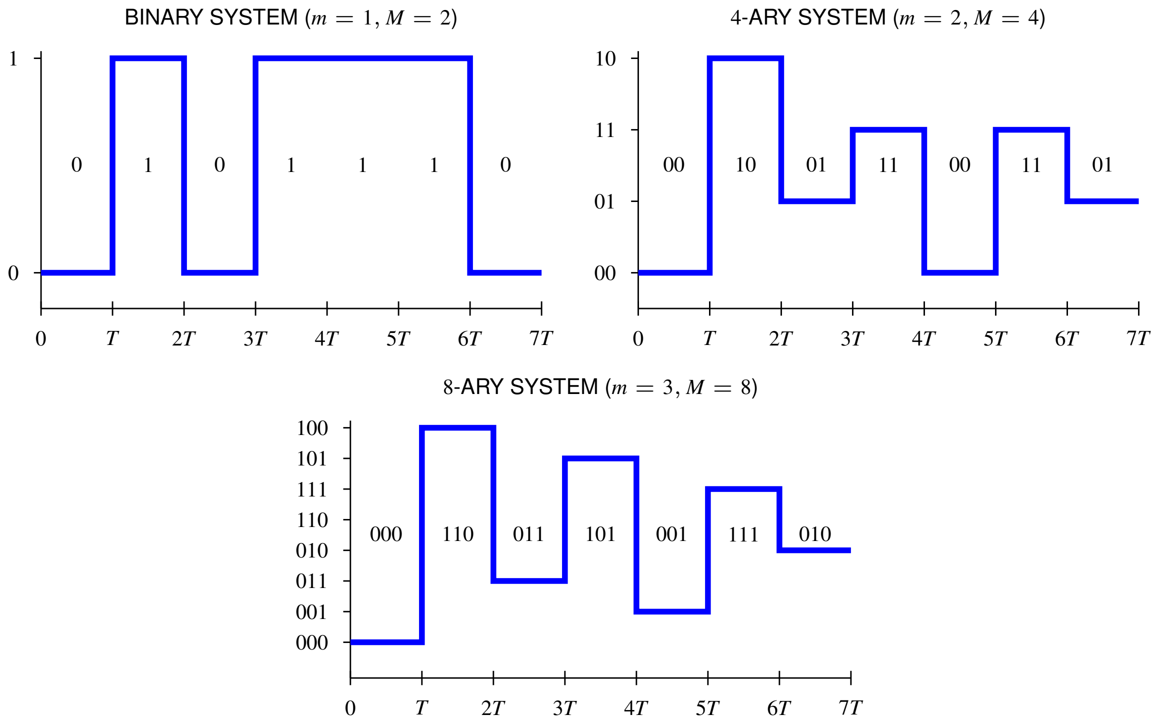
- Symbol: block of m bits
  - ⋆ M-ary systems (with $M = 2^m$)





Digital modulation - Simplest example

- A block of m bits (symbol) is associated to a voltage level
  - M-ary system (with $M = 2^m$ possible symbols)

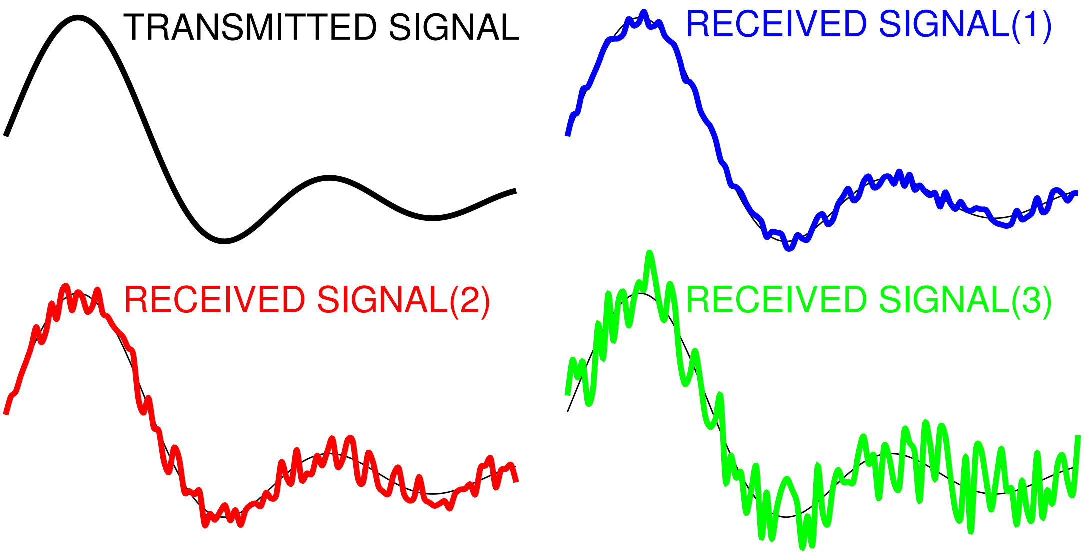


Advantages of digital systems

- **Regeneration** capability
- Existence of techniques for error detection and correction
- Channel distortion can be compensated (equalization)
  - Much easier than in analog systems
- Information can be easily encrypted
- For multiplex/medium access, CDM/CDMA and TDM/TDMA can be used (as well as FDM/FDMA)
- Information format is independent of the nature of information (voice, data, TV, etc.)
  - Nature of information: transmission rate (symbols/s, bits/s)
- In general, circuits are
  - More reliable
  - Cheaper
  - More flexible (programmable)

Distortion in analog signals

- There always exists some distortion during transmission
  - The received signal is different from the transmitted signal
  - Design: to minimize the distortion (maximum fidelity)
- Re-transmission: distortions are accumulated

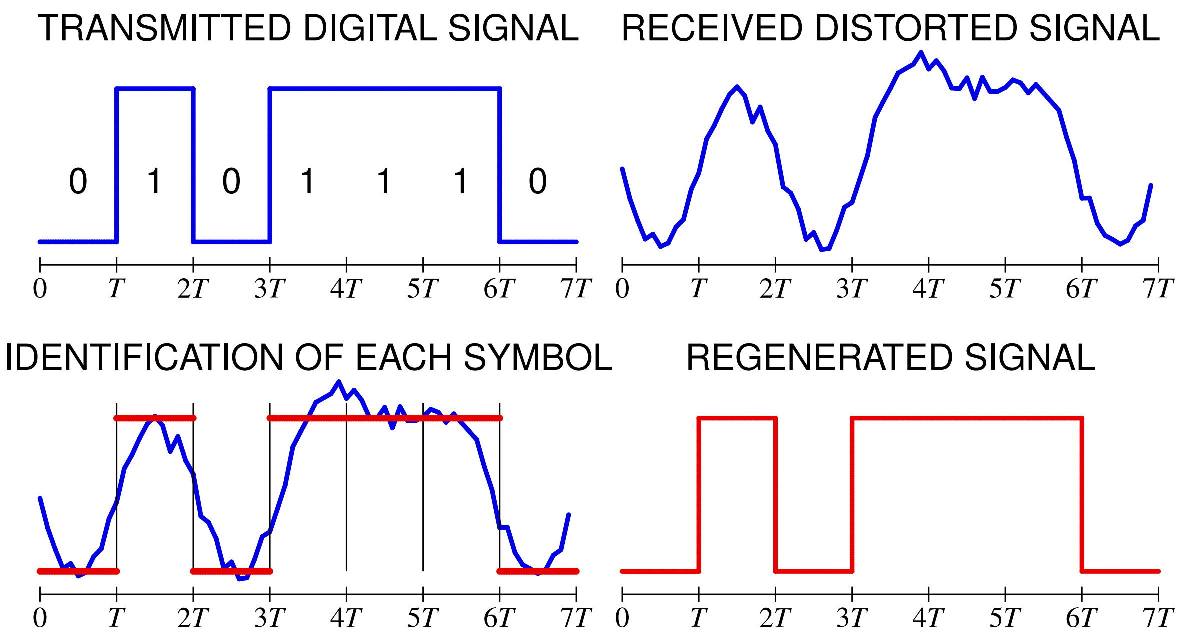


Digital regeneration

BIT ENCODING - Binary system using squared pulses

$1 \equiv$ High level

$0 \equiv$ Low level



Disadvantages of digital systems

- Need for synchronism
  - Identification of the interval for each symbol
- Higher bandwidth
  - Lower as compression techniques improve
- Many information sources are analog in nature
  - A/D conversion
    - Sampling
    - Quantization $\rightarrow$ Quantization error
  - D/A conversion
    - Interpolation
    - Low pass filtering

A/D and D/A are based on the Nyquist sampling theorem (also known as Nyquist-Shannon or Whittaker-Nyquist-Kotelnikov-Shannon theorem)

Analog to digital (A/D) conversion

- Analog sources: continuous amplitude, continuous time
- Analog to digital (A/D) conversion:
  - Discrete time : Sampling at rate f_s samples/s
  - Discrete amplitudes: Quantization with n bits/sample
    - Quantification noise: only 2^n quantification levels
      - Difference between sampled value and quantified value
    - Lower as n increases
  - Binary rate (bits/s):

$$R_b \text{ (bits/s)} = f_s \text{ (samples/s)} \times n \text{ (bits/sample)}$$

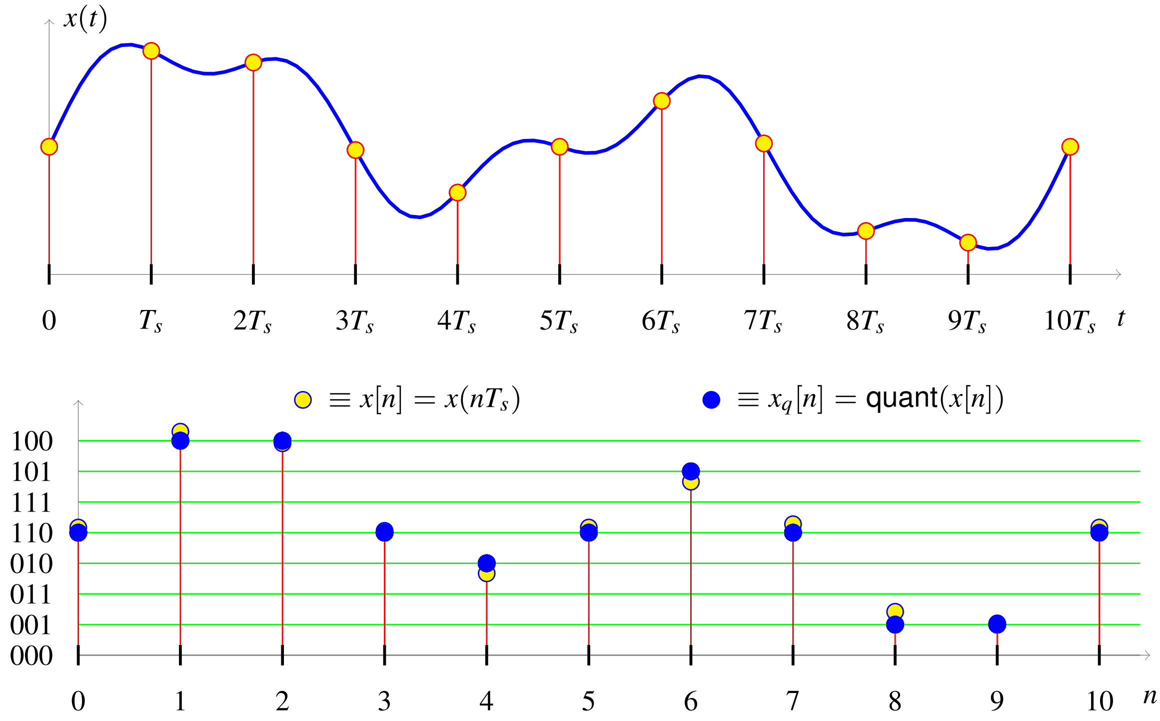
Digital Voice: 64 kbits/s ($B = 4$ kHz, $f_s = 2B = 8000$ samples/s, $n = 8$ bits/sample)

