uc3m | Universidad **Carlos III** de Madrid

Digital Communications

Grades in English

Chapter 1

Pulse amplitude (linear) modulations

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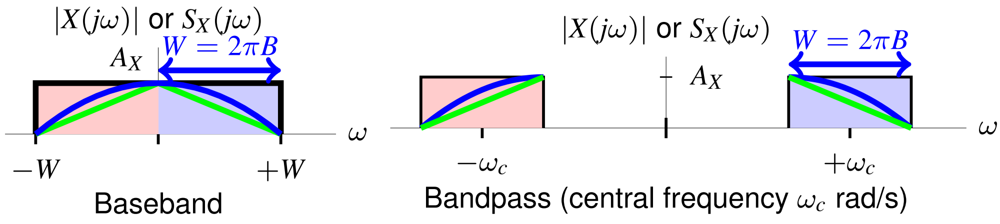
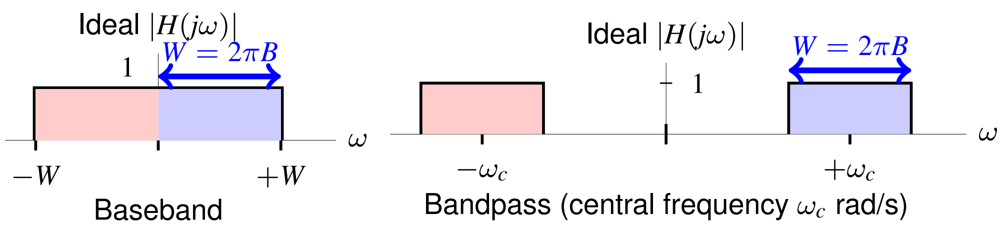
Objectives

- Generation of bandlimited signals: signals with a finite bandwidth
  - Because real channels are bandlimited (finite available bandwidth)
  - Bandwidth of a signal
    - ★ Range of POSITIVE frequencies with non-null components
    - ★ Usual notation: B Hz, $W = 2\pi B$ rad/s
      - Baseband signals
      - Bandpass signals (central frequency ω_c rad/s)



- Design to transmit digital information through non-ideal linear channels



  - Non-ideal channel response (non-flat response in the band): linear distortion

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Communication Theory - Basic model

- Linear modulation in a N-dimensional signal space

$$s(t) = \sum_n \sum_{j=0}^{N-1} A_j[n]\ \phi_j(t - nT)$$

  - Information is linearly conveyed
    - In the amplitude of the set of N functions $\{\phi_j(t)\}_{j=0}^{N-1}$
  - Encoder: $\boldsymbol{A}[n]$
    - Constellation in a space of dimension N
    - Designed considering energy (E_s) and performance (P_e, BER)
      - E_s: mean energy per symbol ($E_s = E[|A[n]|^2]$)
      - P_e: probability of symbol error
      - BER: bit error rate
  - Modulator: $\{\phi_j(t)\}_{j=0}^{N-1}$
    - Designed considering channel characteristics
    - Ideally: the only distortion appearing after the transmission is additive noise (white and Gaussian)

Baseband PAM modulation

- One-dimensional modulation: $N = 1$

$$s(t) = \sum_n A[n]\, g(t - nT) \begin{cases} \text{PAM (\textit{Pulse Amplitude Modulation})} \\ \text{ASK (\textit{Amplitude Shift Keying})} \end{cases}$$

- Symbol length T (inverse of symbol rate $R_s = 1/T$ bauds)
- Sequence $A[n]$ is the sequence of symbols
  - Alphabet is called constellation (1-D plot)
  - Conversion from bits to symbols: encoder
    - M-ary constellations (M-PAM)

$$M = 2^m \text{ symbols} \qquad m = \log_2 M \text{ bits/symbol}$$

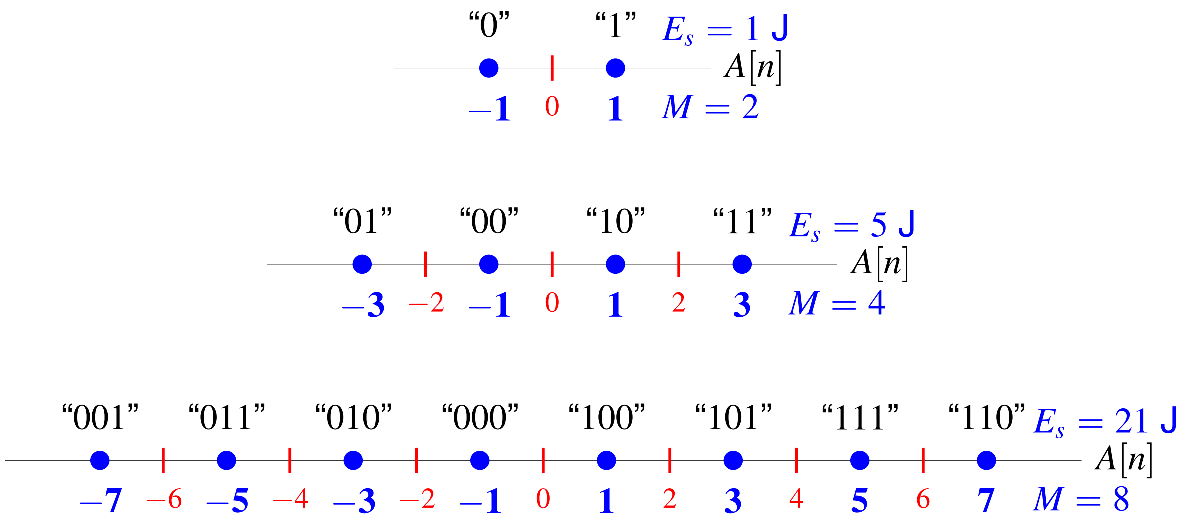
    - Binary assignment: Gray encoding
    - Normalized levels:

$$A[n] \in \{\pm 1, \pm 3, \cdots, \pm(M-1)\}, \qquad E_s = E\left[|A[n]|^2\right] = \frac{M^2 - 1}{3} \text{ J}$$

- Waveform $g(t)$ (one dimensional orthonormal basis):
  - Normalization: unit energy ($\mathcal{E}\{g(t)\} = 1$ J)
  - Tipically receives two names
    - Transmitter filter
    - Shaping pulse (although it is not necessarily a pulse)

Examples of M-PAM constellations

- Normalized levels: $A[n] \in \{\pm 1, \pm 3, \cdots, \pm(M-1)\}$
  - Distance to the decision thresholds for equiprobable symbols is 1
- Binary assignment by Gray encoding
  - Assignments for symbols at minimum distance differ in a single bit
- Examples: 2-PAM, 4-PAM, 8-PAM



Unnormalized M-PAM constellations

- Alphabet of the constellation

$$A[n] \in \{\pm d, \pm 3d, \cdots, \pm(M-1)d\}$$

  - Distance to the decision thresholds for equiprobable symbols is d

- Mean energy per symbol

$$E_s = E\left[|A[n]|^2\right] = d^2 \times \frac{M^2-1}{3} \text{ J}$$

Encoder: Symbol rate vs. bit rate

- Symbol duration (or symbol length): T seconds
  - A symbol of sequence $A[n]$ is transmited each T seconds
- M-ary constellations transmit $m = \log_2 M$ bits per symbol
  - Binary assignment: Gray encoding
- There are two related transmission rates in a digital system
  - Symbol rate (for symbol sequence $A[n]$)

$$R_s = \frac{1}{T} \text{ bauds (symbols/s)}$$

  - Binary rate (for bit sequence $B_b[\ell]$)

$$R_b = \frac{1}{T_b} \text{ bits/s}$$

- Relationship between transmission rates

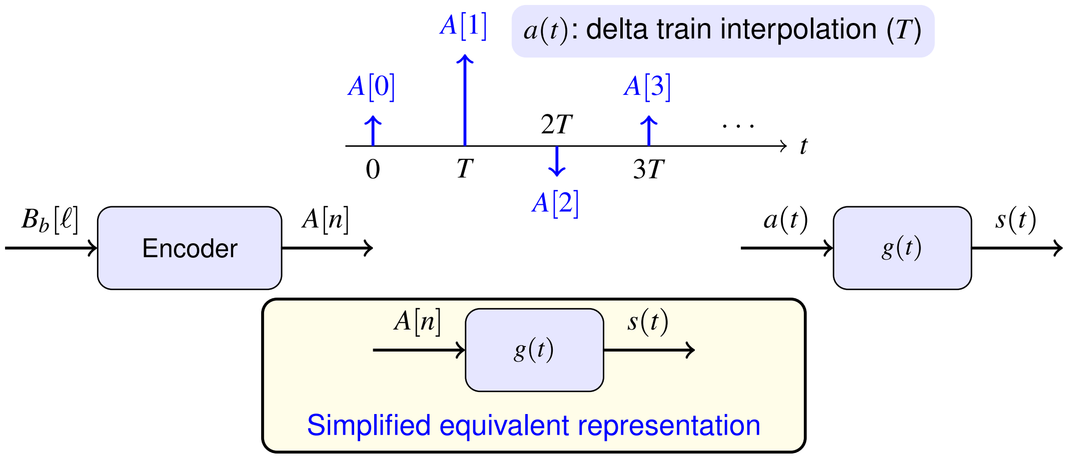
$$R_b = m \times R_s \qquad R_s = \frac{R_b}{m} \qquad T = m \times T_b \qquad T_b = \frac{T}{m}$$

PAM modulation as a filtering process

- Conversion of discrete time sequence $A[n]$ to continuous time signal
  - Signal of symbols: train of impulses (deltas) with amplitudes $A[n]$ at nT

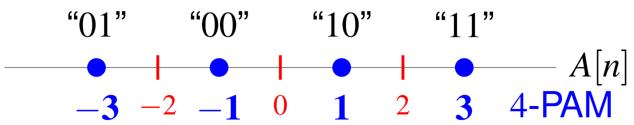
$$a(t) = \sum_n A[n]\ \delta(t - nT)$$

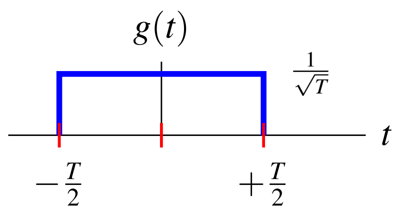
- Generation of PAM signal $s(t) = \sum_n A[n]\ g(t - nT) = a(t) * g(t)$



Example: modulation of a binary sequence (initial 10 bits)

ℓ	0	1	2	3	4	5	6	7	8	9	...
$B_b[\ell]$	1	1	0	0	0	1	1	1	1	0	...

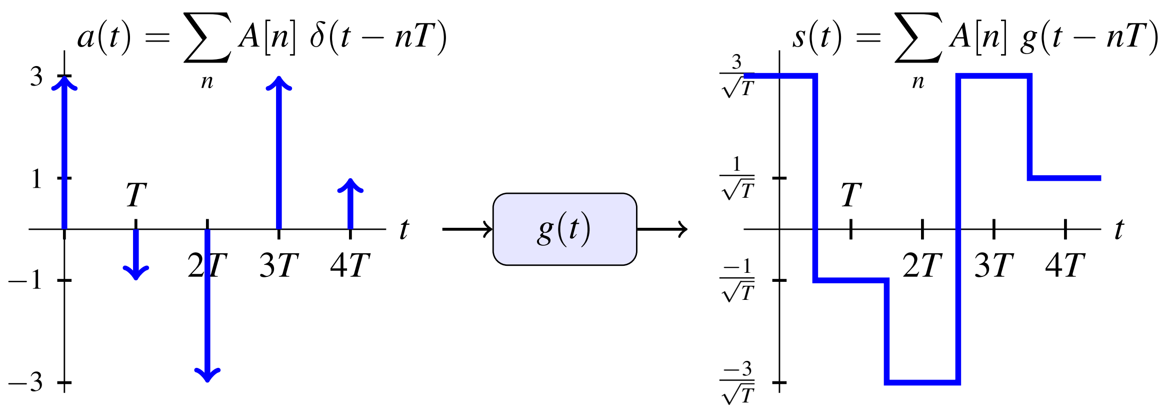




- Encoding: bits to symbols

n	0	1	2	3	4	...
$A[n]$	+3	-1	-3	+3	+1	...

- Delta train $a(t)$ + filtering of $a(t)$ with $g(t)$: PAM modulated signal



Spectrum of a baseband PAM

- PAM baseband signal

$$s(t) = \sum_n A[n]\ g(t - nT)$$

- Let $\{A[n]\}_{n=-\infty}^{\infty}$ be a sequence of random variables (stationary random process):
  - Mean energy per symbol $E_s = E[|A[n]|^2]$
  - Mean $m_A[n] = E[A[n]] = m_A$ ($m_A = 0$ for M-PAM constellations)
  - Autocorrelation function $R_A[n+k, n] = E[A[n+k]\ A^*[n]] = R_A[k]$
  - Power spectral density function of $A[n]$ is

$$S_A(e^{j\omega}) = \mathcal{FT}\{R_A[k]\} = \sum_{k=-\infty}^{\infty} R_A[k]\ e^{-j\omega k}$$

- Let $g(t)$ be any deterministic function with Fourier transform $G(j\omega)$

Review: Wiener-Khinchin theorem

- Power spectral density

$$S_X(j\omega) \stackrel{def}{=} E\left[\lim_{T\to\infty} \frac{|X^{[T]}(j\omega)|^2}{T}\right] = \lim_{T\to\infty} \frac{1}{T}\ E\left[|X^{[T]}(j\omega)|^2\right]$$

Interpretation: average of the squared frequency response of the (truncated) process

- Wiener-Khinchin theorem
If for any finite value τ ans any interval $\mathcal{A}$, of length $|\tau|$, the autocorrelationof random process fulfills

$$\left|\int_{\mathcal{A}} R_X(t+\tau, t)\ dt\right| < \infty$$

power spectral density of $X(t)$ is given by the Fourier transform of

$$S_X(j\omega) = \mathcal{FT}\{\langle R_X(t+\tau, t)\rangle\}$$

$$\langle R_X(t+\tau, t)\rangle \stackrel{def}{=} \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} R_X(t+\tau, t)\ dt$$

Corollary of Wiener-Khinchin theorem

- Corollary 1: If $X(t)$ is an stationary process and $\tau R_X(\tau) < \infty$ for all $\tau < \infty$, then
$$S_X(j\omega) = \mathcal{FT}\left\{R_X(\tau)\right\}$$
- Corollary 2: If $X(t)$ is cyclostationary and
$$\left|\int_0^{T_o} R_X(t+\tau,t)dt\right| < \infty$$
then
$$S_X(j\omega) = \mathcal{FT}\left\{\widetilde{R}_X(\tau)\right\}$$
where
$$\widetilde{R}_X(\tau) = \frac{1}{T_o}\int_{-T_o/2}^{T_o/2} R_X(t+\tau,t)\, dt$$
and T_o is the period of the cyclostationary process

Mean and autocorrelation of a baseband PAM

$$s(t) = \sum_{n=-\infty}^{\infty} A[n]\, g(t-nT)$$

$$m_S(t) = E\left[\sum_n A[n]g(t-nT)\right] = \sum_n \underbrace{E[A[n]]}_{m_A[n]} g(t-nT) = m_A \sum_n g(t-nT)$$

$$\begin{aligned} R_S(t+\tau,t) &= E[s(t+\tau)\, s^*(t)] \\ &= E\left[\left(\sum_k A[k]\, g(t+\tau-kT)\right)\left(\sum_j A^*[j]\, g^*(t-jT)\right)\right] \\ &= \sum_k\sum_j \underbrace{E[A[k]\, A^*[j]]}_{R_A[k-j]}\, g(t+\tau-kT)\, g^*(t-jT) \\ &= \sum_k\sum_j R_A[k-j]\, g(t+\tau-kT)\, g^*(t-jT) \end{aligned}$$

Cyclostationarity

- Mean is a periodical function of t (period T)

$$m_S(t+T) = m_A \sum_n g(t+T-nT) = m_A \sum_n g(t-(n-1)T)$$
$$\stackrel{n'=n-1}{=} m_A \sum_{n'} g(t-n'T) = m_S(t)$$

- Autocorrelation is a periodical function of t (period T)

$$R_S(t+\tau+T, t+T) =$$
$$= \sum_k \sum_j R_A[k-j]\; g(t+\tau+T-kT) g^*(t+T-jT)$$
$$= \sum_k \sum_j R_A[k-j]\; g(t+\tau-(k-1)T) g^*(t-(j-1)T)$$
$$\stackrel{k'=k-1,\; j'=j-1}{=} \sum_{k'} \sum_{j'} R_A[(k'+1)-(j'+1)]\; g(t+\tau-k'T) g^*(t-j'T)$$
$$= \sum_{k'} \sum_{j'} R_A[k'-j']\; g(t+\tau-k'T) g^*(t-j'T+\tau) = R_S(t+\tau, t)$$

Time average of autocorrelation function

$$\tilde{R}_S(\tau) = \frac{1}{T}\int_0^T R_S(t+\tau, t)\; dt$$
$$= \frac{1}{T}\int_0^T \sum_k \sum_j R_A[k-j]\; g(t+\tau-kT) g^*(t-jT)\; dt$$
$$\stackrel{m=k-j}{=} \frac{1}{T} \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} R_A[m] \int_0^T g(t+\tau-kT) g^*(t-(k-m)T)\; dt$$
$$\stackrel{u=t+\tau-kT}{=} \frac{1}{T} \sum_{m=-\infty}^{\infty} R_A[m] \sum_{k=-\infty}^{\infty} \int_{\tau-kT}^{\tau-(k-1)T} g(u) g^*(u-\tau+mT)\; du$$
$$= \frac{1}{T} \sum_{m=-\infty}^{\infty} R_A[m] \int_{-\infty}^{\infty} g(u) g^*(-(\tau-mT-u))\; du$$
$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} R_A[k]\; r_g(\tau-kT)$$

$$r_g(t) = g(t) * g^*(-t)$$

Power spectral density (PSD)

$$\begin{aligned}\tilde{R}_S(\tau) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} R_A[k]\; r_g(\tau - kT)\\ &= \frac{1}{T}\left(\sum_{k=-\infty}^{\infty} R_A[k]\; \delta(\tau - kT)\right) * r_g(\tau)\\ &= \frac{1}{T}\left(\sum_{k=-\infty}^{\infty} R_A[k]\; \delta(\tau - kT)\right) * g(\tau) * g^*(-\tau)\end{aligned}$$

$$\begin{aligned}S_S(j\omega) &= \mathcal{FT}\left\{\tilde{R}_S(\tau)\right\}\\ &= \frac{1}{T}\left(\sum_{k=-\infty}^{\infty} R_A[k]\; e^{-j\omega kT}\right) G(j\omega)\; G^*(j\omega)\\ &= \frac{1}{T}\, S_A(e^{j\omega T})\; |G(j\omega)|^2\end{aligned}$$

Power spectral density - Analysis

$$S_S(j\omega) = \frac{1}{T}\, S_A(e^{j\omega T})\; |G(j\omega)|^2$$

- Three contributions:
  - A constant scale factor given by symbol rate: $\frac{1}{T} = R_s$ bauds
  - A deterministic component given by $g(t)$: $|G(j\omega)|^2$
  - A statistical component given by $A[n]$: $S_A(e^{j\omega})$
    - Evaluated at ωT, i.e. $S_A(e^{j\omega T})$
- For white sequences $A[n]$ (the most typical case)

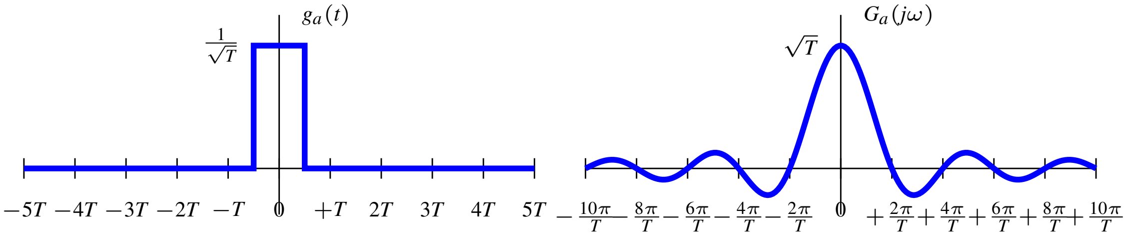
$$R_A[k] = E_s\, \delta[k] \quad \overset{\mathcal{FT}}{\leftrightarrow} \quad S_A(e^{j\omega}) = E_s = E\left[|A[n]|^2\right]$$

$$S_S(j\omega) = \frac{E_s}{T}\, |G(j\omega)|^2 = E_s\, R_s\, |G(j\omega)|^2$$

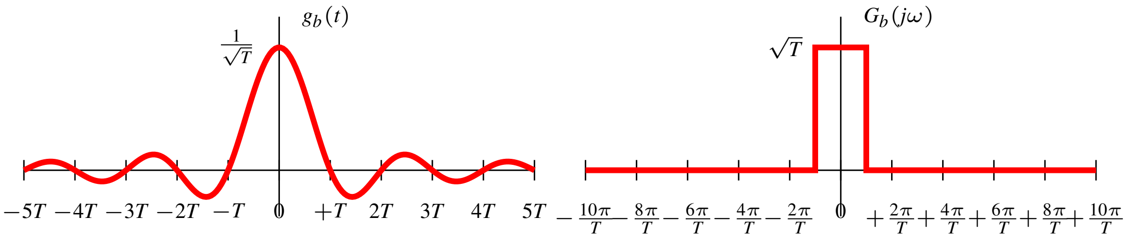
  - $g(t)$: Shaping pulse (determines the shape of spectrum)

Example pulses

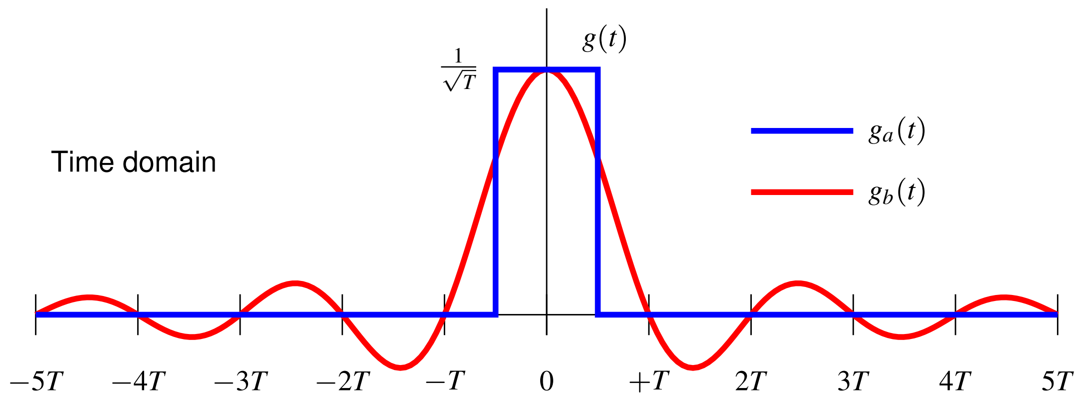
$$g_a(t) = \frac{1}{\sqrt{T}}\,\Pi\left(\frac{t}{T}\right) \quad \overset{\mathcal{FT}}{\leftrightarrow} \quad G_a(j\omega) = \sqrt{T}\ \text{sinc}\left(\frac{\omega T}{2\pi}\right)$$

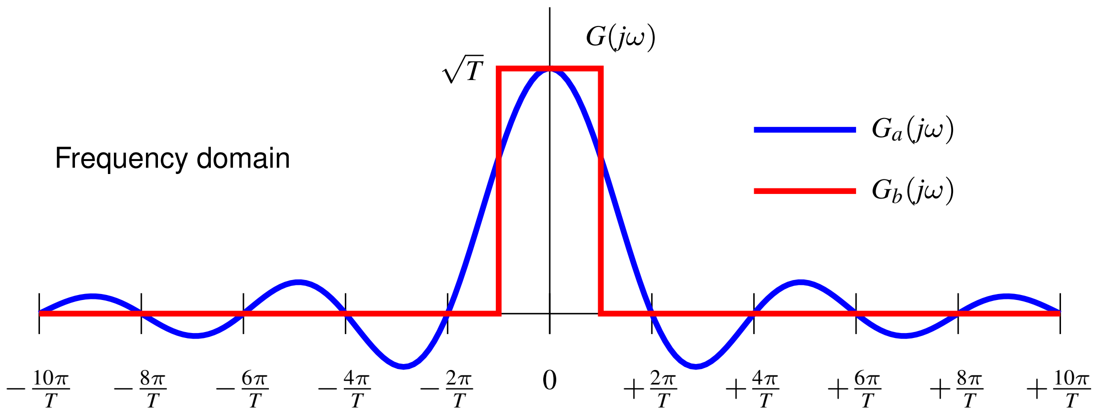


$$g_b(t) = \frac{1}{\sqrt{T}}\ \text{sinc}\left(\frac{t}{T}\right) \quad \overset{\mathcal{FT}}{\leftrightarrow} \quad G_b(j\omega) = \sqrt{T}\,\Pi\left(\frac{\omega T}{2\pi}\right)$$



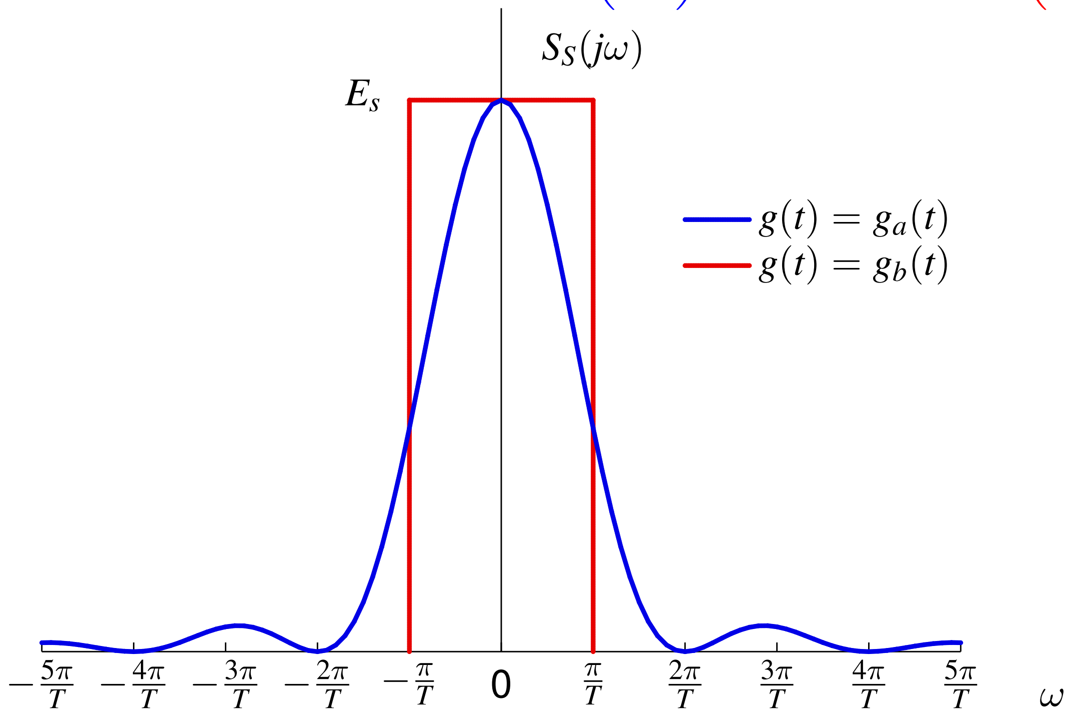
Example pulses (II)





Examples of $S_S(j\omega)$: white data sequence $A[n]$

$$S_S(j\omega) = \frac{E_s}{T}\,|G(j\omega)|^2 \quad G_a(j\omega) = \sqrt{T}\ \text{sinc}\left(\frac{\omega T}{2\pi}\right) \quad G_b(j\omega) = \sqrt{T}\ \Pi\left(\frac{\omega T}{2\pi}\right)$$



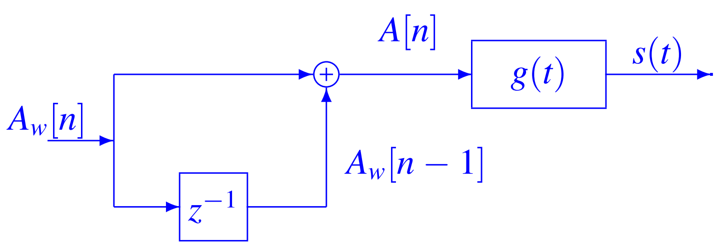
Examples of $S_S(j\omega)$: coloured data sequence $A[n]$