- Digital to analog (D/A) conversion:
  - Conversion of bits to samples (quantized)
  - Reconstruction of the signal from samples (quantized)
    - Interpolation with impulses + low pass filtering

$$x_i(t) = \sum_n x_q[n]\ \delta(t - nT_s) \rightarrow x_r(t) = x_i(t) * h_{LPF}(t)$$

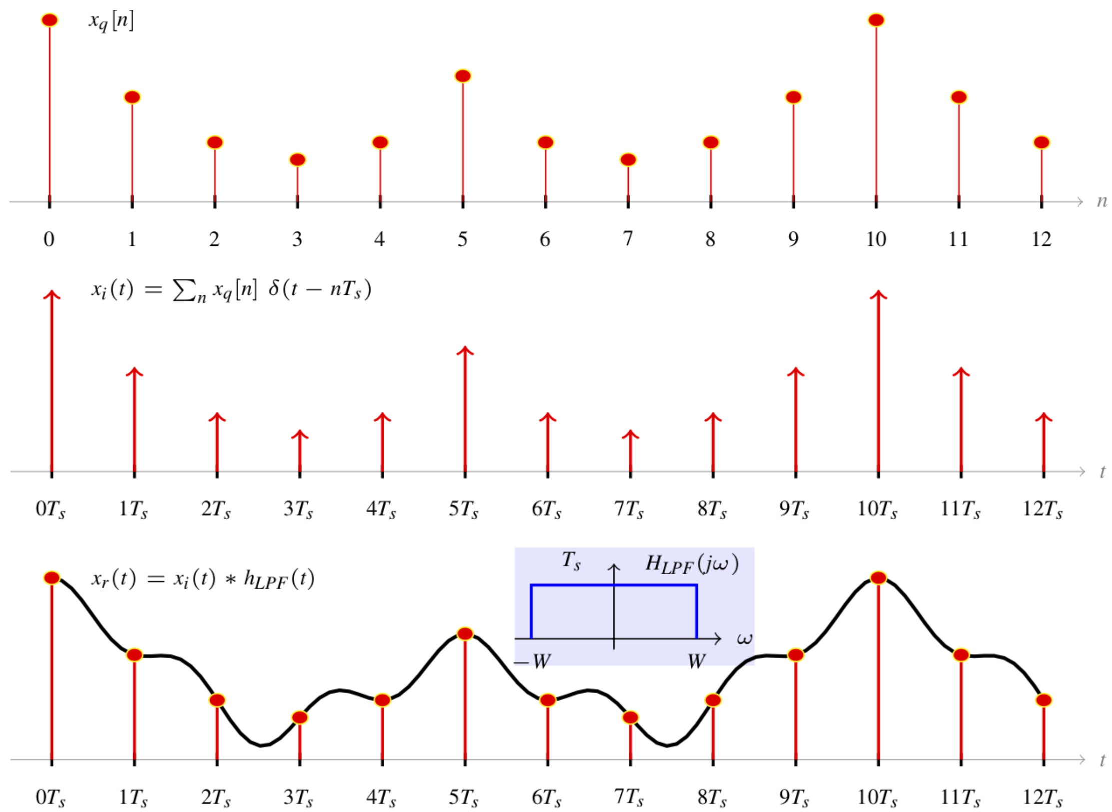
$$x_r(t) = \sum_n x_q[n]\ \text{sinc}\left(\frac{t - nT_s}{T_s}\right)$$

A/D conversion: Sampling + Quantification

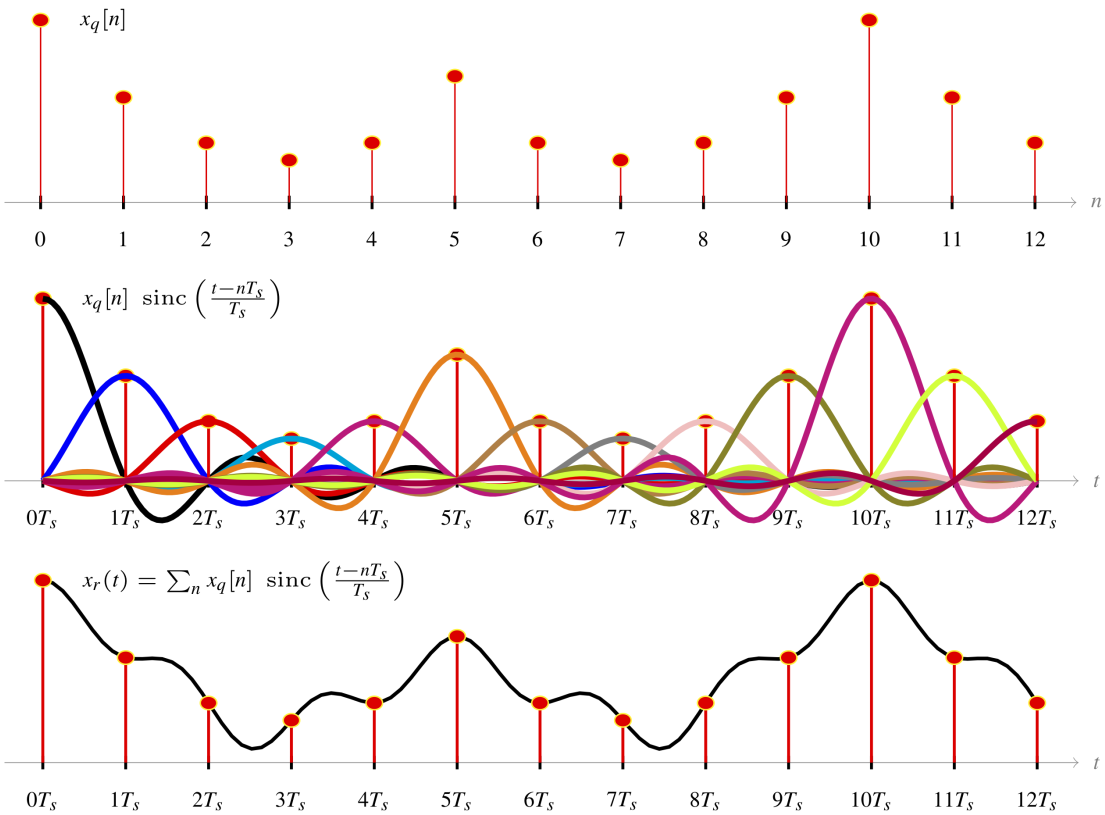


$$B_b[\ell] = 110\ 100\ 100\ 110\ 010\ 110\ 101\ 110\ 001\ 001\ 110 \cdots$$

D/A Conversion: Interpolation with impulses + Filtering



D/A Conversion: Interpolation with sincs at T_s s



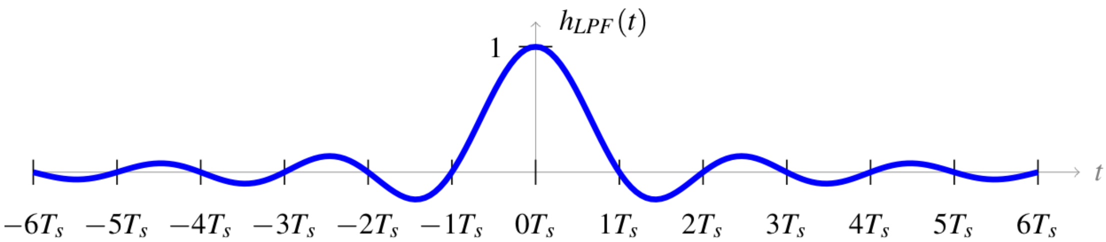
Equivalence of both options

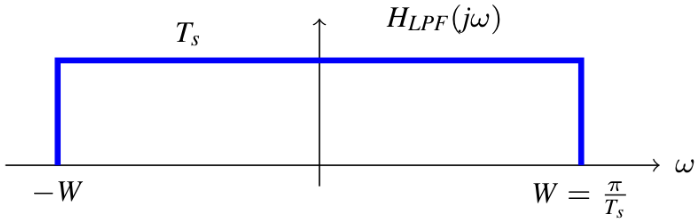
- Sampling at T_s : Sampling frequency

$$f_s = \frac{1}{T_s} = 2B \text{ Hz} \left(\omega_s = \frac{2\pi}{T_s} = 2W \text{ rad/s}\right) \text{ with } W = 2\pi B$$

$$h_{LPF}(t) = \mathrm{sinc}\left(\frac{t}{T_s}\right) \quad \overset{\mathcal{FT}}{\leftrightarrow} \quad H_{LPF}(j\omega) = T_s\,\Pi\left(\frac{\omega}{\omega_s}\right) = T_s\,\Pi\left(\frac{\omega}{2W}\right)$$

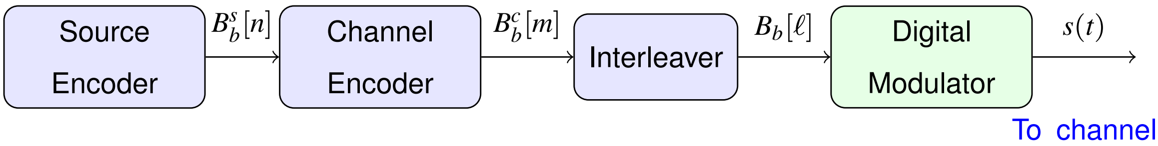
$$\mathrm{sinc}\left(\frac{t}{T_s}\right) * \delta(t - nT_s) = \mathrm{sinc}\left(\frac{t - nT_s}{T_s}\right)$$



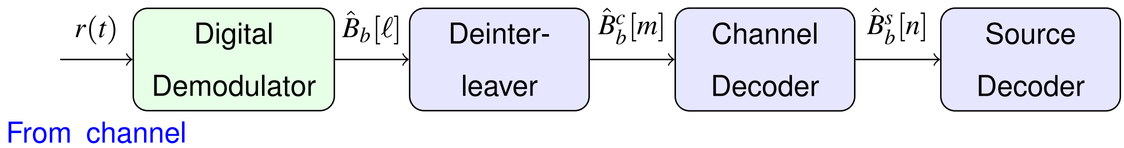


Digital Transmitter/Receiver - Basic functional blocks

- Digital transmitter



  - Digital modulator: Transmission of a sequence of symbols (typically bits, $B_b[\ell]$) through an analog communication channel (electromagnetic signal $s(t)$)
- Digital receiver



  - Demodulador digital: Recovery of the symbol sequence (bits, $\hat{B}_b[\ell]$) from signal received through channel, $r(t)$

Source and channel coding

- Source coding
  - Reduction of redundancy (compression)
  - Lower binary data rate requirements for transmission
  - Examples: MPEG or DivX (video), MP3 or OGG (audio), ZIP or RAR (files),...
- Channel coding
  - Error detection and/or correction
  - Introduction of redundancy (structured)
  - Capability of detection/correction depends on complexity
  - Simplest example: repetition code
    - ★ Repetition code of order 1: $0 \rightarrow 00$ $1 \rightarrow 11$
      - Detects 1 error over a two-bits block
    - ★ Repetition code of order 2: $0 \rightarrow 000$ $1 \rightarrow 111$
      - Detects 2 errors or corrects 1 error (correction based on majority decision) over a three-bits block