- PSD shape can be modified by introducing correlation in the transmitted data sequence
- Typical information data: white sequence $A_w[n]$
  - M-PAM: $A_w[n] \in \{\pm 1, \pm 3, \cdots, \pm(M-1)\}$
  - Mean energy per symbol: $E_s = E\left[|A_w[n]|^2\right] = \frac{M^2-1}{3}$
- Generation of a non-white (*coloured*) sequence $A[n]$

Example: $A[n] = A_w[n] + A_w[n-1]$

- Transmission of the coloured sequence $A[n]$

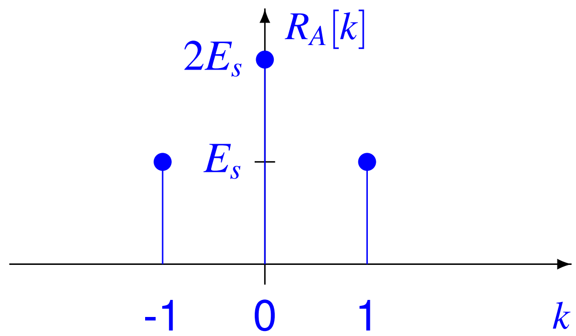
$$s(t) = \sum_{n=-\infty}^{\infty} A[n]\ g(t-nT)$$



Autocorrelation function of $A[n]$

- Autocorrelation of $A_w[n]$: $R_{A_w}[k] = E_s\ \delta[k]$
- Autocorrelation function of $A[n]$

$$\begin{aligned}
R_A[k] &= E\left[A[n+k]\ A^*[n]\right] \\
&= E\left[(A_w[n+k] + A_w[n+k-1])\ \ (A_w^*[n] + A_w^*[n-1])\right] \\
&= E\left[A_w[n+k]\ A_w^*[n]\right] + E\left[A_w[n+k]\ A_w^*[n-1]\right] \\
&\quad + E\left[A_w[n+k-1]\ A_w^*[n]\right] + E\left[A_w[n+k-1]\ A_w^*[n-1]\right] \\
&= R_{A_w}[k] + R_{A_w}[k+1] + R_{A_w}[k-1] + R_{A_w}[k] \\
&= 2R_{A_w}[k] + R_{A_w}[k+1] + R_{A_w}[k-1] \\
&= E_s(2\delta[k] + \delta[k+1] + \delta[k-1])
\end{aligned}$$



Power spectral density

- PSD for sequence $A[n]$

$$\begin{aligned}
S_A\left(e^{j\omega}\right) &= \mathcal{FT}\left\{R_A[k]\right\} = \sum_k R_A[k]\ e^{-j\omega k} \\
&= E_s\left(e^{j\omega} + 2\ e^{j0} + e^{-j\omega}\right) \\
&= 2E_s\ \left[1 + \cos(\omega)\right]
\end{aligned}$$

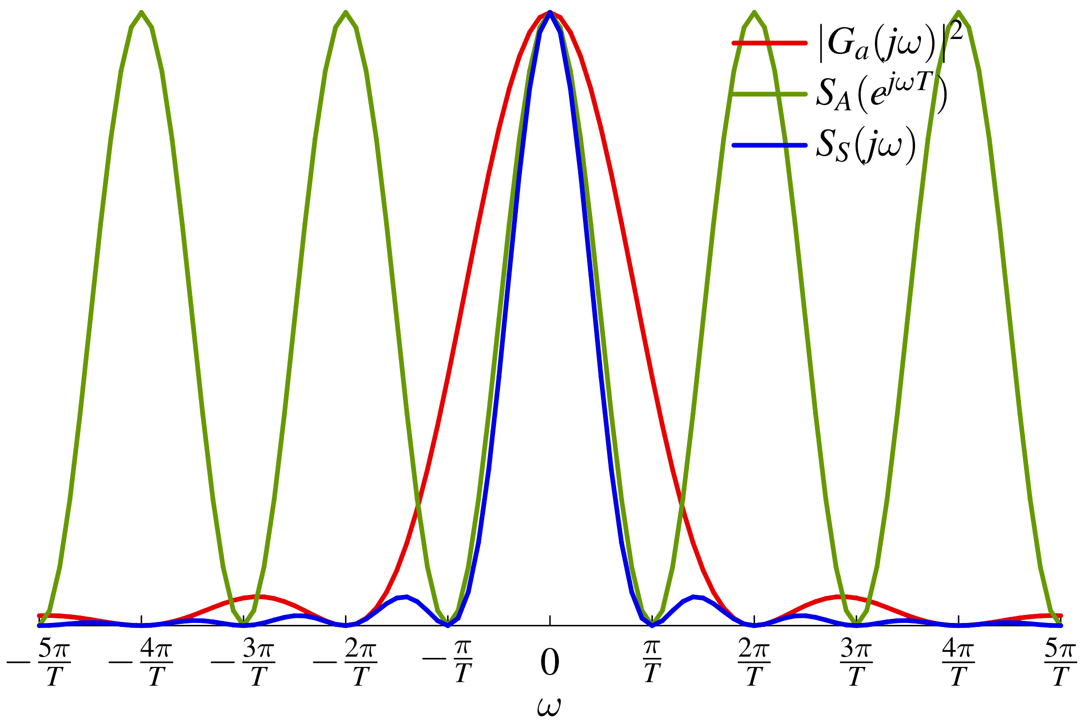
- PSD for baseband PAM signal $s(t)$
  This system transmits coloured data sequence $A[n]$

$$S_S(j\omega) = \frac{1}{T}\ S_A(e^{j\omega T})\ |G(j\omega)|^2$$

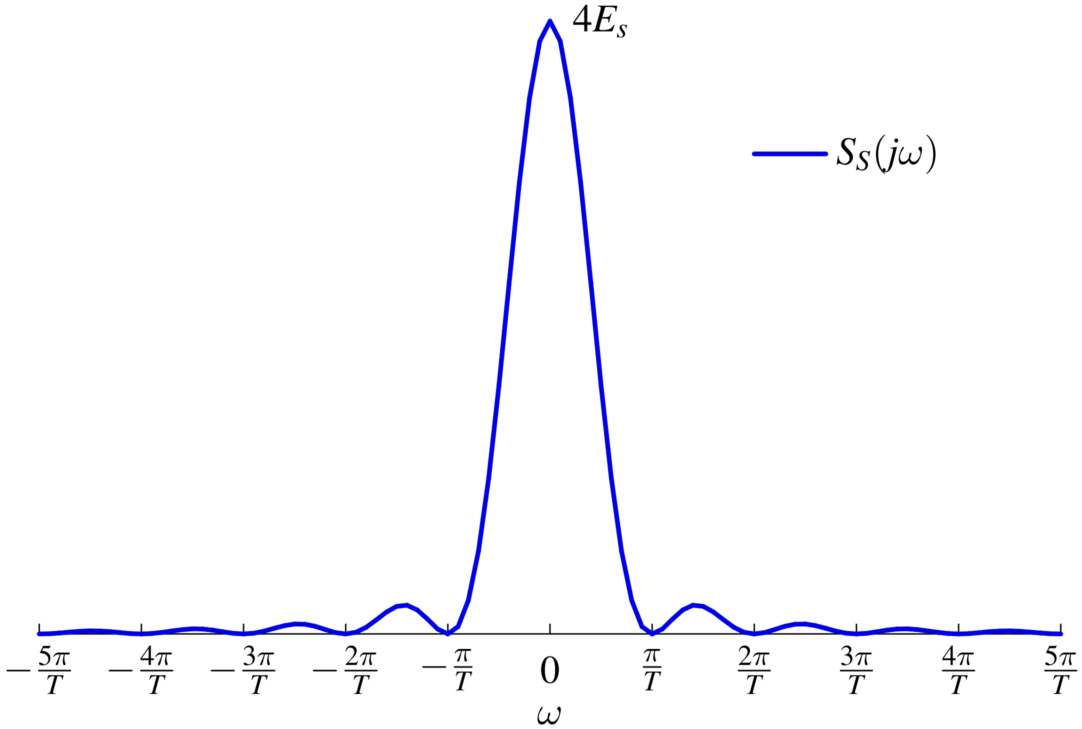
Evaluating the previously obtained expression for $S_A(e^{j\omega})$ in ωT we have

$$S_S(j\omega) = \frac{2E_s}{T}\ [1 + \cos(\omega T)]\ \ |G(j\omega)|^2$$

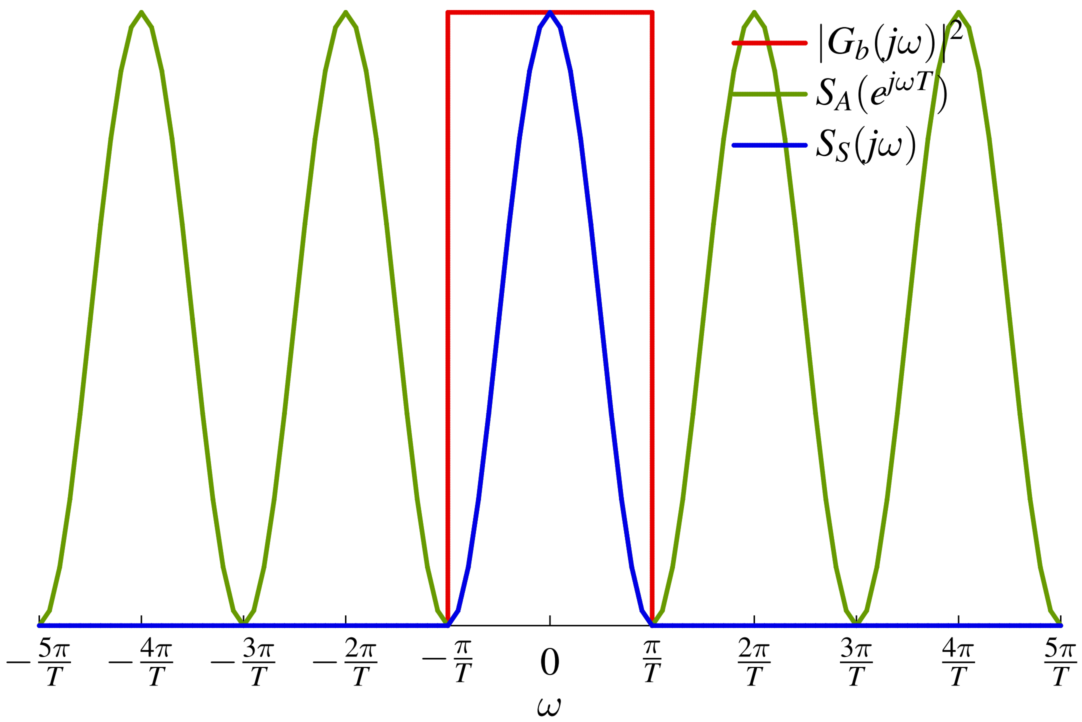
Power spectral density with $g_a(t)$



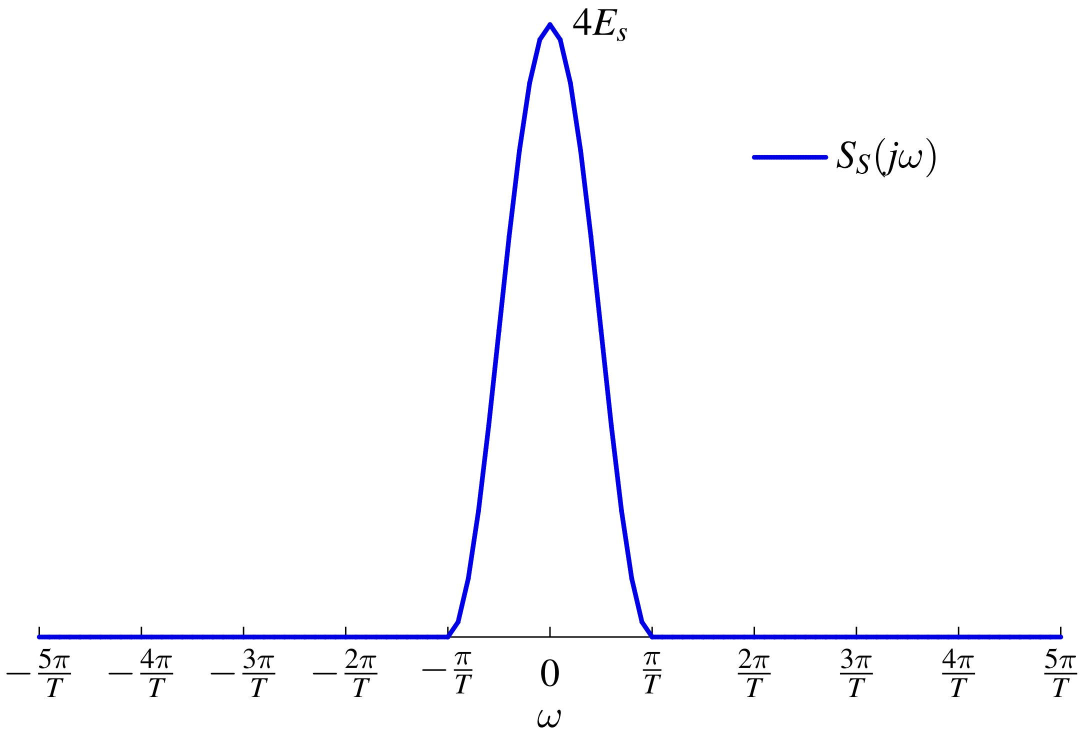
Power spectral density with $g_a(t)$



Power spectral density with $g_b(t)$



Power spectral density with $g_b(t)$



Power of a baseband PAM modulation

- Power can be obtained from $S_S(j\omega)$

$$P_S = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_S(j\omega)\, d\omega$$

- For white symbol sequences $A[n]$: $S_S(j\omega) = \frac{E_s}{T}\,|G(j\omega)|^2$

$$P_S = \underbrace{\frac{E_s}{T}}_{E_s \times R_s}\ \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} |G(j\omega)|^2\, d\omega}_{\mathcal{E}\{g(t)\}}$$