Interleaving

- Protection for burst errors
  - In combination with channel encoder
- Re-arrangement of data in a non-contiguous way
  - Goal: to transform burst error in several isolated errors
    - Channel decoder deals with relatively few errors per block
- Kinds of interleavers
  - Block interleavers
  - Convolutional interleavers

Interleaving - One example (block interleaver)

Uncoded Data

1 0 1 1 1 0

Coded Data

111 000 111 111 111 000

Channel Code

Repetition code (order 2)

Interleaved data

101 110 101 110 101 110

Transmission

101 110 101 1 10 101 1 10

Burst of errors

De-interleaved data

11 1 00 0 11 1 11 1 1 1 1 0 0 0

Isolated errors

1	0	1	1	1	0
1	0	1	1	1	0
1	0	1	1	1	0

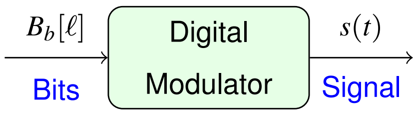
Interleaver
$N_c \times N_b$

Block interleaver
Data input: per column
Data output: per row

1	1	1
0	0	0
1	1	1
1	1	1
1	1	1
0	0	0

De-interleaver
$N_b \times N_c$

Digital modulator - Conversion of bits ($B_b[\ell]$) into a signal



- Transmission of sequence of bits $B_b[\ell]$ at rate $R_b = \frac{1}{T_b}$ bits/s
  - Conversion in electrical signal $s(t)$
- Block-wise bit transmission - Sequence of symbols
  - Segmentation of sequence $B_b[\ell]$ in blocks of m bits
  - Each block of m bits is a symbol
    - ★ 1 symbol $\equiv m$ bits
    - ★ Alphabet of possible symbols: $M = 2^m$ symbols: $B \in \{b_i\}_{i=0}^{N-1}$
  - Sequence of symbols $B[n]$
    - ★ Symbol rate $R_s = \frac{1}{T}$ symbols/s (bauds)
    - ★ Relationship between rates R_b / R_s: $R_b = m \times R_s$ (or $T = m \times T_b$)
  - Transmission of a symbol (block of m bits) each T seg.
- Simplest conversion from bits/symbols sequence to a signal $s(t)$
  - Piecewise generation: "pieces" of T seconds (corresponding to 1 symbol)
    - ★ Symbol interval for $B[n]$: interval $nT \leq t < (n+1)T$

Symbol / signal conversion - Simplest model

- Analyzed in *Communication Theory*
- Symbol / signal conversion
  - Alphabet of M possible symbols: $B \in \{b_0, b_1, \cdots, b_{M-1}\}$
  - Definition of M waveforms with a duration of T seconds

$$\{s_0(t), s_1(t), \cdots, s_{M-1}(t)\}, \text{ defined in interval } 0 \leq t < T$$

  - Symbol / waveform association: $b_i \leftrightarrow s_i(t)$
  - Generation of the piece of signal to be transmitted
    - ★ If $B = b_i$, then $s(t) = s_i(t)$
- Transmission of symbol $B[n]$
  - Symbols interval: $nT \leq t < (n+1)T$
  - Value of the symbol: $B[n] = b_j$
    - ★ The waveform associated to b_j is moved to symbol interval

$$s(t) = s_j(t - nT), \text{ in } nT \leq t < (n+1)T$$

- Gaussian channel model (a simplification)

$$r(t) = s(t) + n(t)$$

$n(t)$: white and Gaussian thermal noise

Design: selection of the M waveforms

- Constraints: energy, performance, channel characteristics
- To simultaneously consider these 3 constraints is difficult in the continuous time domain
- Discrete time representation of signals
  - Points in a N-dimensional Hilbert space
    - ★ Coordinates: N-dimensional vector (Encoder)

$$s_i(t) \rightarrow \boldsymbol{a}_i = \begin{bmatrix} a_{i,0} \\ a_{i,1} \\ \vdots \\ a_{i,N-1} \end{bmatrix}$$

    - ★ Orthonormal basis: N orthonormal signals (Modulator)

$$\{\phi_0(t), \phi_1(t), \cdots, \phi_{N-1}(t)\}, \quad \mathcal{E}\{\phi_j(t)\} = 1, \quad \int_{-\infty}^{\infty} \phi_j(t)\ \phi_k^*(t)\ dt = 0, \text{ if } k \neq j$$

    - ★ Definition of signals in this discrete time representation

$$s_i(t) = \sum_{j=0}^{N-1} a_{i,j}\ \phi_j(t),\ 0 \leq t < T$$

Basic digital communication model

- Two step conversion
  - Encoder: Converts each m bits block (b_i) in a point in a N-dimensional space ($\boldsymbol{a}_i$)
    - ★ Energy and performance can be obtained from discrete representation

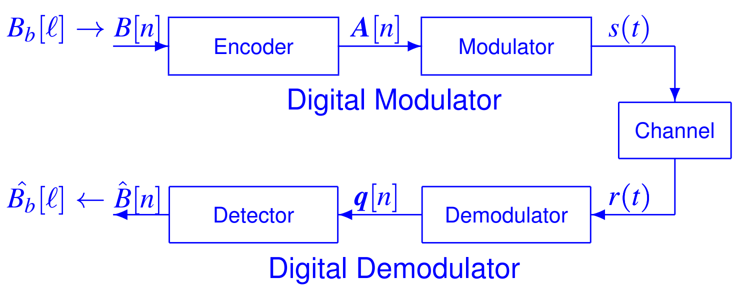
$$\mathcal{E}\{s_i(t)\} \equiv \mathcal{E}\{\boldsymbol{a}_i\} = ||\boldsymbol{a}_i||^2$$

$$P_e \rightarrow d(s_i(t), s_k(t)) = \sqrt{\mathcal{E}\{s_i(t) - s_j(t)\}} \equiv d(\boldsymbol{a}_i, \boldsymbol{a}_k) = ||\boldsymbol{a}_i - \boldsymbol{a}_k||$$

  - Modulator: Generates the waveform associated to each symbol ($s_i(t)$) by using the basis
    - ★ Adaptation to the channel requires adaptation of each element in the base

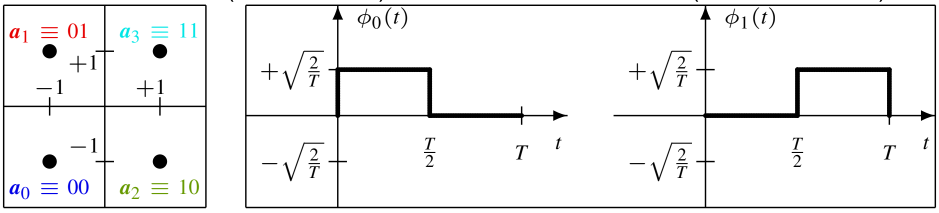
$$\text{Ideal adaptation}: \phi_j(t) * h(t) = \phi_j(t),\ \forall j$$

- Basic digital communication model



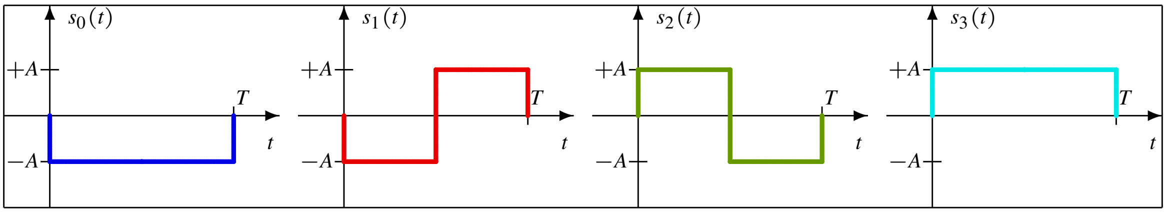
Signal Generation - Example A

Constellation (ENCODER) and orthonormal basis (MODULATOR)



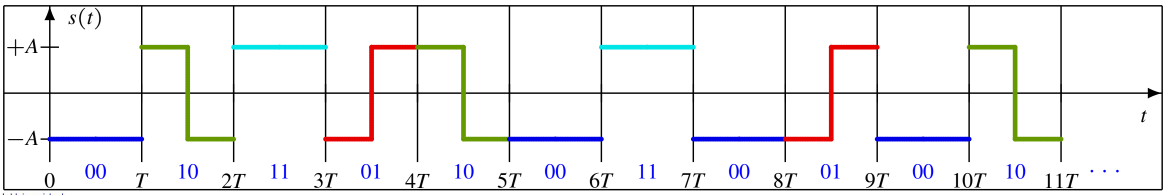
Signals associated to each symbol

$s_0(t) = -1 \times \phi_0(t) - 1 \times \phi_1(t), s_1(t) = -1 \times \phi_0(t) + 1 \times \phi_1(t), s_2(t) = +1 \times \phi_0(t) - 1 \times \phi_1(t), s_3(t) = +1 \times \phi_0(t) + 1 \times \phi_1(t)$



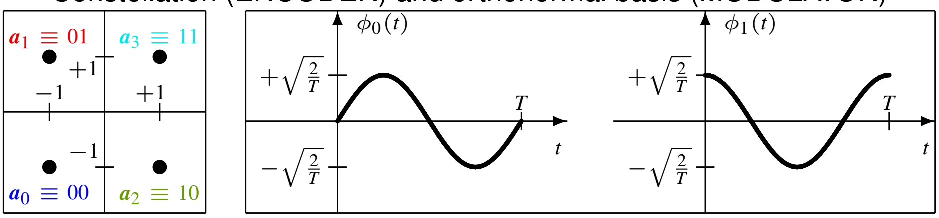
Modulated signal for the following binary sequence

$$B_b[\ell] = 00\ 10\ 11\ 01\ 10\ 00\ 11\ 00\ 01\ 00\ 10\ \cdots$$



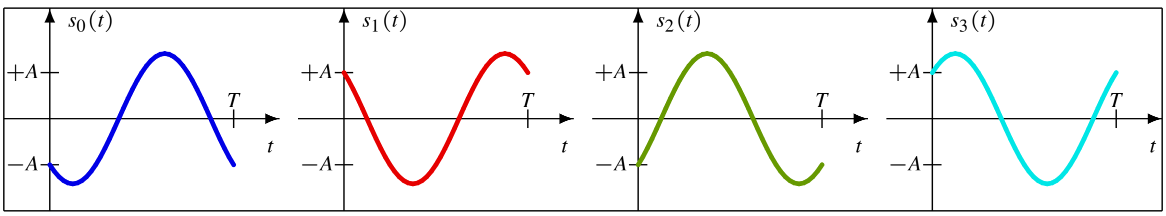
Signal Generation - Example B

Constellation (ENCODER) and orthonormal basis (MODULATOR)



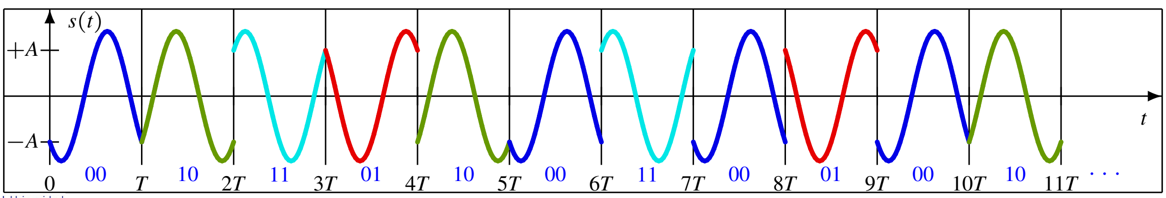
Signals associated to each symbol

$s_0(t) = -1 \times \phi_0(t) - 1 \times \phi_1(t), s_1(t) = -1 \times \phi_0(t) + 1 \times \phi_1(t), s_2(t) = +1 \times \phi_0(t) - 1 \times \phi_1(t), s_3(t) = +1 \times \phi_0(t) + 1 \times \phi_1(t)$