  - If $g(t)$ is normalized, by applying Parseval's relationship

$$P_S = \frac{E_s}{T} = E_s \times R_s \text{ Watts}$$

Selection of $g(t)$ waveforms

- Selection to be able to identify sequence $A[n]$ by sampling $s(t)$
  - **(a)** Pulses with duration limited to symbol period: T seconds
    - ⋆ No overlapping between waveforms delayed nT seconds

$$\text{Example}: g_a(t) = \frac{1}{\sqrt{T}}\,\Pi\left(\frac{t}{T}\right)$$

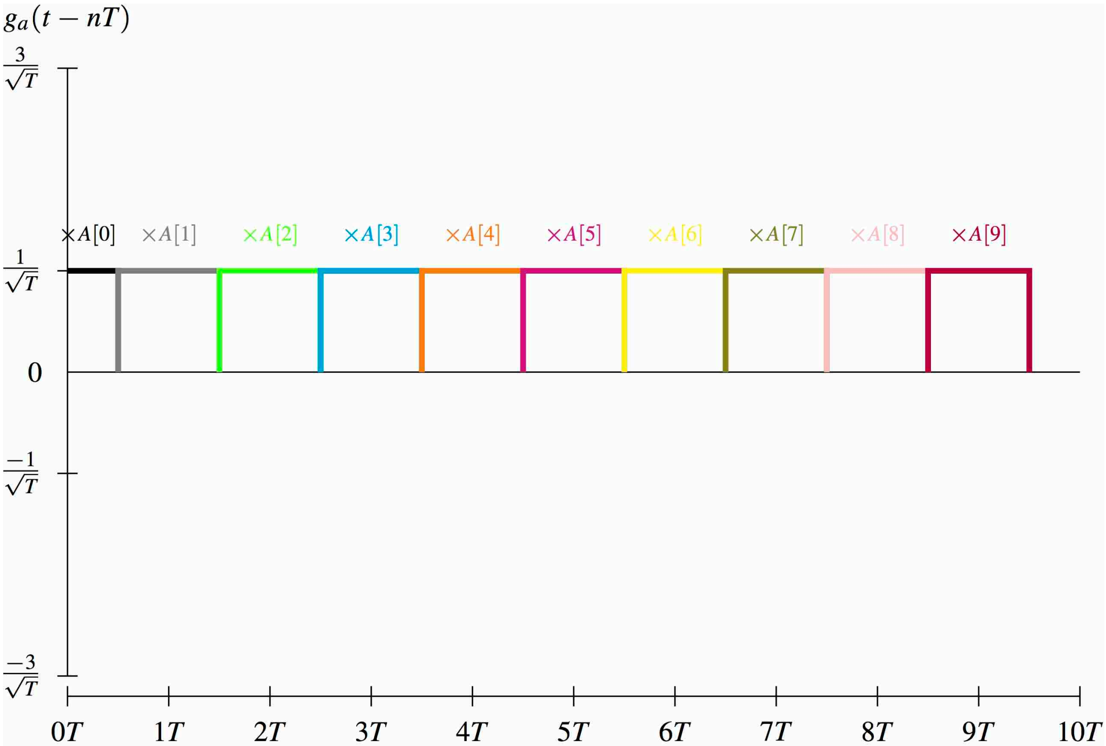
    - ⋆ Symbol $A[n]$ determines signal amplitude in its associated symbol interval
    - ⋆ Drawback: infinite bandwidth
  - **(b)** Pulses with infinite length: finite bandwidth
    - ⋆ Overlapping: non-destructive interference at some point each T seconds (periodical zeros)

$$g(nT) = 0,\ \forall n \neq 0;\ \text{Example}: g_b(t) = \frac{1}{\sqrt{T}}\ \mathrm{sinc}\left(\frac{t}{T}\right)$$

    - ⋆ Symbol $A[n]$ determines signal amplitude at the nondestructive point associated to its symbol interval

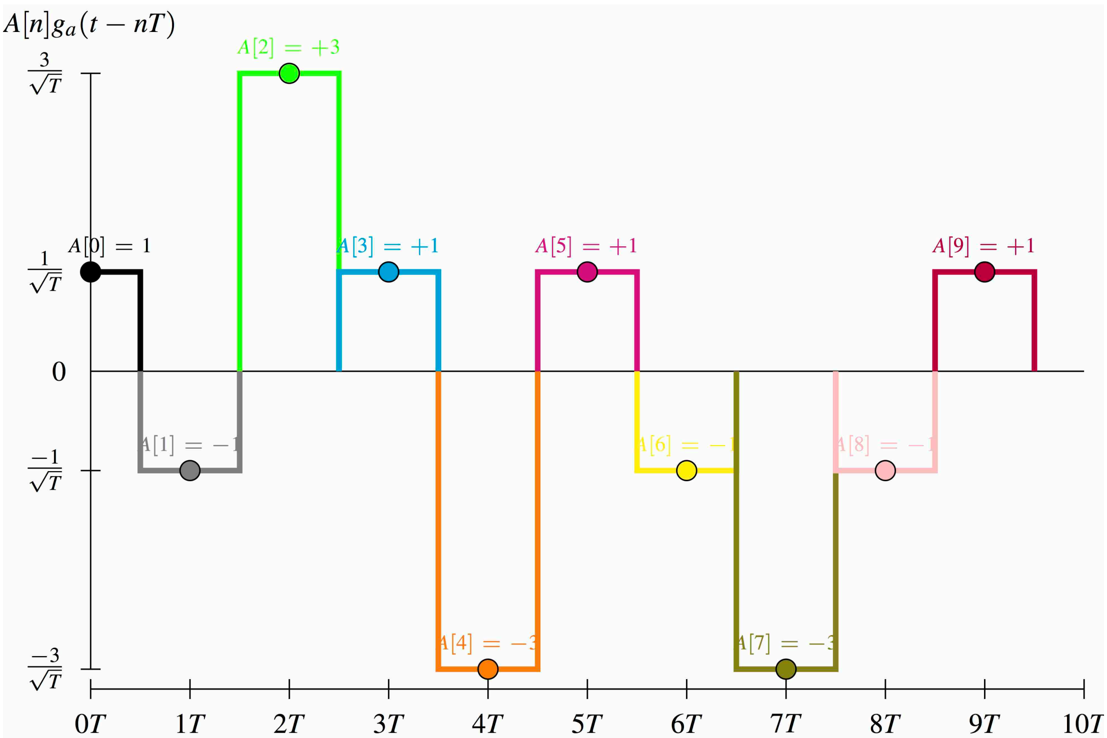
Rectangle : pulses delayed nT ($n \in \{0, 1, 2, \cdots\}$)

n	0	1	2	3	4	5	6	7	8	9
$A[n]$	$+1$	-1	$+3$	$+1$	-3	$+1$	-1	-3	-1	$+1$



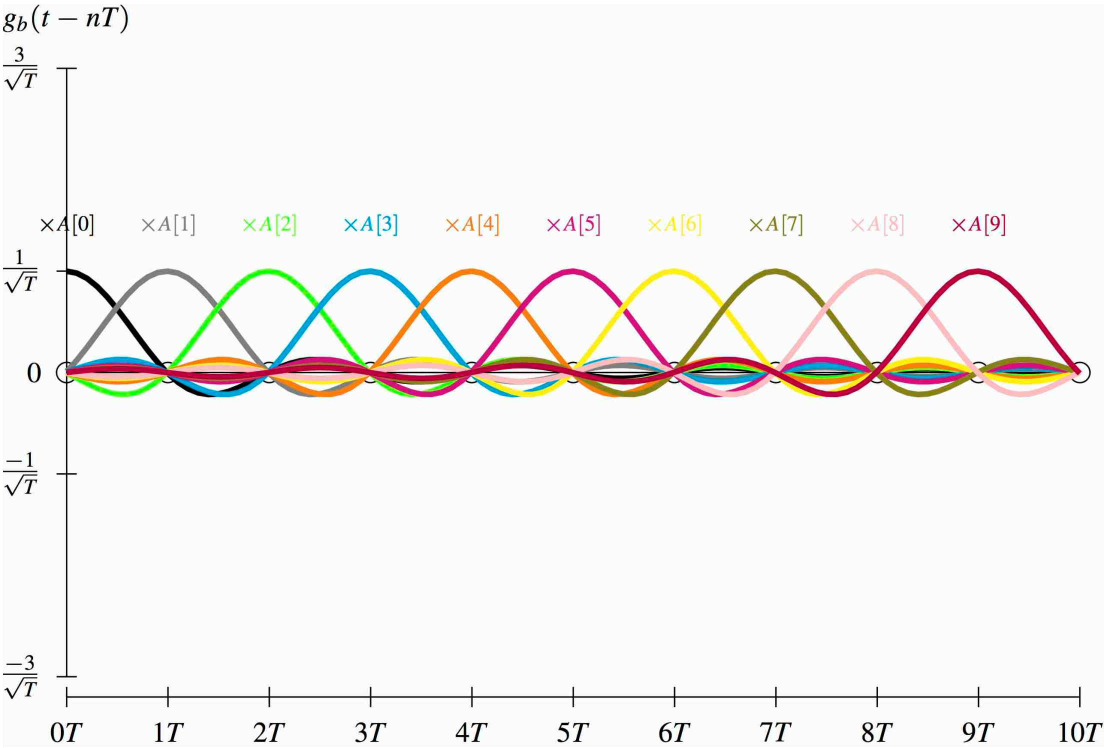
Rectangle : Contribution of each symbol

n	0	1	2	3	4	5	6	7	8	9
$A[n]$	$+1$	-1	$+3$	$+1$	-3	$+1$	-1	-3	-1	$+1$



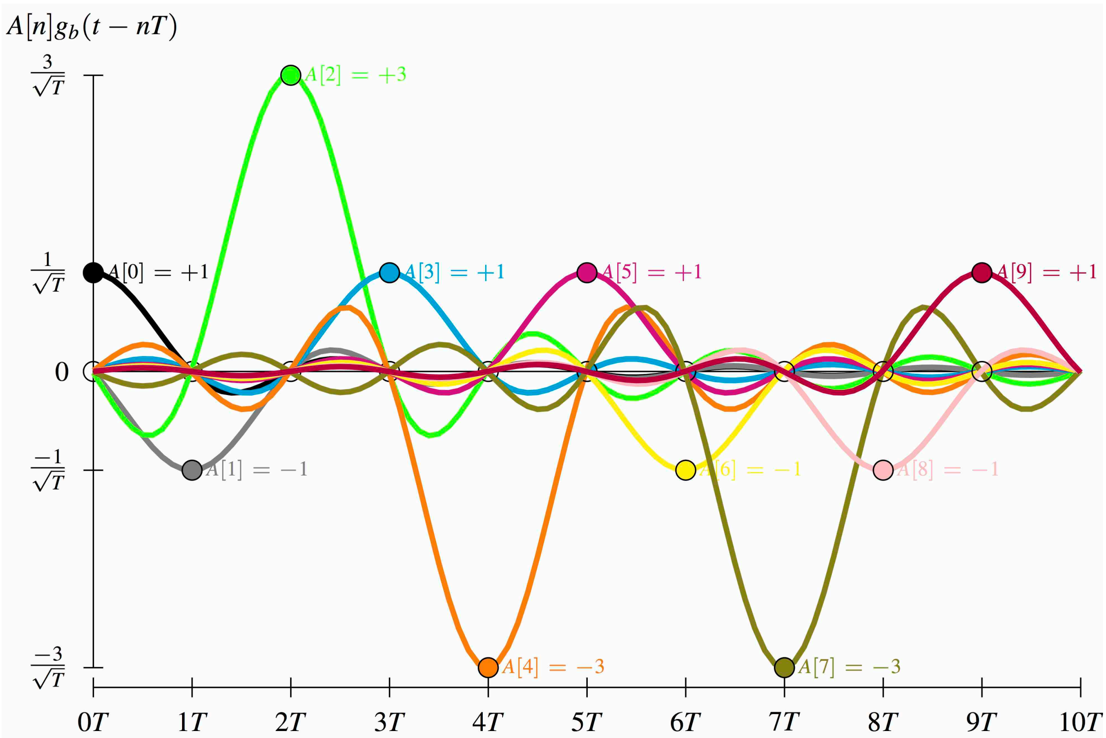
Sinc : pulses delayed nT ($n \in \{0, 1, 2, \cdots\}$)

n	0	1	2	3	4	5	6	7	8	9
$A[n]$	$+1$	-1	$+3$	$+1$	-3	$+1$	-1	-3	-1	$+1$



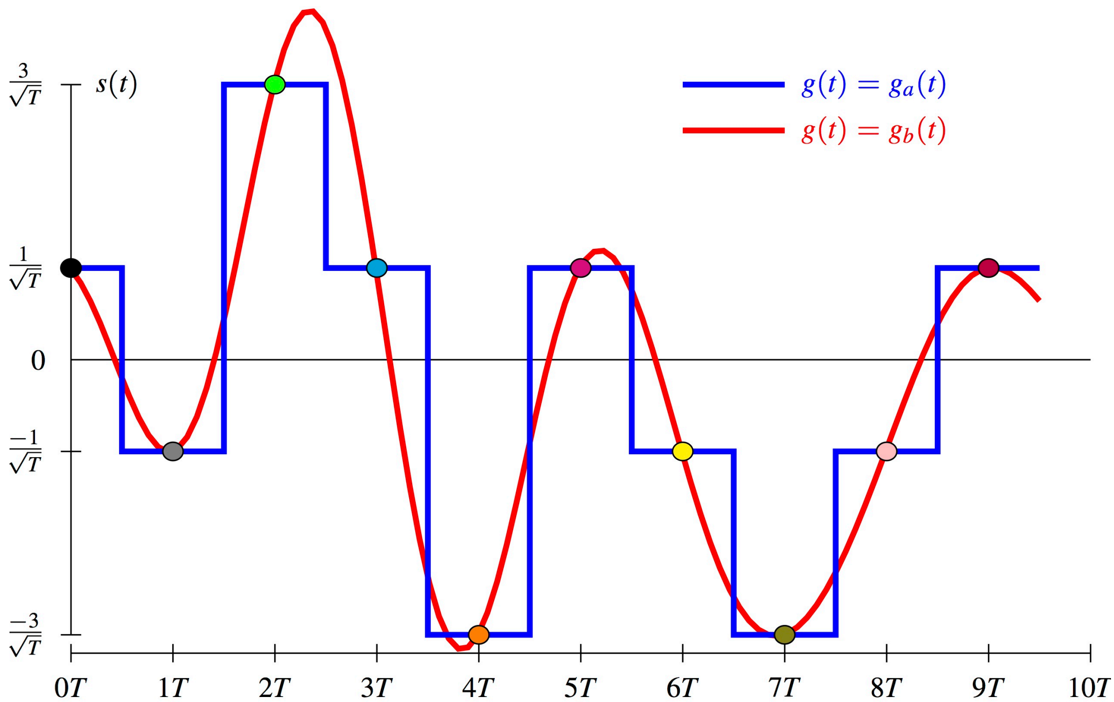
Sinc : Contribution of each symbol

n	0	1	2	3	4	5	6	7	8	9
$A[n]$	$+1$	-1	$+3$	$+1$	-3	$+1$	-1	-3	-1	$+1$



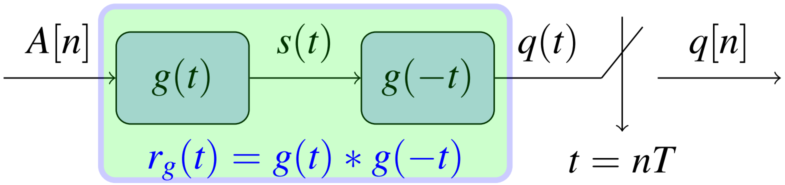
Modulated PAM signal $s(t)$

n	0	1	2	3	4	5	6	7	8	9
$A[n]$	$+1$	-1	$+3$	$+1$	-3	$+1$	-1	-3	-1	$+1$



Recovery of $A[n]$ from $s(t)$ using a matched filter

- Recovery of $A[n]$ in an ideal scenario
  - There is no distortion over $s(t)$
  - A matched filter (matched to $g(t)$) is applied on $s(t)$
  - Recovery of $A[n]$ sampling $q(t)$ (output of the filter)