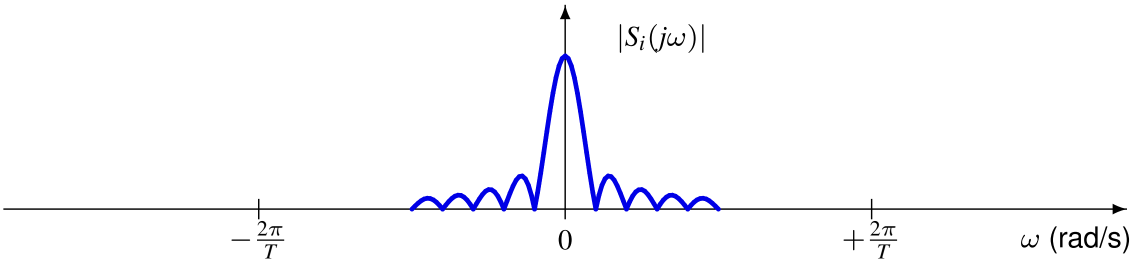
Modulated signal for the following binary sequence

$$B_b[\ell] = 00\ 10\ 11\ 01\ 10\ 00\ 11\ 00\ 01\ 00\ 10\ \cdots$$

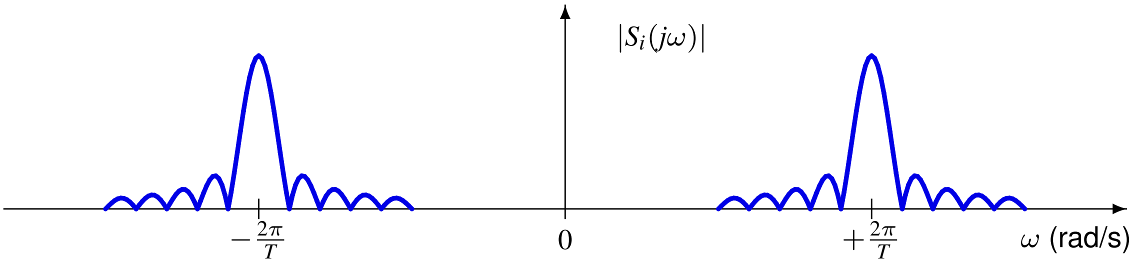


Design of modulator: channel characteristics

- Frequency response of the basis elements
- Example A



  - Signals are appropriated for channels with “*usable range of frequencies*” in low frequencies (baseband channel)
- Example B

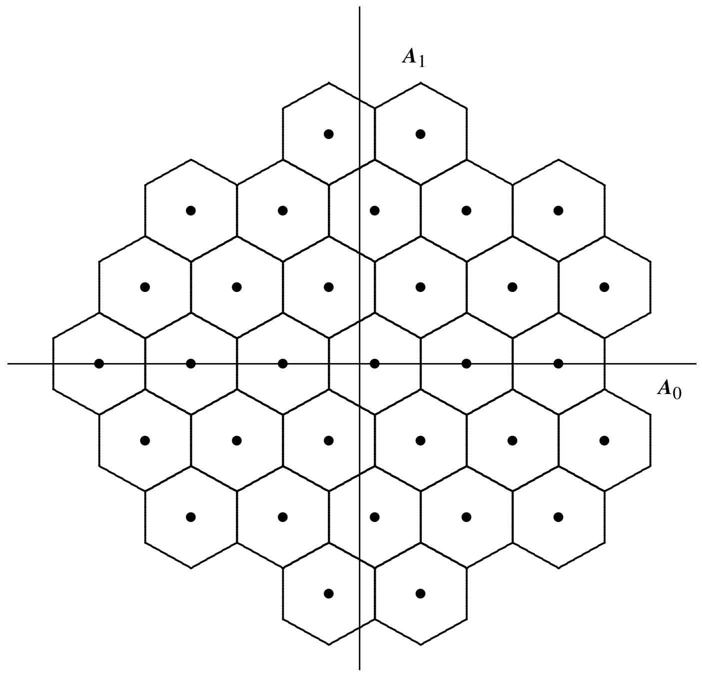


  - Signals are appropriated for channels with “*usable range of frequencies*” centered around a given frequency, in this case $\frac{2\pi}{T}$ radians/s (bandpass channel)

Design of encoder: performance / energy tradeoff

- Constellation design: sphere packing
  - Optimal in the sense of performance / energy tradeoff
    - ★ Minumum P_e for a given E_s
  - 1D: Symmetric equispaced constellations
  - 2D: Hexagonal constellations
  - Practical considerations
    - ★ Simplicity of the transmitter
    - ★ Peak energy limitation
    - ★ Peak to average power ratio
    - ★ Simplicity of the receiver
    - $\Rightarrow$ Constellations: QAM, PSK, unipolar, orthogonal, $\cdots$
- Bit assignment
  - M symbols $\rightarrow m = \log_2 M$ bits/symbol
  - Gray encoding (minimizes BER)

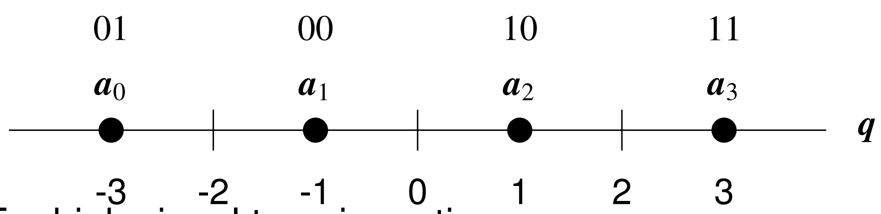
Hexagonal constellations



Hexagonal constellation of 32 points

Binary assignment - Gray encoding

- Binary assignment
  - Association of each possible m bits combination to a constellation point
- Minimum BER for a given P_e: Gray code
  - The m bits assigned to adjacent symbols (at minimum distance) differ in only one bit

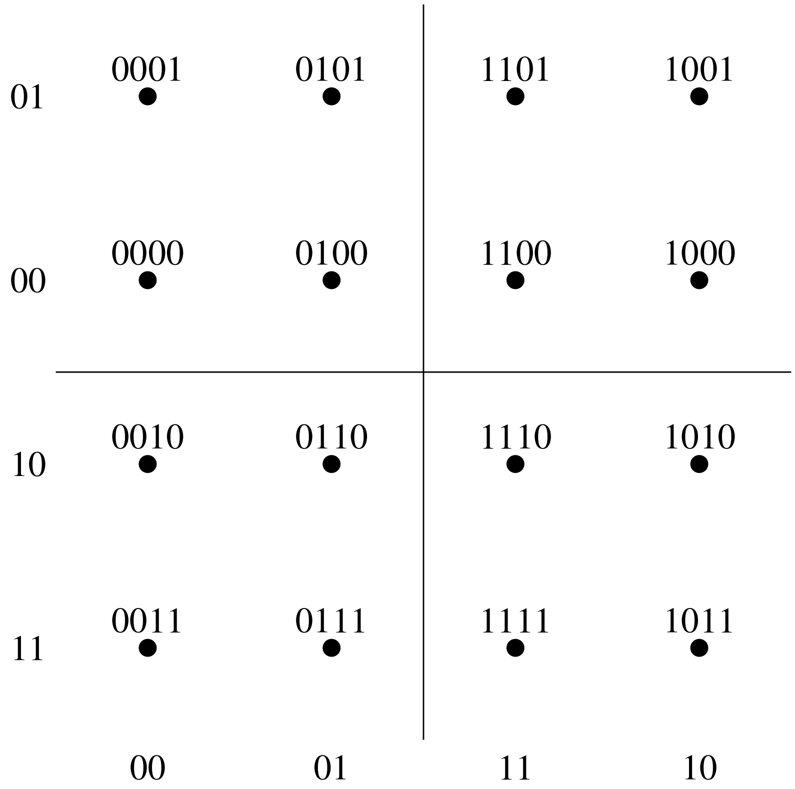


  - For high signal to noise ratio

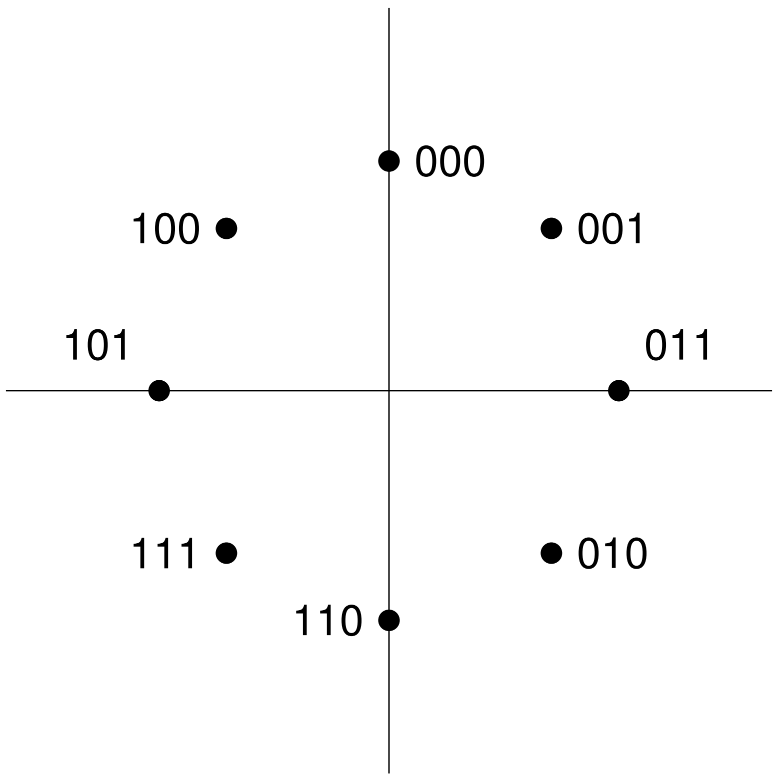
$$BER \approx \frac{1}{m} P_e$$

$m = \log_2(M)$: number of bits per symbol

Gray code for QAM

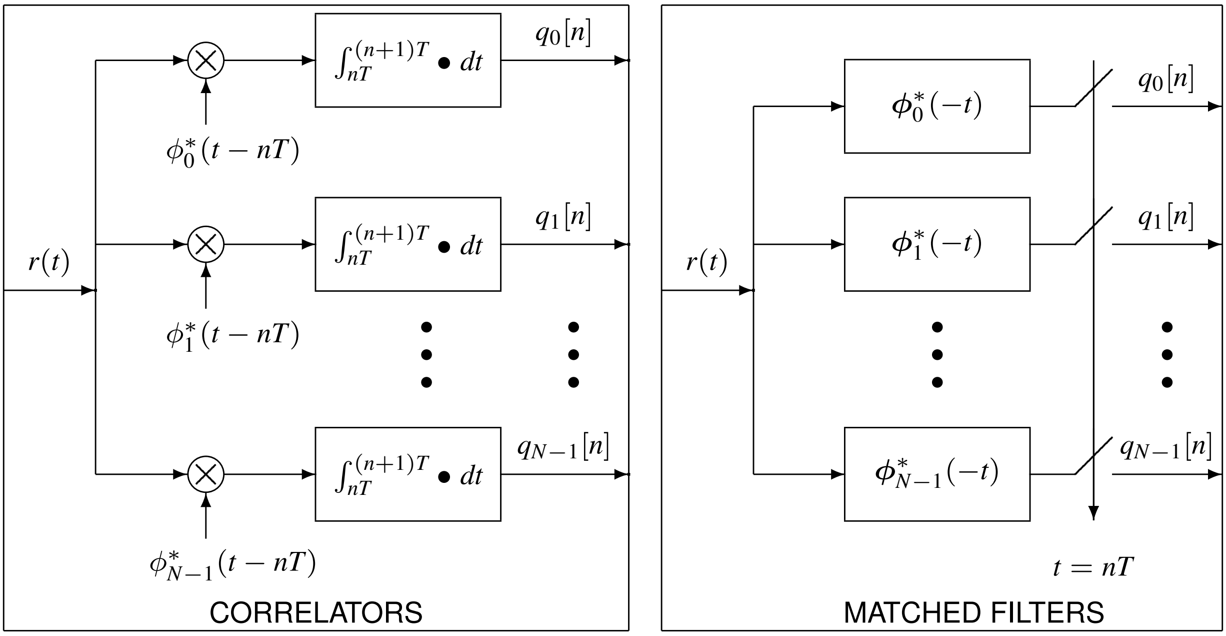


Gray code for PSK



Demodulator

- Discrete time representation of the received signal
  - Process of each symbol interval: $nT \leq t < T \rightarrow \boldsymbol{q}[n]$