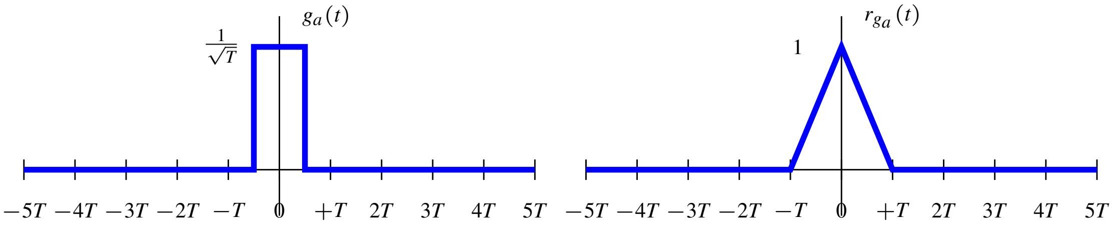
$$s(t) = \sum_n A[n]\; g(t - nT) \qquad q(t) = \sum_n A[n]\; r_g(t - nT)$$

- Conditions to recover $A[n]$ from $q[n]$ (by sampling $q(t)$)
  - The same as before, but applied on $r_g(t)$ instead of on $g(t)$
    - ⋆ Conditions for pulses of kind (a)
      - $r_g(t)$ of duration T
    - ⋆ Conditions for pulses of kind (b)
      - Periodical zeros on $r_g(t)$ ($r_g(nT) = 0\ \forall n \neq 0$)

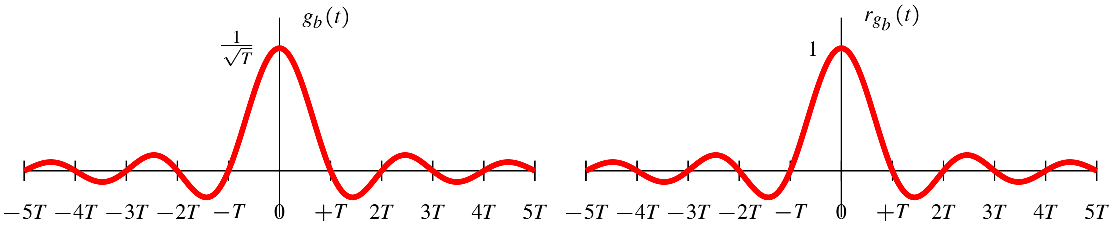
REMARK: If duration of $g(t)$ is lower than T, $r_g(t)$ satisfies conditions (b)

Shape of $r_g(t)$ for pulses of previous examples

$$g_a(t) = \frac{1}{\sqrt{T}}\,\Pi\left(\frac{t}{T}\right) \quad \leftrightarrow \quad r_{g_a}(t) = \Lambda\left(\frac{t}{T}\right)$$

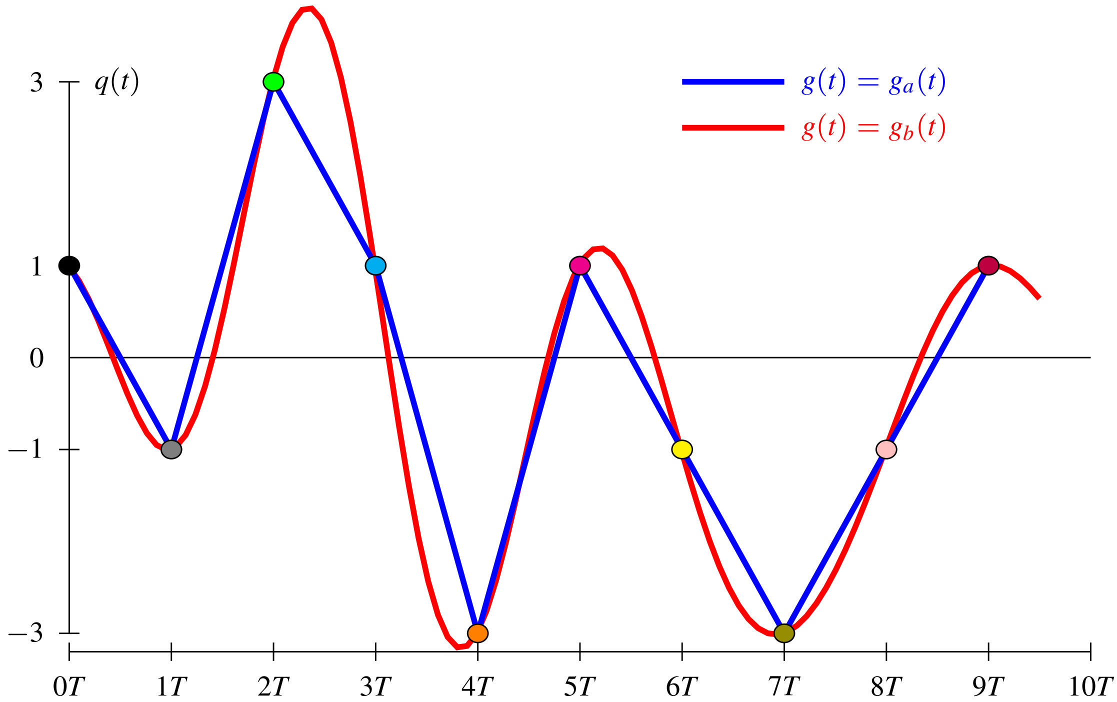


$$g_b(t) = \frac{1}{\sqrt{T}}\,\text{sinc}\left(\frac{t}{T}\right) \quad \leftrightarrow \quad r_{g_b}(t) = \text{sinc}\left(\frac{t}{T}\right)$$



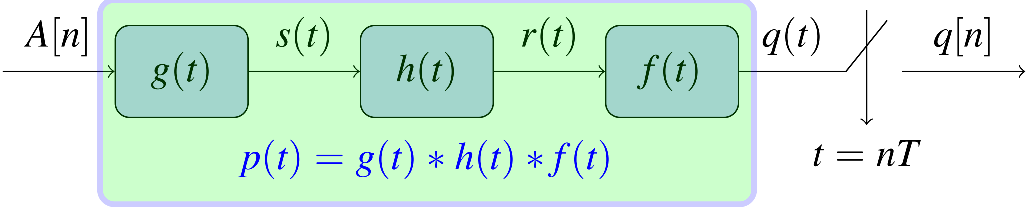
Received signal $q(t)$

n	0	1	2	3	4	5	6	7	8	9
$A[n]$	$+1$	-1	$+3$	$+1$	-3	$+1$	-1	-3	-1	$+1$



Recovery of $A[n]$ transmitting through a channel (noiseless)

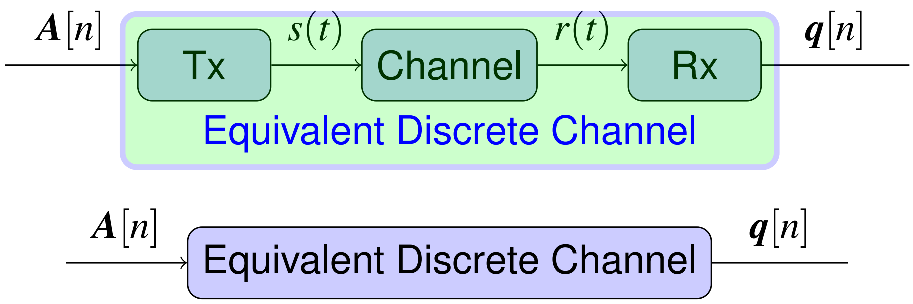
- Recovery of $A[n]$ transmitting through a channel
  - For the sake of simplicity, noise is neglected
  - A receiver filter $f(t)$ is applied at the channel output
    - ★ Usual choice: $f(t) = g(-t)$ (matched filter)



$$s(t) = \sum_n A[n]\, g(t - nT) \qquad q(t) = \sum_n A[n]\, p(t - nT)$$

- Now conditions have to be assessed on $p(t)$
  - Duration limited to T seconds
  - Cyclic zero values each T seconds
- Design to satisfy these conditions
  - Transmitter $g(t)$ and receiver $g(-t)$ can be designed
  - Channel response $h(t)$ is given: it is not a design parameter

Equivalent discrete channel

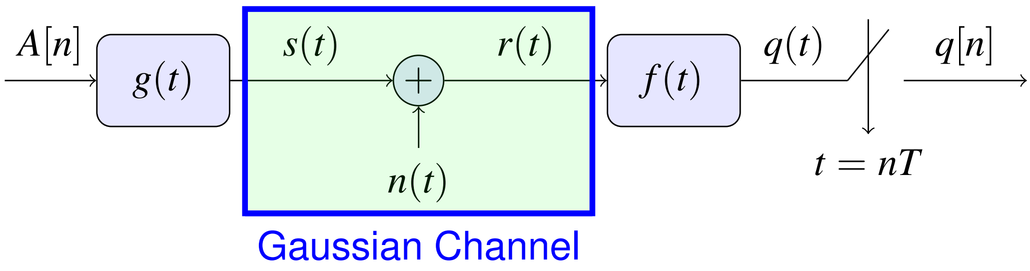


- Provides the discrete time expression for observations at the output of the demodulator $\boldsymbol{q}[n]$ as a function of the transmitted sequence $\boldsymbol{A}[n]$
  - In ideal systems: $\boldsymbol{q}[n] = \boldsymbol{A}[n] + \mathbf{z}[n]$
    If $z[n]$ is Gaussian, conditional distributions for observations (given $\boldsymbol{A}[n] = \boldsymbol{a}_i$)

$$f_{\boldsymbol{q}[n]|\boldsymbol{A}[n]}(\boldsymbol{q}|\boldsymbol{a}_i) = \frac{1}{(\pi N_o)^{N/2}} e^{-\frac{||\boldsymbol{q}-\boldsymbol{a}_i||^2}{N_0}}$$

- Expressions will now be obtained for two channel models
  - Gaussian channel
  - Linear channel

Transmission of PAM signals over Gaussian channels



- Gaussian channel model
  - Distortion during transmission is limited to noise addition

$$r(t) = s(t) + n(t)$$

$n(t)$: stationary random process, white, Gaussian, zero mean, $S_n(j\omega) = N_0/2$

- Receiver filter $f(t)$
  - Typical set up: matched filter

$$f(t) = g^*(-t) = g(-t), \text{ because } g(t) \text{ is real}$$

- Signal at the input of the sampler

$$q(t) = s(t) * f(t) + n(t) * f(t)$$

Equivalent discrete channel for Gaussian channels

- Signal before sampling

$$q(t) = \underbrace{\left(\overbrace{\sum_k A[k]\, g(t-kT)}^{s(t)}\right) * f(t)}_{\text{Noiseless output } o(t)} + \underbrace{n(t) * f(t)}_{\text{Filtered noise } z(t)}$$

$$o(t) = \sum_k A[k] \left(g(t-kT) * f(t)\right) = \sum_k A[k]\, p(t-kT)$$

- $p(t) = g(t) * f(t)$: joint transmitter-receiver response
  - This joint response determines the noiseless output at the receiver

- Observation at demodulator output

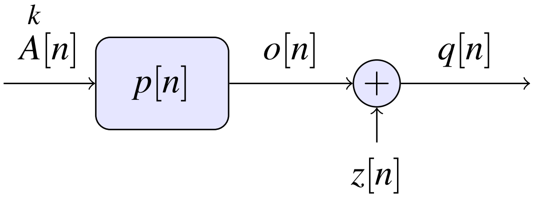
$$q[n] = q(t)|_{t=nT} = q(nT) = \sum_k A[k]\, p\left((n-k)T\right) + z(nT)$$

Equivalent discrete channel for Gaussian channels (II)

- Definition of equivalent discrete channel $p[n]$

$$p[n] = p(t)\big|_{t=nT}$$

$$q[n] = \sum_k A[k]\, p[n-k] + z[n] = A[n] * p[n] + z[n]$$



- Definition por joint response $p(t)$ (or $P(j\omega)$)

$$p(t) = g(t) * f(t) \quad \overset{\mathcal{FT}}{\leftrightarrow} \quad P(j\omega) = G(j\omega)\, F(j\omega)$$

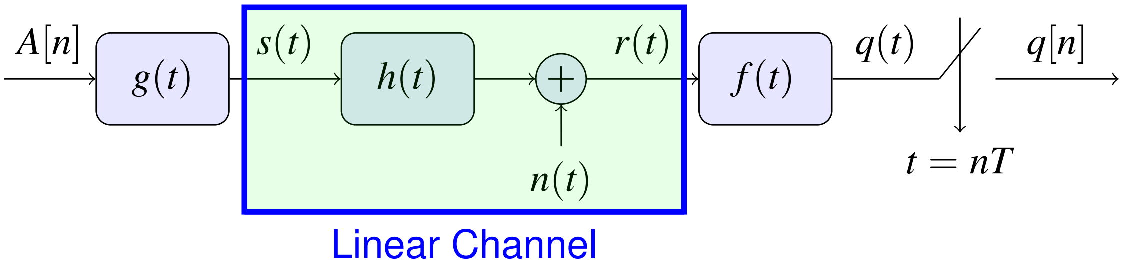
  - Using matched filters:

$$f(t) = g(-t) \quad \overset{\mathcal{FT}}{\leftrightarrow} \quad F(j\omega) = G^*(j\omega)$$

$$p(t) = g(t) * g(-t) = r_g(t) \quad \overset{\mathcal{FT}}{\leftrightarrow} \quad P(j\omega) = G(j\omega)\, G^*(j\omega) = |G(j\omega)|^2$$

$r_g(t)$: continuous time autocorrelation of $g(t)$ (or time ambiguity function of $g(t)$)

Transmission of PAM through linear channels



- Linear channel model
  - PAM signal $s(t)$ suffers a linear distortion during transmission
  - Gaussian noise is also added

$$r(t) = s(t) * h(t) + n(t)$$

$h(t)$: linear system impulse response modeling linear distortion

$n(t)$: stationary random process, white, Gaussian, zero mean, $S_n(j\omega) = N_0/2$

- Receiver filter $f(t)$
  - Typical set up: matched filter $f(t) = g^*(-t) = g(-t)$
- Signal at the input of the sampler

$$q(t) = r(t) * f(t) = s(t) * h(t) * f(t) + n(t) * f(t)$$

Equivalent discrete channel for linear channels

- Signal before sampling

$$q(t) = \left(\overbrace{\sum_k A[k]\, g(t-kT)}^{s(t)} \right) * h(t) * f(t) + n(t) * f(t)$$
$$= \sum_k A[k] \left(g(t-kT) * h(t) * f(t) \right) + n(t) * f(t)$$
$$= \sum_k A[k]\, p(t-kT) + z(t)$$

- $p(t) = g(t) * h(t) * f(t)$: joint transmitter-channel-receiver response
  - For a matched filter at the receiver

$$p(t) = g(t) * h(t) * g^*(-t) = r_g(t) * h(t)$$

$r_g(t)$: time autocorrelation of $g(t)$ (or time ambiguity function of $g(t)$)

- Observation at demodulator output

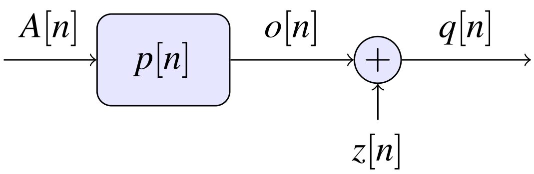
$$q[n] = q(t)|_{t=nT} = q(nT) = \sum_k A[k]\, p\left((n-k)T\right) + z(nT)$$

Equivalent discrete channel for linear channels (II)

- Definition of equivalent discrete channel $p[n]$

$$\boxed{p[n] = p(t)|_{t=nT}}$$

$$q[n] = \sum_k A[k]\, p[n-k] + z[n] = A[n] * p[n] + z[n]$$



- Same basic model as for Gaussian channels
  - New definition por $p(t)$: it includes the effect of $h(t)$

$$\boxed{p(t) = g(t) * h(t) * f(t) \quad \overset{\mathcal{FT}}{\leftrightarrow} \quad P(j\omega) = G(j\omega)\, H(j\omega)\, F(j\omega)}$$

  - Using matched filters: $f(t) = g(-t) \overset{\mathcal{FT}}{\leftrightarrow} F(j\omega) = G^*(j\omega)$

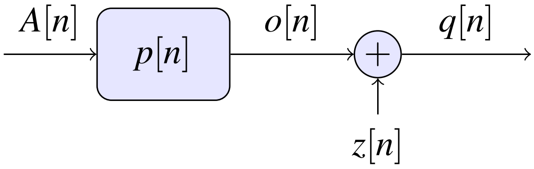
$$\boxed{p(t) = r_g(t) * h(t) \quad \overset{\mathcal{FT}}{\leftrightarrow} \quad P(j\omega) = |G(j\omega)|^2\ H(j\omega)}$$

Inter-Symbol Interference (ISI)

- Definition of equivalent discrete channel $p[n]$

$$p[n] = p(t)\big|_{t=nT} \qquad q[n] = o[n] + z[n]$$

$$\text{Noiseless output } o[n] = \sum_k A[k]\, p[n-k] = A[n] * p[n]$$



- Ideal

$$p[n] = \delta[n] \rightarrow o[n] = A[n]$$

- Real: Intersymbol interference (ISI)

$$o[n] = A[n] * p[n] = \sum_k A[k]\, p[n-k] = \underbrace{A[n]}_{Ideal}\ \underbrace{p[0]}_{scaling} + \underbrace{\sum_{\substack{k \\ k \neq n}} A[k]\, p[n-k]}_{ISI}$$

Inter-Symbol Interference - Analysis

- Intersymbol interference for equivalent discrete channel $p[n]$

$$o[n] = \underbrace{\underbrace{A[n]}_{Ideal}\ \underbrace{p[0]}_{scaling}}_{desired} + \underbrace{\sum_{\substack{k \\ k \neq n}} A[k]\, p[n-k]}_{ISI\ interference}$$

  - Effect of intersymbol interference

$$\text{ISI} = \sum_{\substack{k \\ k \neq n}} A[k]\, p[n-k]$$

Contribution at discrete instant n of previous and posterior symbols

$$o[n] = \underbrace{\cdots + A[n-2]\, p[2] + A[n-1]\, p[1]}_{precursor\ ISI} + \underbrace{A[n]\, p[0]}_{cursor} + \underbrace{A[n+1]\, p[-1] + A[n+2]\, p[-2] + \cdots}_{postcursor\ ISI}$$

Inter-Symbol Interference - Effect : Extended constellation

- ISI produces an extended constellation at the receiver side

Values of noiseless discrete output $o[n] = A[n] * p[n]$

- Example: 2-PAM modulation

Channel A

$$p[n] = \delta[n] + \frac{1}{2}\delta[n-1]$$
$$o[n] = A[n] + \frac{1}{2}A[n-1]$$

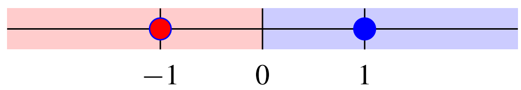
Channel B

$$p[n] = \delta[n] + \frac{1}{2}\delta[n-1] + \frac{1}{4}\delta[n-2]$$
$$o[n] = A[n] + \frac{1}{2}A[n-1] + \frac{1}{4}A[n-2]$$

$A[n]$	$A[n-1]$	$o[n]$
$+1$	$+1$	$+\frac{3}{2}$
$+1$	-1	$+\frac{1}{2}$
-1	$+1$	$-\frac{1}{2}$
-1	-1	$-\frac{3}{2}$

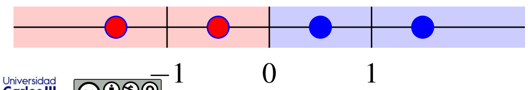
$A[n]$	$A[n-1]$	$A[n-2]$	$o[n]$
$+1$	$+1$	$+1$	$+\frac{7}{4}$
$+1$	$+1$	-1	$+\frac{5}{4}$
$+1$	-1	$+1$	$+\frac{3}{4}$
$+1$	-1	-1	$+\frac{1}{4}$
-1	$+1$	$+1$	$-\frac{1}{4}$
-1	$+1$	-1	$-\frac{3}{4}$
-1	-1	$+1$	$-\frac{5}{4}$
-1	-1	-1	$-\frac{7}{4}$

2-PAM Constellation

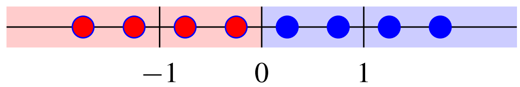


● $\equiv A[n] = +1$
● $\equiv A[n] = -1$

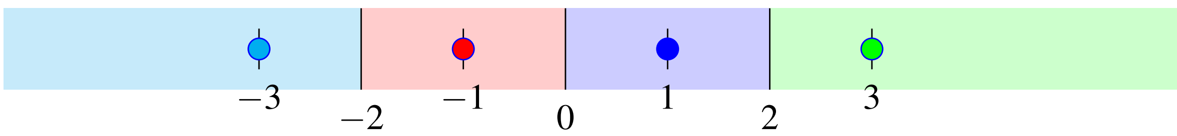
Extended Constellation (Channel A)



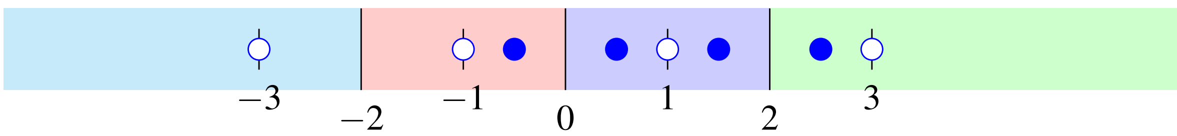
Extendided Constellation (Channel B)



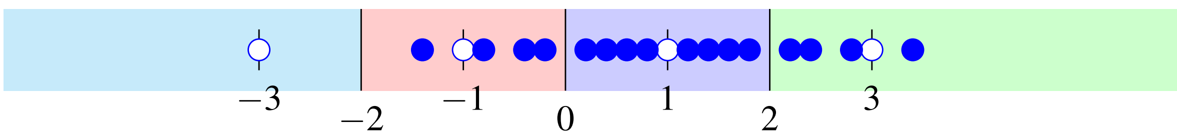
Inter-Symbol Interference - Effect : Extended constellation (II)



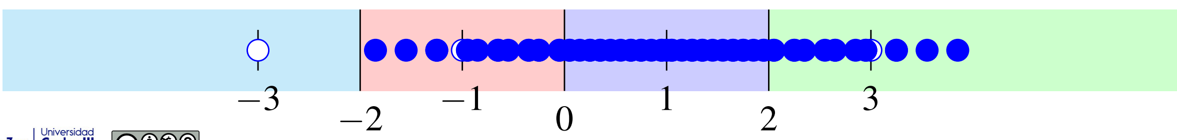
$A[n] = +1,\ K_p = 1 : p[n] = \delta[n] + 0.5\delta[n-1]$



$A[n] = +1,\ K_p = 2 : p[n] = \delta[n] + 0.5\delta[n-1] + 0.3\delta[n-2]$



$A[n] = +1,\ K_p = 3 : p[n] = \delta[n] + 0.5\delta[n-1] + 0.3\delta[n-2] + 0.15\delta[n-3]$



ISI : Joint transmitter-channel-receiver response $p(t)$

- Response $p(t)$ determines the ISI behavior
  - Noiseless output depends on the value of $p[n]$
    - Sampling the joint transmitter-channel-receiver response $p(t)$
    - Samling at symbol rate (at nT_s)
- Definition of joint transmitter-channel-receiver response
  - Gaussian channel

$$p(t) = g(t) * f(t) \quad \stackrel{\mathcal{FT}}{\leftrightarrow} \quad P(j\omega) = G(j\omega)\; F(j\omega)$$

  - Lineal channel

$$p(t) = g(t) * h(t) * f(t) \quad \stackrel{\mathcal{FT}}{\leftrightarrow} \quad P(j\omega) = G(j\omega)\; H(j\omega)\; F(j\omega)$$

- Usual receiver: matched filter $f(t) = g^*(-t)$
  - Gaussian channel

$$p(t) = r_g(t) \quad \stackrel{\mathcal{FT}}{\leftrightarrow} \quad P(j\omega) = |G(j\omega)|^2$$

  - Lineal channel

$$p(t) = r_g(t) * h(t) \quad \stackrel{\mathcal{FT}}{\leftrightarrow} \quad P(j\omega) = |G(j\omega)|^2\; H(j\omega)$$

Nyquist criterion for zero ISI

- Conditions for avoiding ISI written in the time domain

$$p[n] = p(t)\Big|_{t=nT} = \delta[n] \qquad \underbrace{(\times C)}_{scaling/gain}$$

- Equivalent condition in the frequency domain

$$P\left(e^{j\omega}\right) = 1 \quad (\times C)$$

- Equivalent continuous-time expressions

$$p(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) = \delta(t) \quad (\times C)$$

$$\frac{1}{2\pi} P(j\omega) * \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(j\omega - j\frac{2\pi}{T}k\right) = 1 \quad (\times C)$$

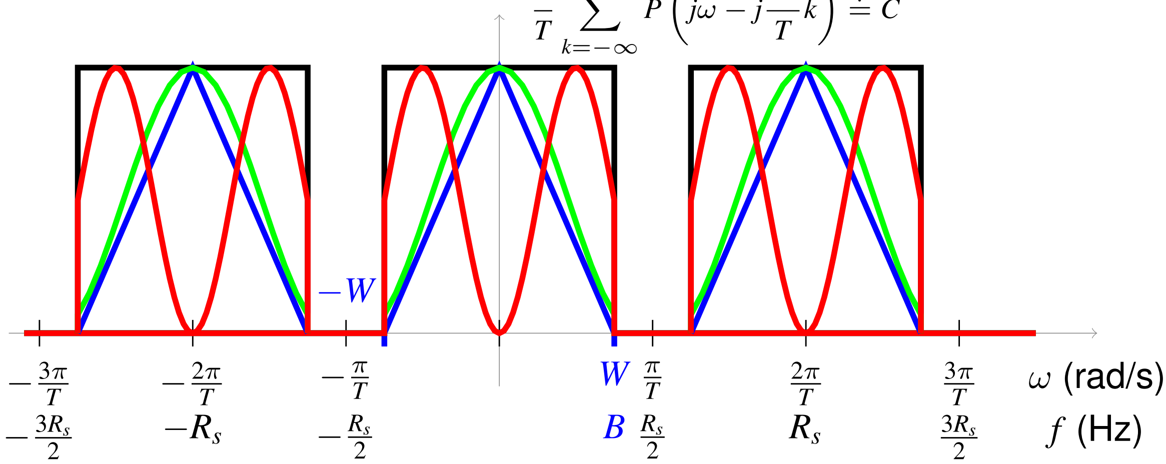
$$\frac{1}{T} \sum_{k=-\infty}^{\infty} P\left(j\omega - j\frac{2\pi}{T}k\right) = 1 \quad (\times C)$$

Replicas of $P(j\omega)$ shifted multiples of $\frac{2\pi}{T}$ rad/s sum a constant