Design of detector

- Design: Decision regions - $\hat{B} = b_j$ si $\boldsymbol{q}_0 \in I_j$
- Minimization of the probability of symbol error P_e
  - Assignment of $\boldsymbol{q}_0$: decision region of symbol maximizing posterior probability $p_{B|\boldsymbol{q}}(b_j|\boldsymbol{q}_0)$
- Design rules: $\boldsymbol{q}_0 \in I_i$ if for all $j \neq i$
  - General case: *Maximum a posteriori* (MAP) criterion

$$p_A(\boldsymbol{a}_i)\, f_{\boldsymbol{q}|A}(\boldsymbol{q}_0|\boldsymbol{a}_i) > p_A(\boldsymbol{a}_j)\, f_{\boldsymbol{q}|A}(\boldsymbol{q}_0|\boldsymbol{a}_j)$$

  - Equiprobable symbols ($p_A(\boldsymbol{a}_i) = 1/M$): *Maximum Likelihood* (ML) criterion

$$f_{\boldsymbol{q}|A}(\boldsymbol{q}_0|\boldsymbol{a}_i) > f_{\boldsymbol{q}|A}(\boldsymbol{q}_0|\boldsymbol{a}_j)$$

  - Equiprobable symbols ($p_A(\boldsymbol{a}_i) = 1/M$) and Gaussian noise: *Minimum euclidean distance* criterion

$$d(\boldsymbol{q}_0, \boldsymbol{a}_i) < d(\boldsymbol{q}_0, \boldsymbol{a}_j)$$

- Detection rule depends on probabilities $p_A(\boldsymbol{a}_i)$ and conditional distributions $f_{\boldsymbol{q}|A}(\boldsymbol{q}|\boldsymbol{a}_i)$

Conditional distribution $f_{q|A}(q|a_i)$ in Gaussian channels

- Gaussian channel model

$$r(t) = s(t) + n(t)$$

Noise $n(t)$ is stationnary, white and Gaussian, zero mean and PSD $S_n(j\omega) = N_0/2$

- Observation in a Gaussian channel

$$\boldsymbol{q}[n] = \boldsymbol{A}[n] + \mathbf{z}[n]$$

  - Noise $\mathbf{z}[n]$: N-dimensional Gaussian distribution, zero mean and variance $N_0/2$.

$$f_{\mathbf{z}}(\mathbf{z}) = \mathcal{N}^N\left(\mathbf{0}, \frac{N_0}{2}\right) = \frac{1}{(\pi N_0)^{N/2}} e^{-\frac{||\mathbf{z}||^2}{N_0}}$$

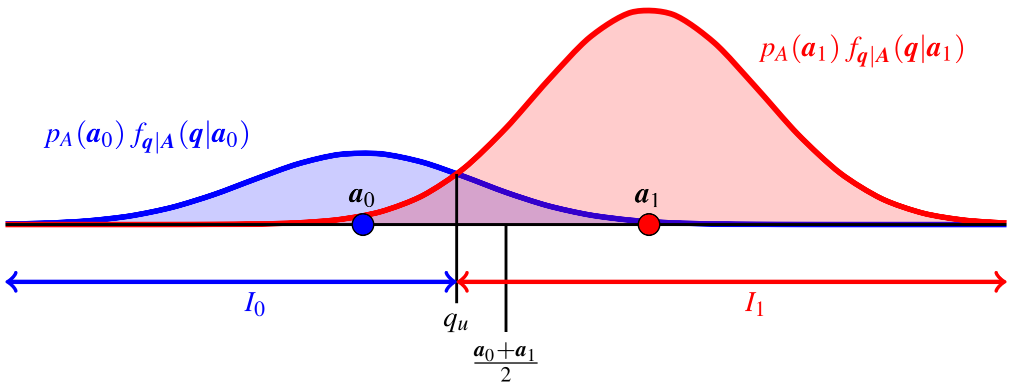
- Conditional distribution of the observation

$$f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}|\boldsymbol{a}_i) = \mathcal{N}^N\left(\boldsymbol{a}_i, \frac{N_0}{2}\right) = \frac{1}{(\pi N_0)^{N/2}} e^{-\frac{||\boldsymbol{q}-\boldsymbol{a}_i||^2}{N_0}}$$

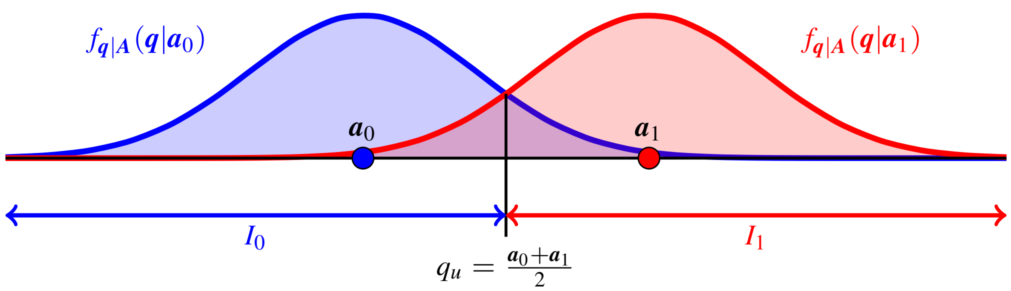
N-dimensional Gaussian centered at the transmitted symbol

MAP, ML and minimum distance criteria for Gaussian noise

MAP Criterion (with $p_A(\boldsymbol{a}_1) = 3 \times p_A(\boldsymbol{a}_0)$ in this case)

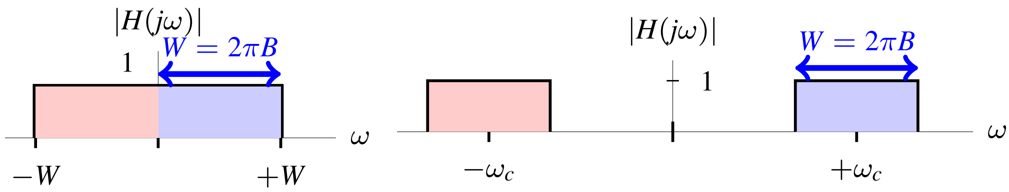


ML and minimum distance criteria are equivalent under Gaussian noise



Characteristics of real channels

- Limited bandwidth
  - The assigned channel typically has a limited available bandwidth (B Hz, $W = 2\pi B$ rad./s)
    - ★ Baseband channels
    - ★ Bandpass channels (central frequency ω_c rad./s)



Baseband   Bandpass (central frequency ω_c rad/s)

  - Trasmitted signals (their freq. response) have to be fitted to this constraint in the available bandwidth
- Introduction of distortions (non ideal channels)
  - Noise (Gaussian) *(Communication Theory)*
  - Linear distortion: linear time invariant (LTI) model: $h(t)$, $H(j\omega)$

$$\boldsymbol{q}[n] \neq \mathbf{A}[n] + \mathbf{z}[n]$$

  - Non linear distortion (not considered here): intermodulation distortion (IMD)

Main objectives of *Digital Communications*

- To extend the basic digital communication model *(Communication Theory)* to consider the realistic constraints introduced by real channels
  - To analyze the mechanisms that are necessary to generate band-limited signals (digital modulations)
    - ★ Baseband
    - ★ Bandpass
  - To analyze the effect of linear distortion and the necessary mechanisms to handle it at the receiver
    - ★ Optimum receiver
    - ★ Sub-optimum receivers (with lower implementation requirements)
- To analyze techniques allowing to control the probability of error of the system
  - Channel coding techniques

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Digital Communications

Grades in English

Chapter 0

Introduction

Part II - Review of basic concepts

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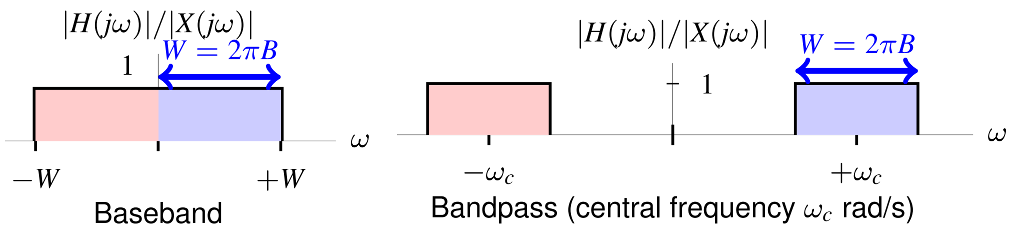
Notation - Frequency and bandwidth

- Linear / angular frequency

$$f \text{ Hz (cycles/s) } / \ \omega = 2\pi f \text{ rad/s}$$

- Bandwidth of a system (or signal)
  - Range of positive available (or non-null) frequencies
  - Usual notation: B Hz, $W = 2\pi B$ rad/s
    - Baseband channels (signals)
    - Bandpass (central frequency ω_c rad/s) channels (signals)