Nyquist in the freq. domain: an important implication

- Transmission at $R_s = \frac{1}{T}$ bauds
- $P(j\omega)$: bandwidth W rad/s, with $W < \frac{\pi}{T} = \pi\, R_s$ rad/s
  - Equivalent to bandwidth $B = \frac{W}{2\pi}$ Hz with $B < \frac{R_s}{2}$ Hz

$$\frac{1}{T}\sum_{k=-\infty}^{\infty} P\left(j\omega - j\frac{2\pi}{T}k\right) \stackrel{?}{=} C$$



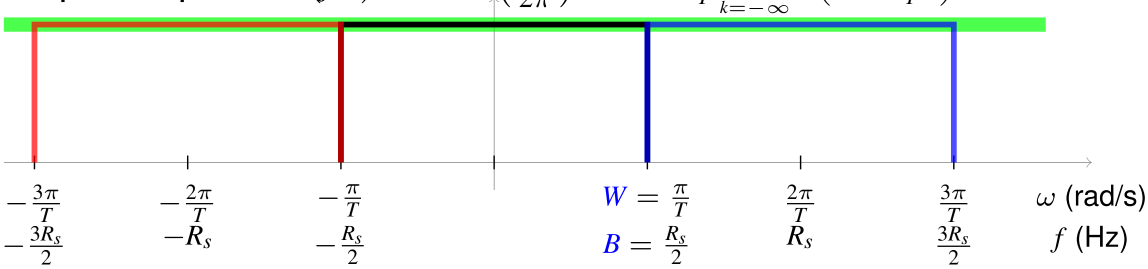
- It is **NOT possible** to satisfy Nyquist with bandwidth

$$W < \frac{\pi}{T} = \pi\, R_s \text{ rad/s or, equivalently, } B < \frac{R_s}{2} \text{ Hz}$$

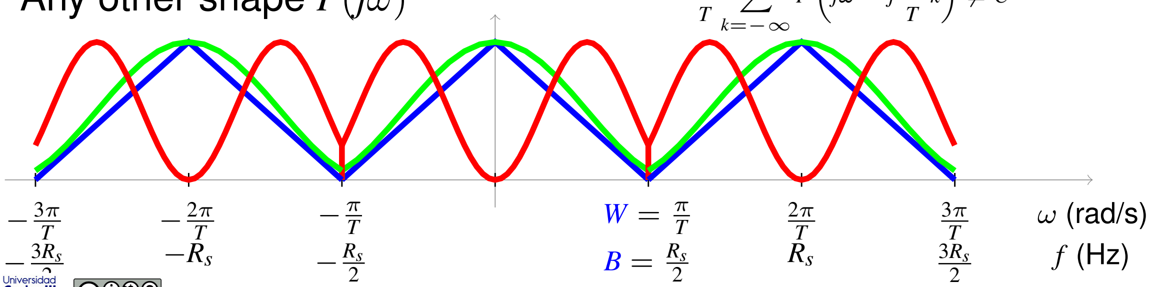
Nyquist in the freq. domain: an important implication (II)

- Transmission at $R_s = \frac{1}{T}$ bauds
- $P(j\omega)$: bandwidth W rad/s, with $W = \frac{\pi}{T} = \pi\, R_s$ rad/s
  - Equivalent to bandwidth $B = \frac{W}{2\pi}$ Hz with $B = \frac{R_s}{2}$ Hz

Squared pulse $P(j\omega) = a\,\Pi\left(\frac{\omega T}{2\pi}\right)$ $\qquad \frac{1}{T}\sum_{k=-\infty}^{\infty} P\left(j\omega - j\frac{2\pi}{T}k\right) = C$

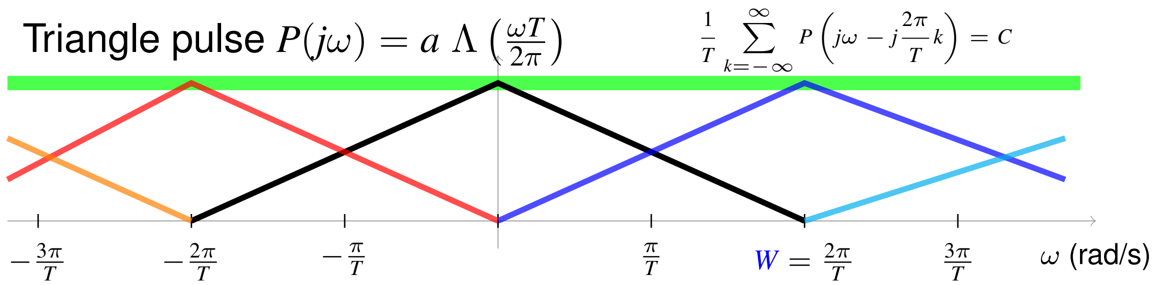


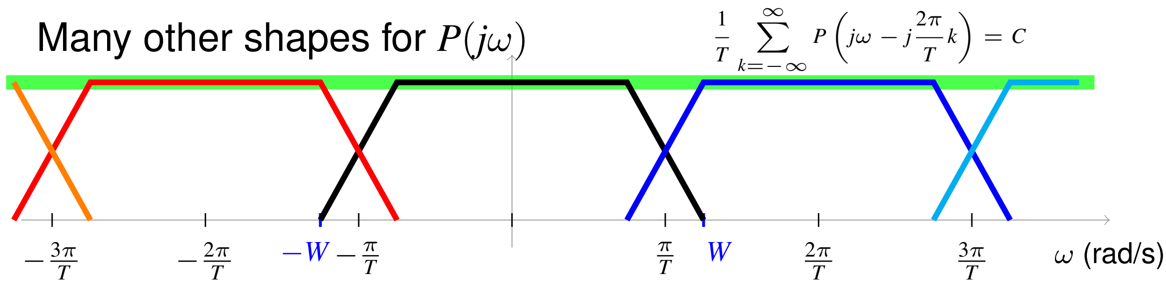
Any other shape $P(j\omega)$ $\qquad \frac{1}{T}\sum_{k=-\infty}^{\infty} P\left(j\omega - j\frac{2\pi}{T}k\right) \neq C$



Nyquist in the freq. domain: an important implication (III)

- Transmission at $R_s = \frac{1}{T}$ bauds
- $P(j\omega)$: bandwidth W rad/s, with $W > \frac{\pi}{T} = \pi\ R_s$ rad/s
  - Equivalent to bandwidth $B = \frac{W}{2\pi}$ Hz with $B > \frac{R_s}{2}$ Hz





Optimal shape for $p(t) \overset{\mathcal{FT}}{\leftrightarrow} P(j\omega)$ to transmit without ISI

Best bandwidth vs transmission rate trade-off

- Minimum bandwidth to transmit without ISI at rate $R_s = \frac{1}{T}$ bauds

$$W_{min} = \frac{\pi}{T} = \pi\ R_s \text{ rad/s} \quad \left(B_{min} = \frac{R_s}{2} \text{ Hz}\right)$$

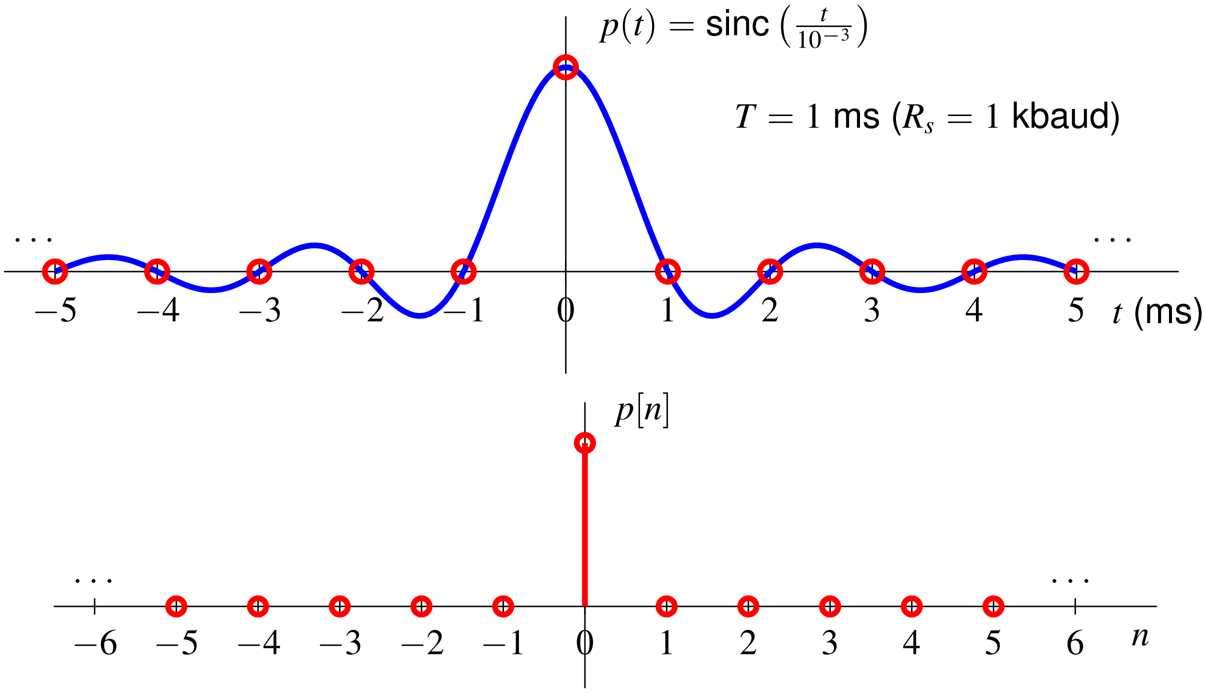
- Maximum rate without ISI through a bandwidth W rad/s (B Hz)

$$R_s\big|_{max} = \frac{W}{\pi} = 2\ B \text{ bauds (symbols/s)}$$

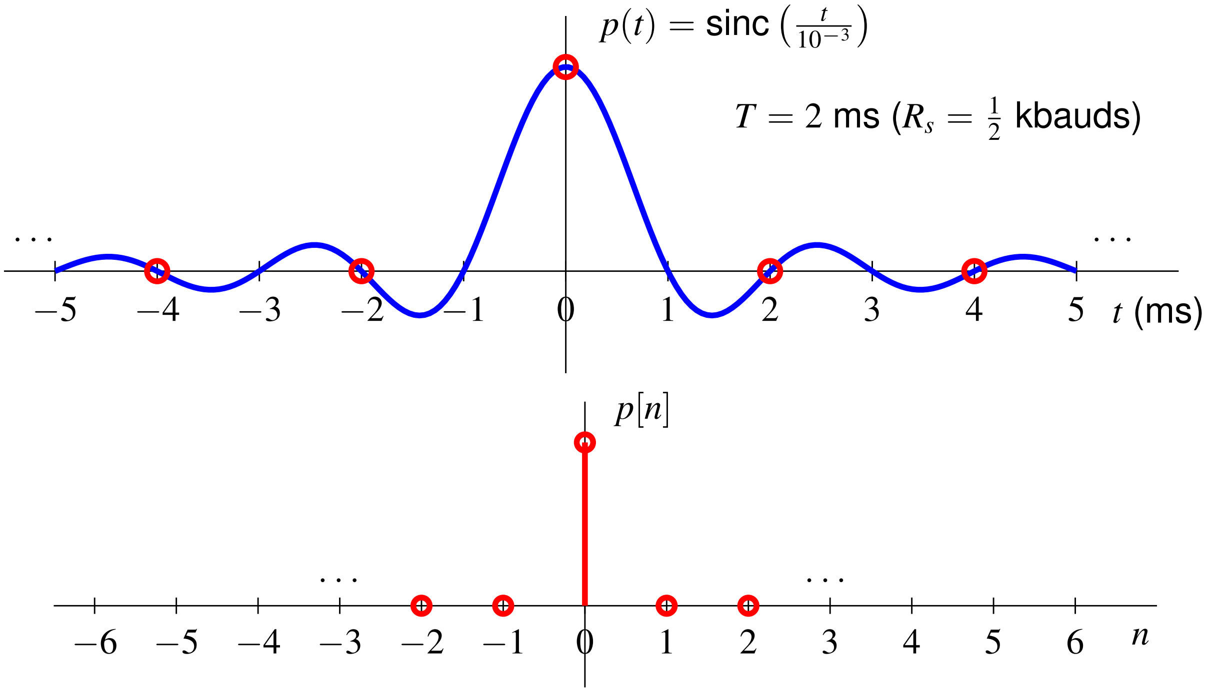
- Optimal joint transmitter-channel-receiver response

$$p(t) = \text{sinc}\left(\frac{t}{T}\right) \quad \overset{\mathcal{FT}}{\leftrightarrow} \quad P(j\omega) = T\ \Pi\left(\frac{\omega T}{2\pi}\right)$$