Notation - Signals

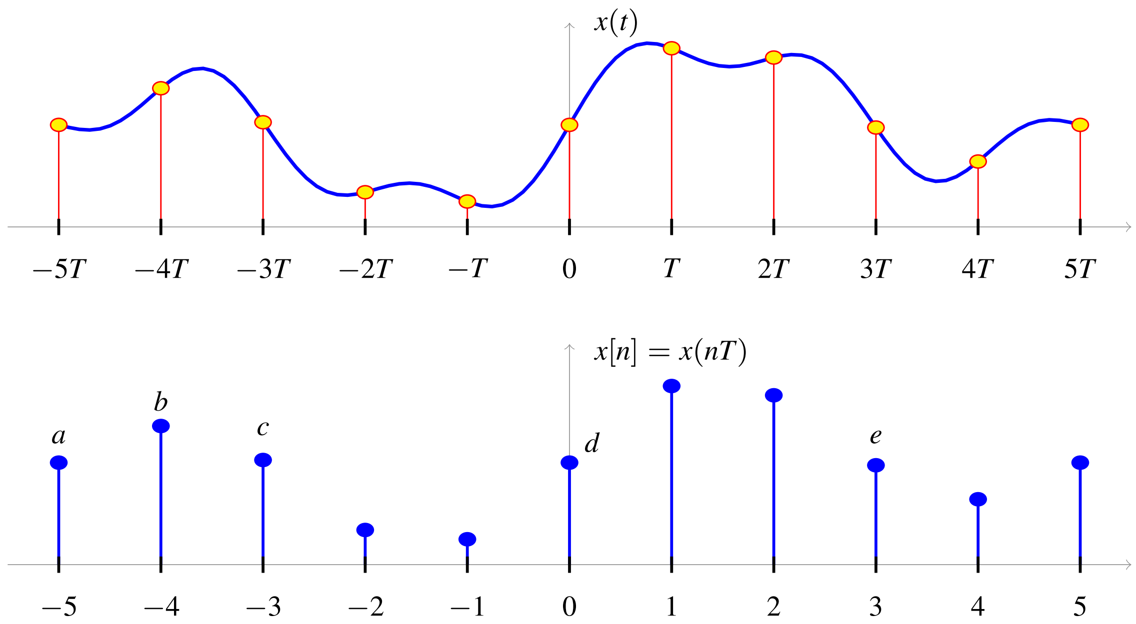
- Continuous time signals
  - Time domain: $x(t)$
  - Frequency domain
    - ★ Fourier transform (deterministic): $X\left(j\omega\right)$
    - ★ Power spectral density (random): $S_X\left(j\omega\right)$
- Discrete time signals
  - Time domain: $x[n]$
  - Frequency domain
    - ★ Fourier transform (deterministic): $X\left(e^{j\omega}\right)$
    - ★ Power spectral density (random): $S_X\left(e^{j\omega}\right)$
- Discrete time signal by sampling at $R_s = \frac{1}{T}$ a continuous time signal

$$x[n] = x(t)\big|_{t=nT} = x(nT)$$

$$X\left(e^{j\omega}\right) = \frac{1}{T}\sum_k X\left(j\frac{\omega}{T} - j\frac{2\pi}{T}k\right) \qquad X(j\omega) = T\,X\left(e^{j\omega T}\right),\ |\omega| \leq \frac{\pi}{T}$$

$$S_X\left(e^{j\omega}\right) = \frac{1}{T}\sum_k S_X\left(j\frac{\omega}{T} - j\frac{2\pi}{T}k\right) \qquad S_X(j\omega) = T\,S_X\left(e^{j\omega T}\right),\ |\omega| \leq \frac{\pi}{T}$$

Sampling and notation for discrete time signals



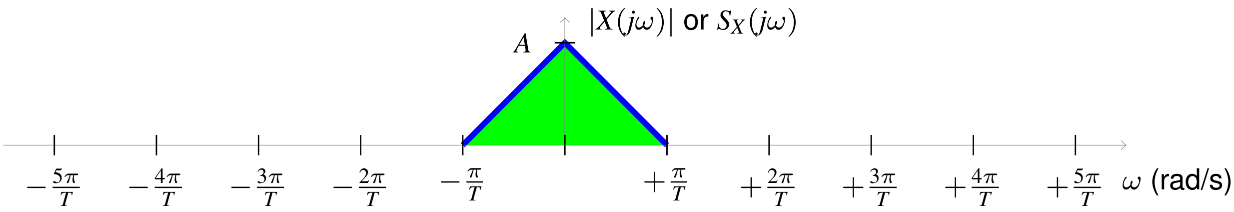
Notation using the $\delta[n]$ function

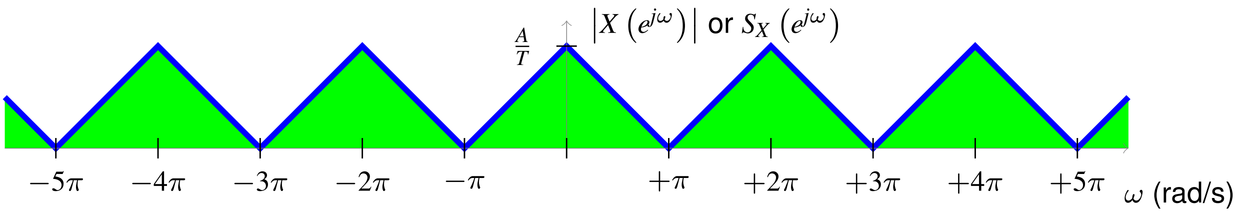
$$x[n] = \cdots + a\ \delta[n+5] + b\ \delta[n+4] + c\ \delta[n+3] + \cdots + d\ \delta[n] + \cdots + e\ \delta[n-3] + \cdots$$

Sampling in the frequency domain

$$X\left(e^{j\omega}\right) = \frac{1}{T}\sum_k X\left(j\frac{\omega}{T} - j\frac{2\pi}{T}k\right) \qquad X(j\omega) = T\, X\left(e^{j\omega T}\right),\ |\omega| \leq \frac{\pi}{T}$$

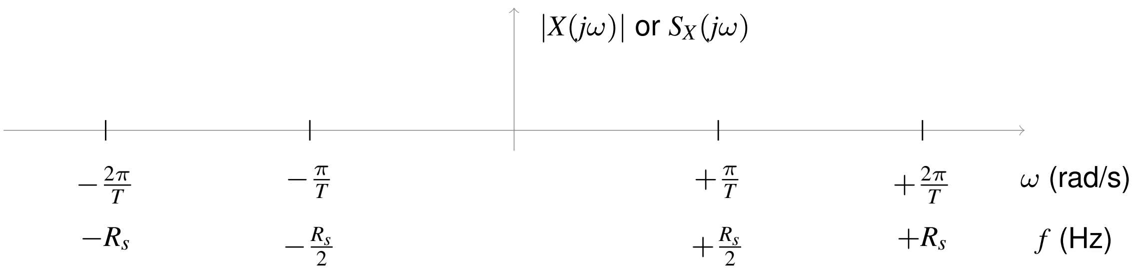
$$S_X\left(e^{j\omega}\right) = \frac{1}{T}\sum_k S_X\left(j\frac{\omega}{T} - j\frac{2\pi}{T}k\right) \qquad S_X(j\omega) = T\, S_X\left(e^{j\omega T}\right),\ |\omega| \leq \frac{\pi}{T}$$

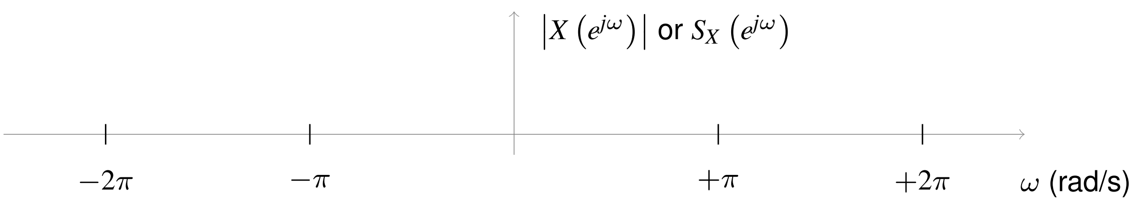




Frequency representation - Some relevant frequencies

- Transmission of symbols at $R_s = \frac{1}{T}$ symbols/s (bauds)
  - Symbol rate: R_s
  - Symbol duration time (symbol interval): T
- Some signals will be sampled at R_s (sampling at symbol rate)
- Some important frequencies





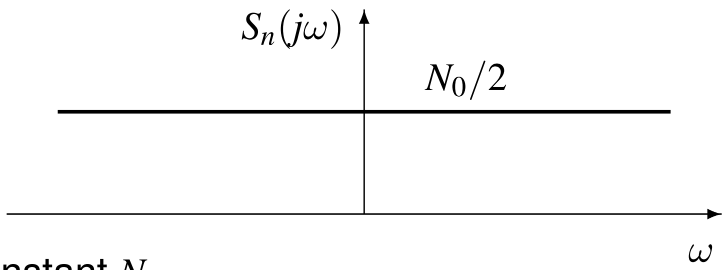
Model for thermal noise

- Stationnary ergodic white and Gaussian random process, $n(t)$
  - Zero mean ($m_n = 0$)
  - Autocorrelation function

$$R_n(\tau) = \frac{N_0}{2}\,\delta(\tau)$$

  - Power spectral density

$$S_n(j\omega) = \frac{N_0}{2}$$



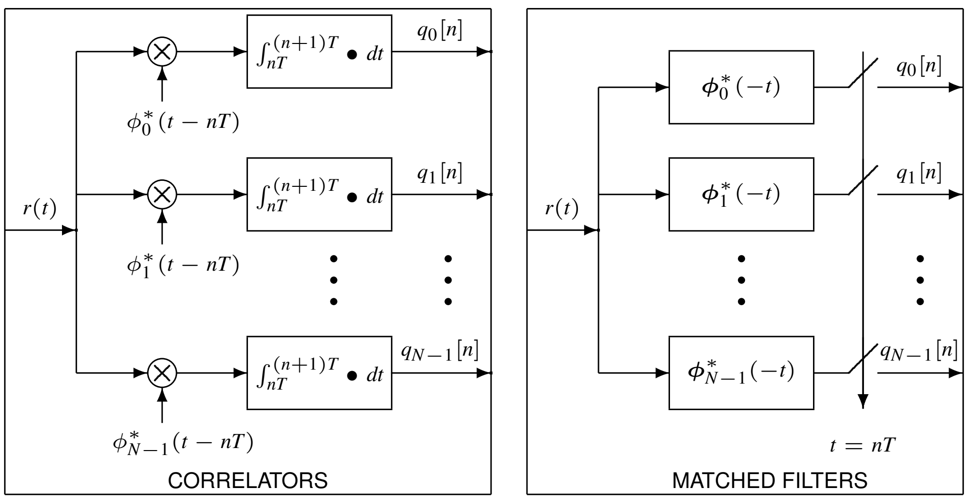
- Value for constant N_0

$$N_0 = k\,T \text{ W/Hz or } J \begin{cases} k & \text{Boltzmann constant } (1.38 \times 10^{-23} \text{ J/K}) \\ T & \text{Temperature (K)} \end{cases}$$

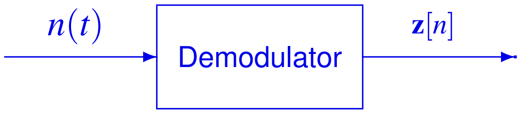
Demodulator

- Obtains the discrete time representation of received signal $r(t)$

$$\boldsymbol{q}[n] = \begin{bmatrix} q_0[n] \\ q_1[n] \\ \vdots \\ q_{N-1}[n] \end{bmatrix} \equiv r(t) \text{ in orthonormal basis } \{\phi_0(t), \phi_1(t), \cdots, \phi_{N-1}(t)\}$$



Discrete time noise at the output of the demodulator



- Demodulator is normalized ($\phi_k(t)$ with unitary energy)
- Distribution for each noise component (independent)
  - Gaussian with zero mean and variance $\frac{N_0}{2}$

$$f_{z_k}(z_k) = \mathcal{N}\left(0, \frac{N_0}{2}\right)$$

- Conditional distribución for $\boldsymbol{q}[n]$ in the ideal case

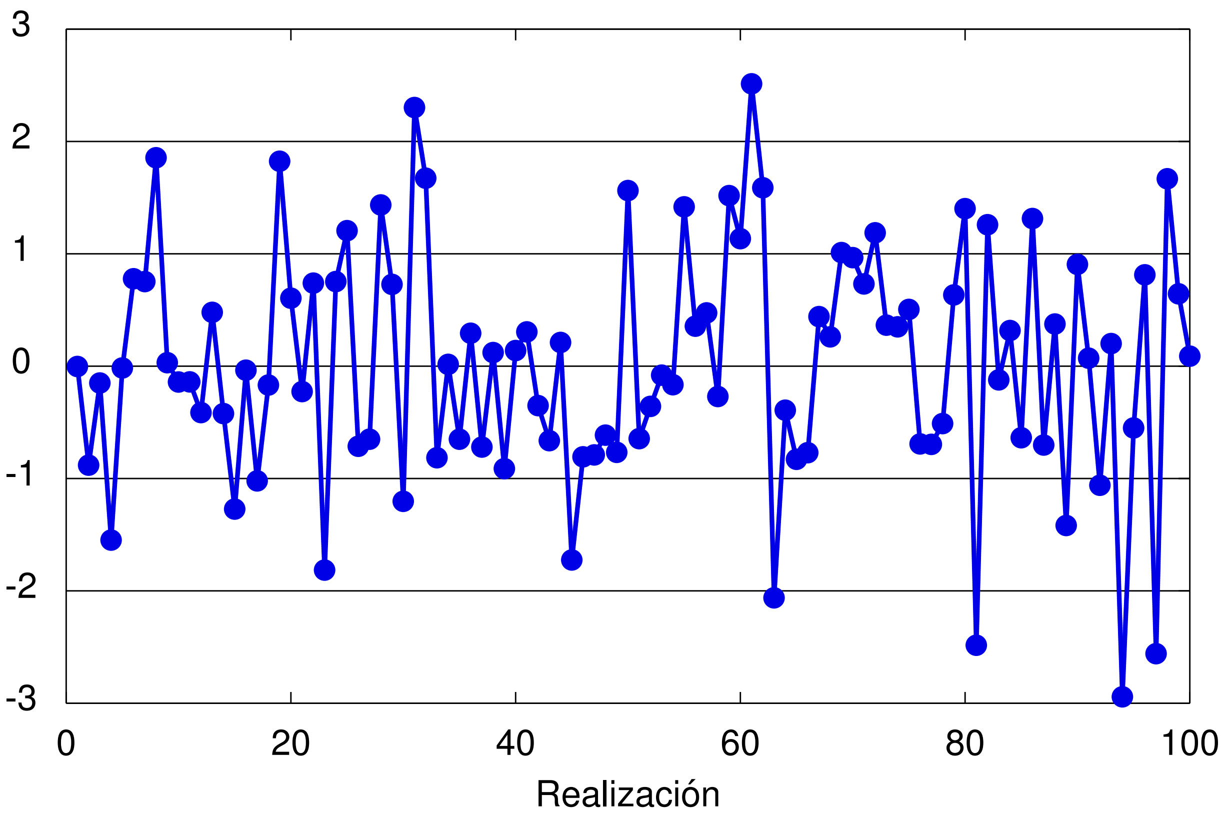
$$\boldsymbol{q}[n] = \boldsymbol{A}[n] + \mathbf{z}[n]$$

  - If $\boldsymbol{A}[n] = \boldsymbol{a}_i$, each component of $\boldsymbol{q}[n]$ is

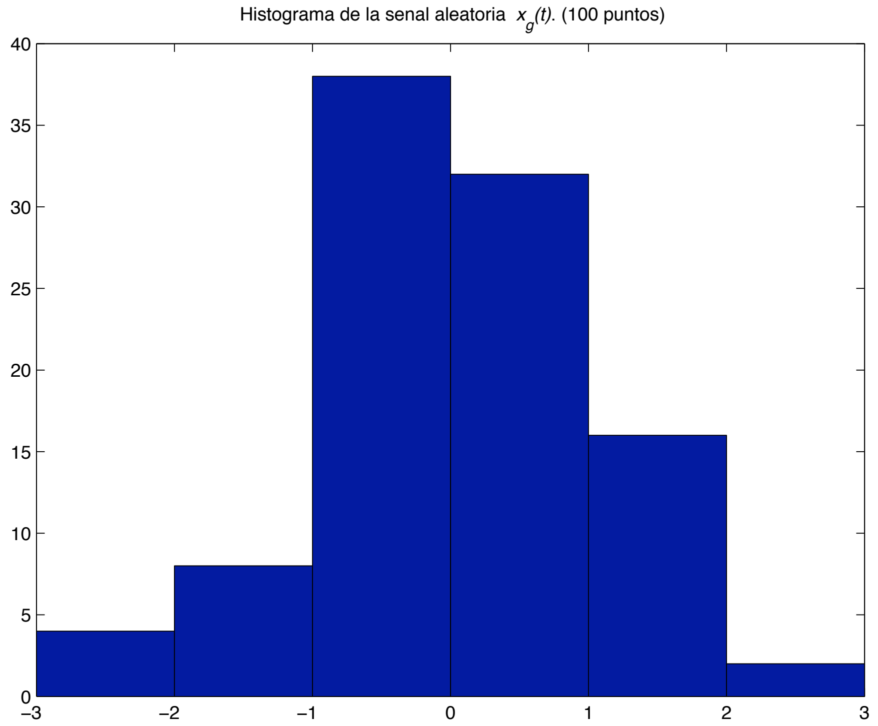
$$q_k = a_{i,k} + z_k$$

$$f_{q_k|\boldsymbol{A}}(q_k|\boldsymbol{a}_i) = \mathcal{N}\left(a_{i,k}, \frac{N_0}{2}\right)$$

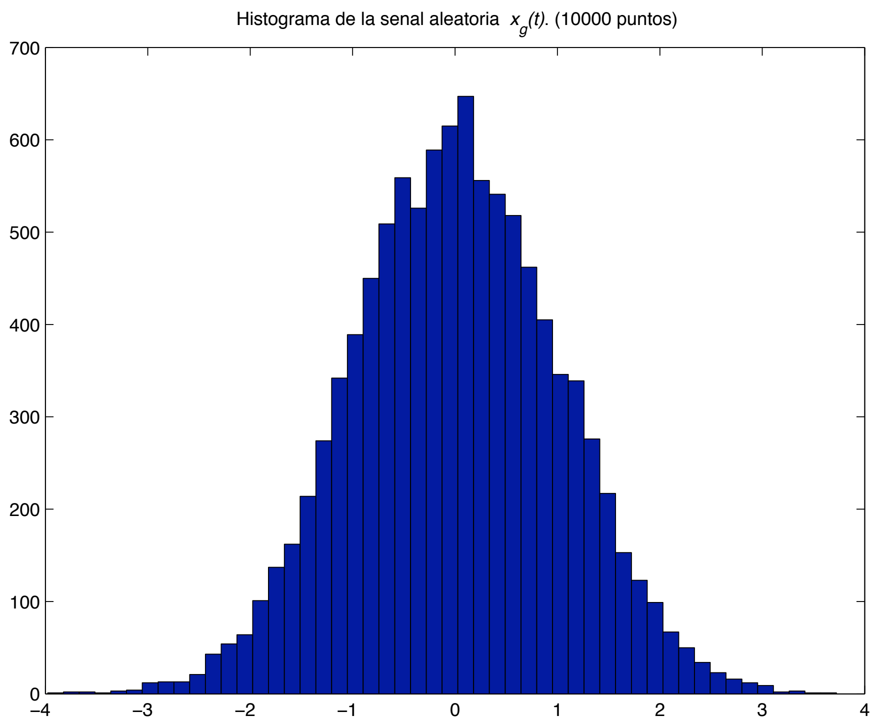
Realizations of a Gaussian random variable



Histogram with these 100 realizations of the random variable



Histogram with 10000 realizations of a Gaussian random variable



Evaluation of the symbol error probability

- When symbol $\boldsymbol{A} = \boldsymbol{a}_i$ is transmitted
  - Distribution of observation is $f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}|\boldsymbol{a}_i)$
  - Conditional probability of error

$$P_{e|\boldsymbol{A}=\boldsymbol{a}_i} \equiv P_{e|\boldsymbol{a}_i}$$

  When symbol $\boldsymbol{A} = \boldsymbol{a}_i$ is transmitted
    - ★ An error happens when decision is $\hat{\boldsymbol{A}} = \boldsymbol{a}_j \neq \boldsymbol{a}_i$
    - ★ This happens when transmitting $\boldsymbol{a}_i$ the observation $\boldsymbol{q} \notin I_i$

$$P_{e|\boldsymbol{a}_i} = \int_{q \notin I_i} f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}|\boldsymbol{a}_i)\, d\boldsymbol{q}$$

- Symbol error probability
  - Conditional error probabilities are averaged

$$P_e = \sum_{i=0}^{M-1} p_{\boldsymbol{A}}(\boldsymbol{a}_i)\, P_{e|\boldsymbol{a}_i}$$

    - ★ For equiprobable symbols

$$p_A(\boldsymbol{a}_i) = \frac{1}{M} \rightarrow P_e = \frac{1}{M} \sum_{i=0}^{M-1} P_{e|\boldsymbol{a}_i}$$

Evaluation of the Bit Error Rate (BER)

- Conditional BER for $\boldsymbol{a}_i$ are averaged

$$BER = \sum_{i=0}^{M-1} p_A(a_i)\, BER_{\boldsymbol{a}_i}$$

- Evaluation of conditional BER

$$BER_{\boldsymbol{a}_i} = \sum_{\substack{j=0 \\ j \neq i}}^{M-1} P_{e|\boldsymbol{a}_i \rightarrow \boldsymbol{a}_j} \frac{m_{e|\boldsymbol{a}_i \rightarrow \boldsymbol{a}_j}}{m}$$

  - $P_{e|a_i \rightarrow a_j}$: probability of transmitting $\boldsymbol{A} = \boldsymbol{a}_i$ and decide $\hat{A} = \boldsymbol{a}_j$

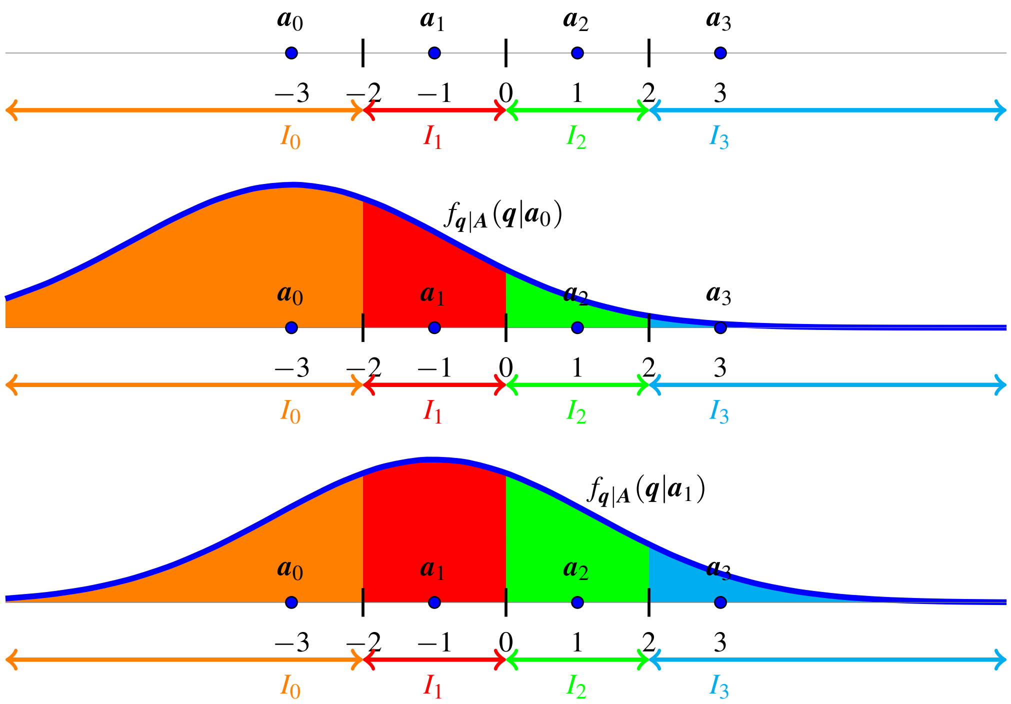
$$P_{e|\boldsymbol{a}_i \rightarrow \boldsymbol{a}_j} = \int_{\boldsymbol{q}_0 \in I_j} f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}_0|\boldsymbol{a}_i)\, d\boldsymbol{q}_0$$

  - $m_{e|\boldsymbol{a}_i \rightarrow \boldsymbol{a}_j}$: number of erroneous bits associated to this erroneous decision
  - m: number of bits per symbol in the constellation

Conditional distributions and error probabilities

Example: 1-D constellation with 4 equiprobable symbols

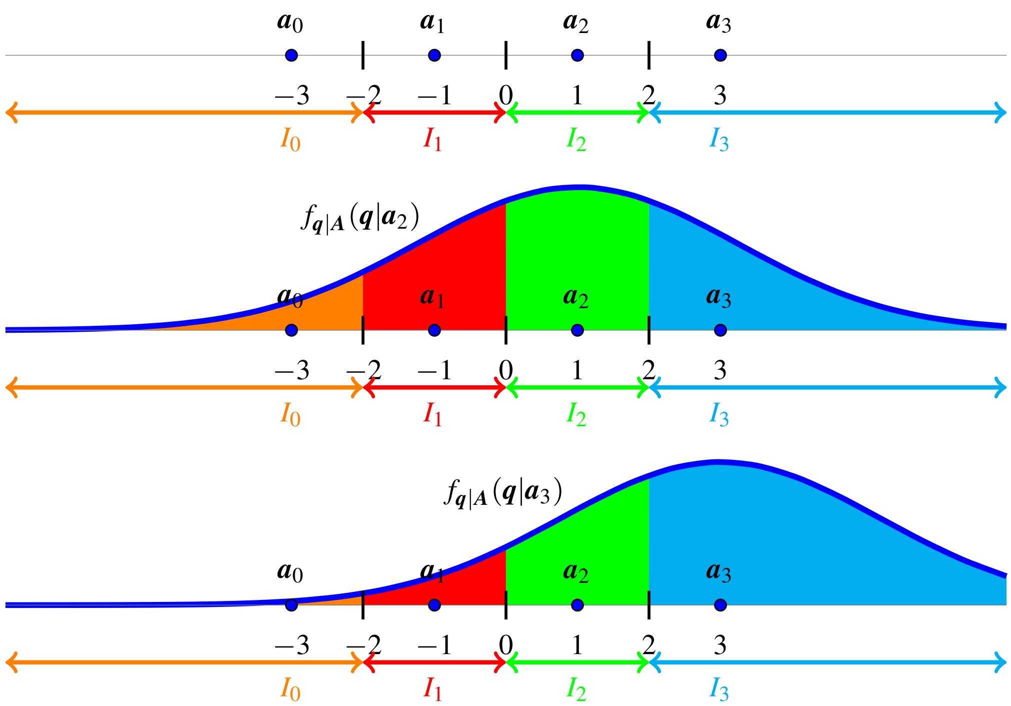
Ideal case: $q[n] = A[n] + z[n]$ with $z[n] \sim \mathcal{N}(0, N_0/2) \Rightarrow f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}|\boldsymbol{a}_i) = \mathcal{N}(\boldsymbol{a}_i, N_0/2)$



Conditional distributions and error probabilities (II)

Example: 1-D constellation with 4 equiprobable symbols

Ideal case: $q[n] = A[n] + z[n]$ with $z[n] \sim \mathcal{N}(0, N_0/2) \Rightarrow f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}|\boldsymbol{a}_i) = \mathcal{N}(\boldsymbol{a}_i, N_0/2)$



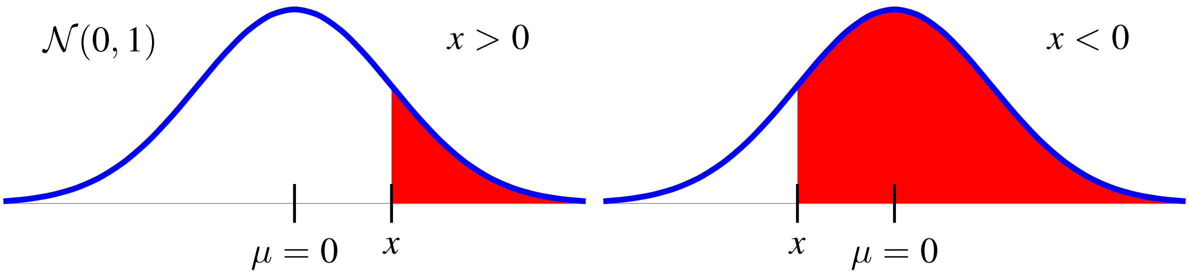
$Q(x)$ function

- Numerically obtained function related with the integral of a Gaussian distribution
- Definition: probability of a zero mean and unit variance Gaussian random variable taking values greater than argument

$$\text{Si } X \sim \mathcal{N}(0,1) \rightarrow f_X(x) = \mathcal{N}(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \rightarrow Q(x) = P(X > x)$$

$$Q(x) = \int_x^{+\infty} f_X(z)\, dz = \int_x^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz$$

- Graphic interpretation
  - Only tabulated for positive arguments $x \geq 0$
  - For $x < 0$, given the symmetry of $f_X(x)$: $Q(-x) = 1 - Q(x)$



$Q(x)$ function - Properties

- Relationship with the distribution function of a Gaussian random variable (with $\mu = 0, \sigma^2 = 1$)

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{t^2}{2}} dt$$

- Function $Q(x) = 1 - F_X(x) = P(X > x)$ for $\mu = 0, \sigma^2 = 1$
- Some properties of $Q(x)$ function
  - $Q(-x) = 1 - Q(x)$
  - $Q(0) = \frac{1}{2}$
  - $Q(\infty) = 0$

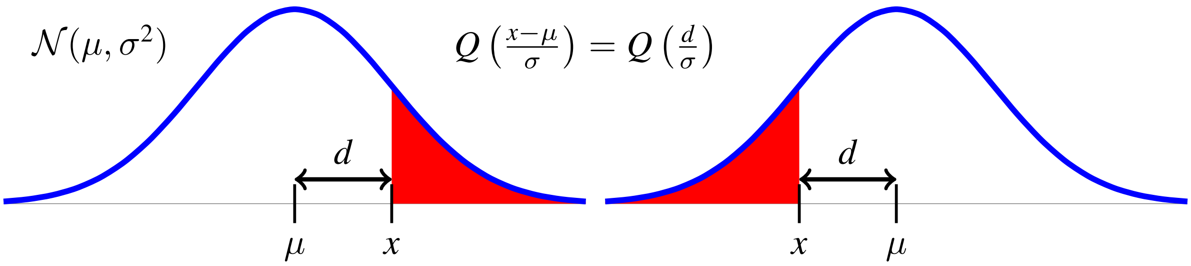
Integrals over Gaussian distributions $\mathcal{N}(\mu, \sigma^2)$

- If Gaussian distribution has mean μ and variance σ^2

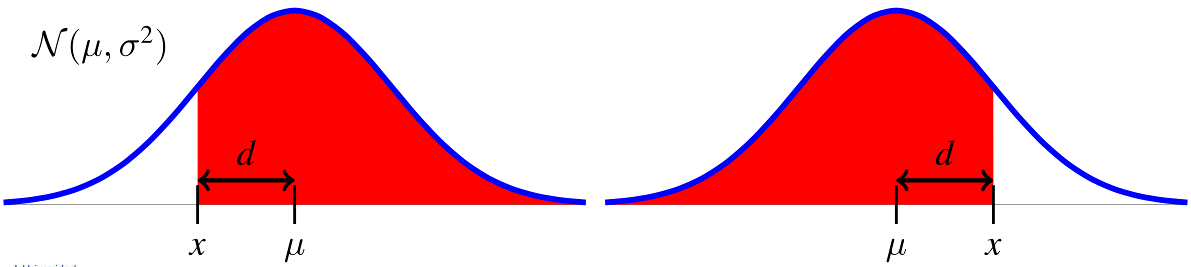
$$P(X > x) = Q\left(\frac{x-\mu}{\sigma}\right)$$

- Graphic interpretation (considering definition and symmetry)

$$Q\left(\frac{x-\mu}{\sigma}\right) = Q\left(\frac{d}{\sigma}\right)$$

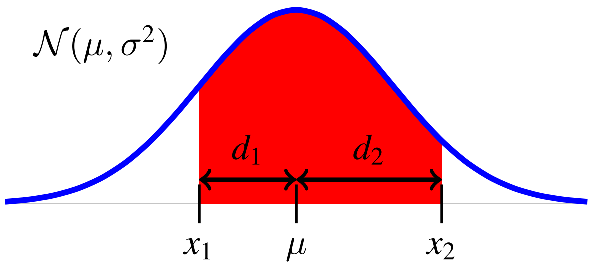


$$Q\left(\frac{x-\mu}{\sigma}\right) = Q\left(-\frac{d}{\sigma}\right) = 1 - Q\left(\frac{d}{\sigma}\right)$$

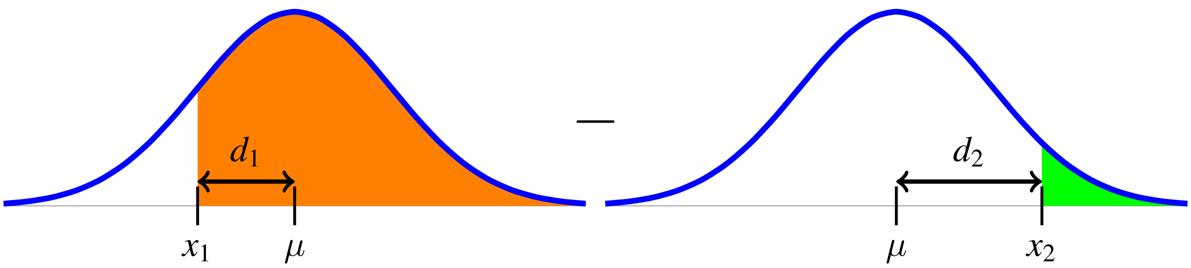


Integrals over intervals in $\mathcal{N}(\mu, \sigma^2)$

- In general can be written as additions or differences of terms involving integrals from one point to $\pm\infty$, which can be obtained using $Q(x)$
- An illustrative example



$$\int_{x_1}^{x_2} \mathcal{N}(\mu, \sigma^2) = \int_{x_1}^{\infty} \mathcal{N}(\mu, \sigma^2) - \int_{x_2}^{\infty} \mathcal{N}(\mu, \sigma^2) = \left[1 - Q\left(\frac{d_1}{\sigma}\right)\right] - \left[Q\left(\frac{d_2}{\sigma}\right)\right]$$



Time ambiguity function (time autocorrelation)

- Definition for a signal $x(t)$

$$r_x(t) = x(t) * x^*(-t)$$

Convolution of the signal with its matched signal, which in frequency becomes

$$R_x(j\omega) = |X(j\omega)|^2$$

- Properties
  - Symmetric function with maximun value at zero
  - Allows to obtain the energy of $x(t)$

$$\mathcal{E}\{x(t)\} = r_x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_x(j\omega)\, d\omega$$

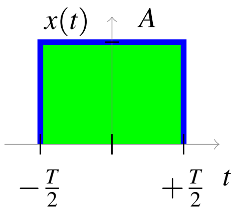
    This is evident from definition of energy (Parseval identity)

$$\mathcal{E}\{x(t)\} = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2\, d\omega$$

  - Shift-invariant function (shifts in $x(t)$)

$$y(t) = x(t - t_0) \rightarrow r_y(t) = r_x(t)$$

Example



$$\mathcal{E}\{x(t)\} = \int_{-\infty}^{\infty} |x(t)|^2\, dt = A^2 T$$