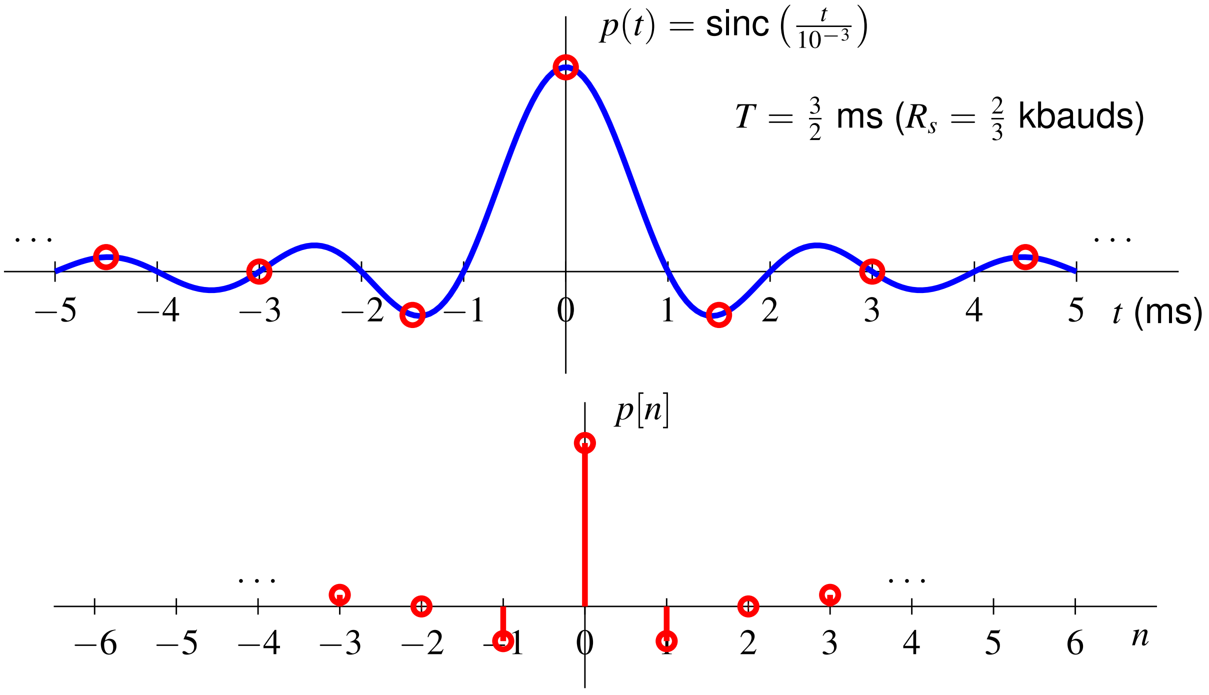
Example: $p(t)$



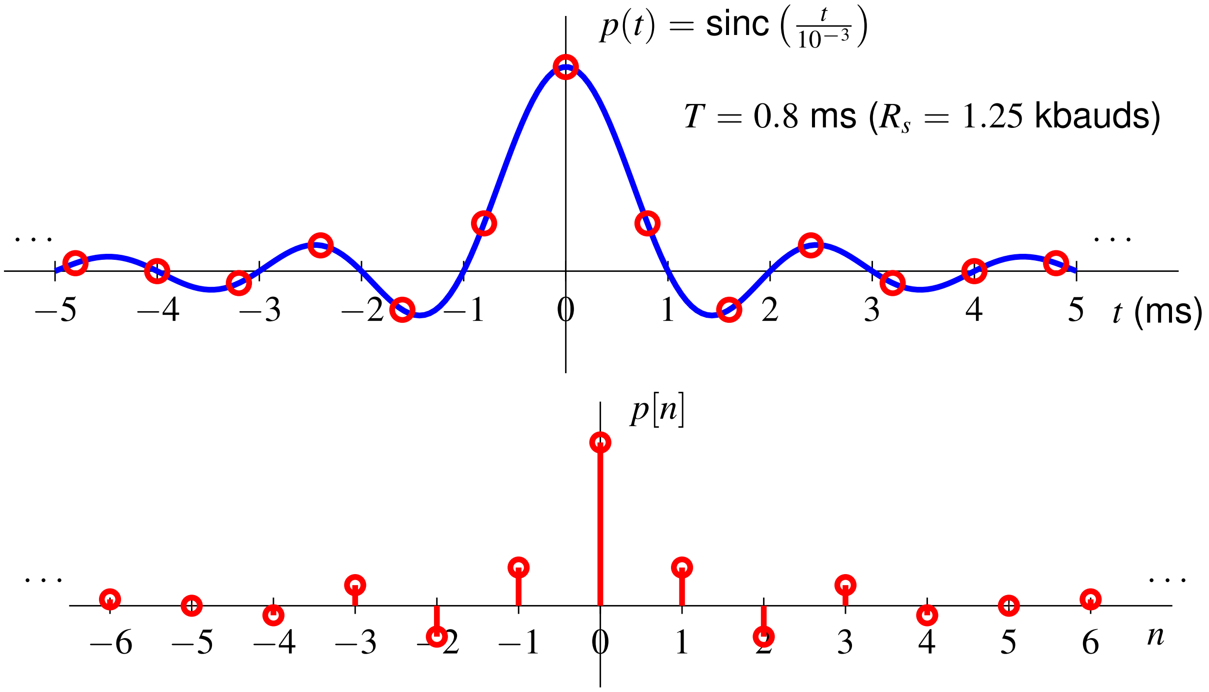
Example: $p(t)$



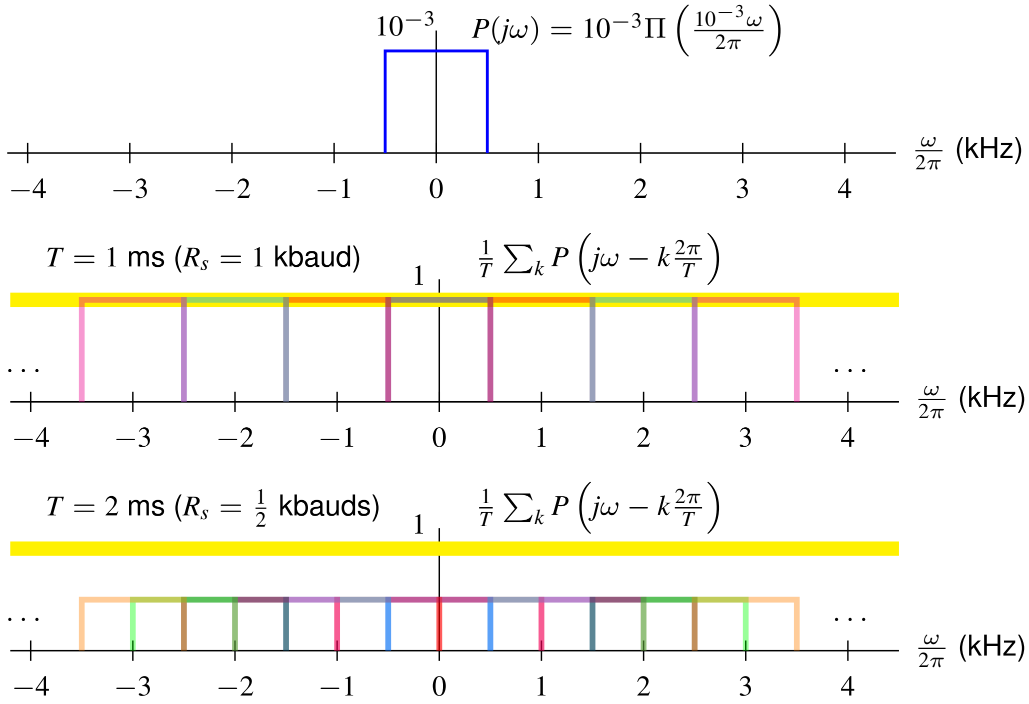
Example: $p(t)$



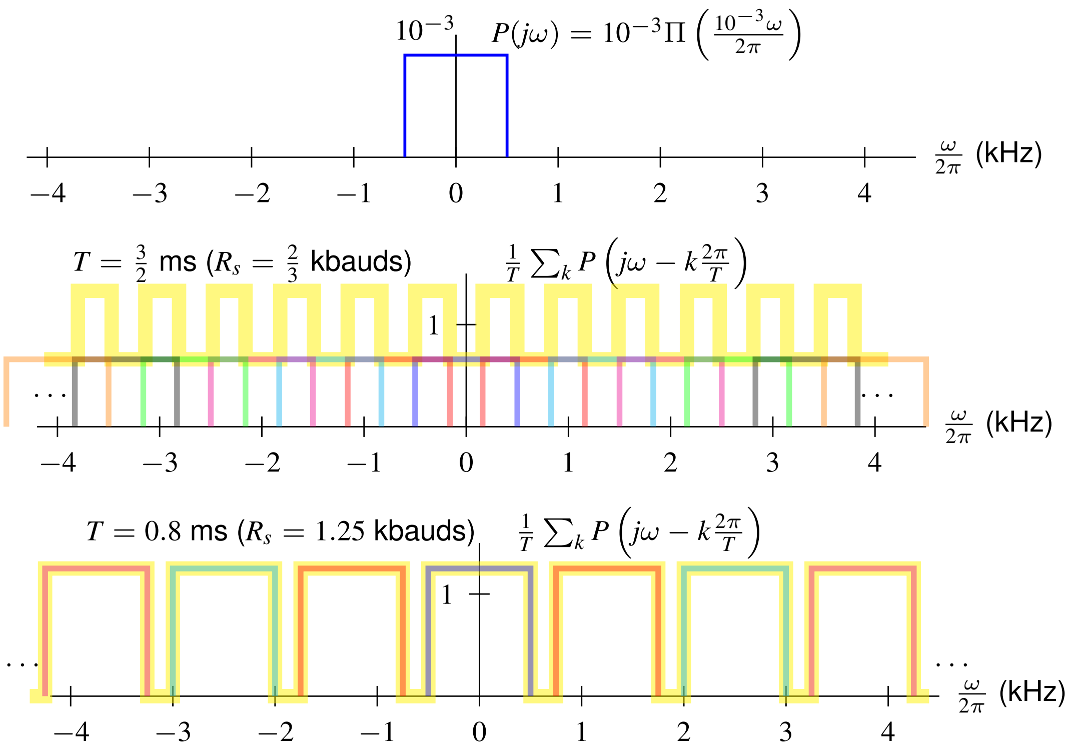
Example: $p(t)$



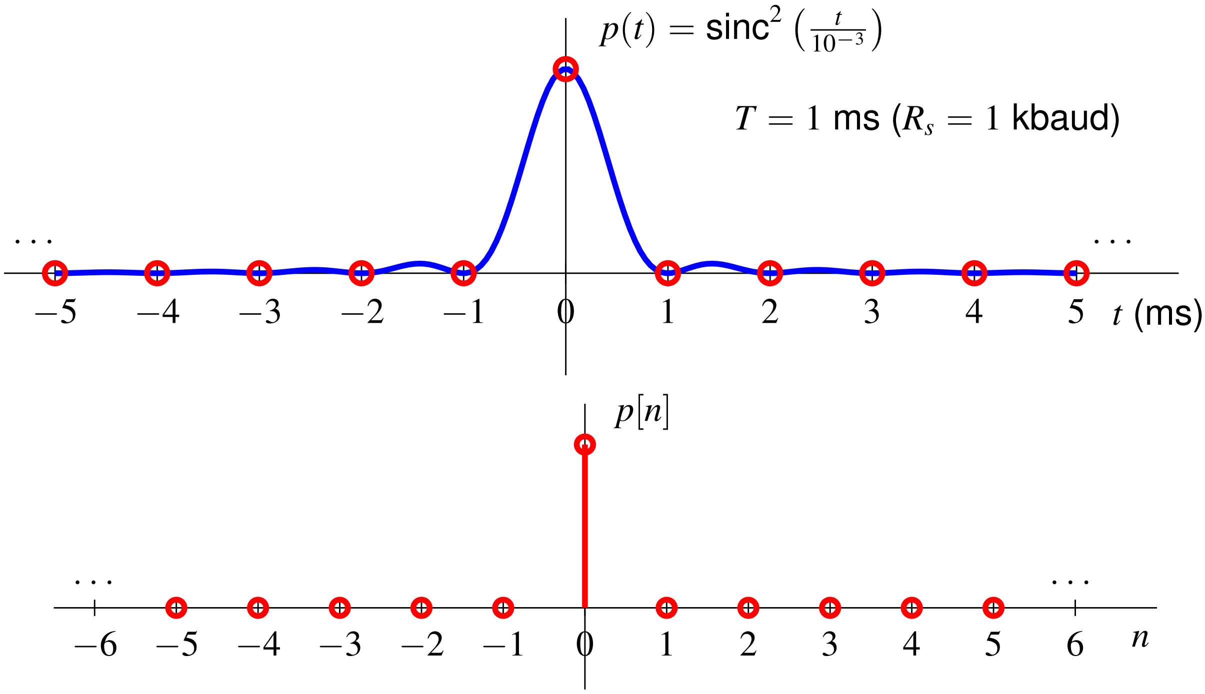
Example: $P(j\omega)$



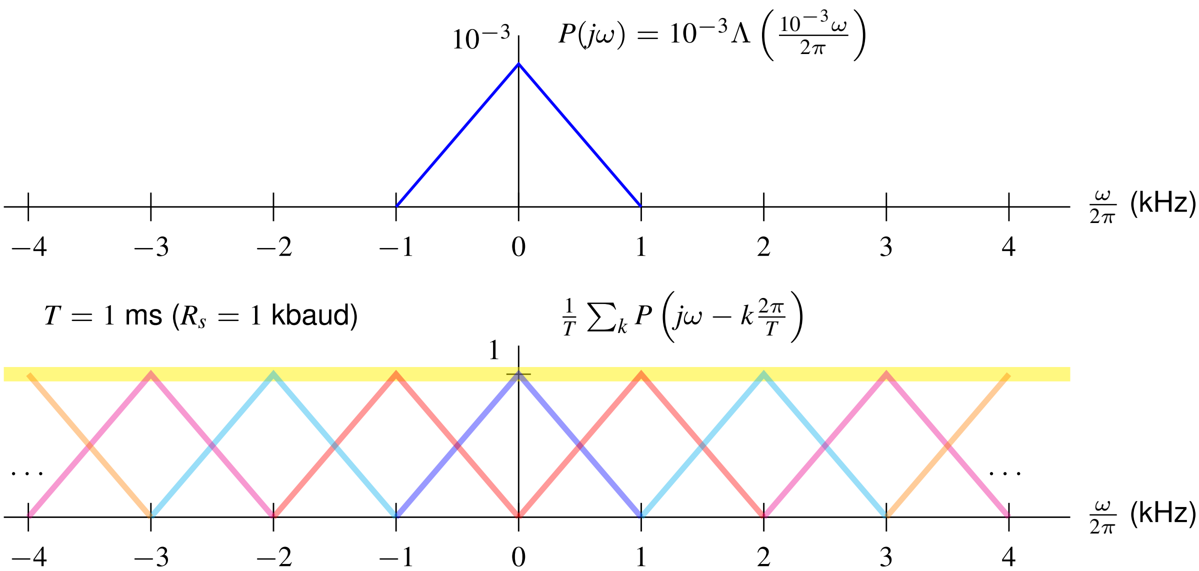
Example: $P(j\omega)$



Example: $p(t)$



Example: $P(j\omega)$



Raised cosine pulses

- Family of bandlimited pulses
- Parameters
  - Symbol lenght (or rate): T seconds (or $R_s = \frac{1}{T}$ bauds)
  - Roll-off factor: α
    - Range for roll-off factor: $\alpha \in [0, 1]$
    - Particular case $\alpha = 0$: $h_{RC}^{0,T}(t) = \text{sinc}\left(\frac{t}{T}\right) \stackrel{\mathcal{FT}}{\longleftrightarrow} H_{RC}^{0,T}(j\omega) = T\,\Pi\left(\frac{\omega T}{2\pi}\right)$
- Analytic expressions (time and freq. domains)

$$h_{RC}^{\alpha,T}(t) = \left(\frac{\sin(\pi t/T)}{\pi t/T}\right)\left(\frac{\cos(\alpha\pi t/T)}{1-(2\alpha t/T)^2}\right)$$

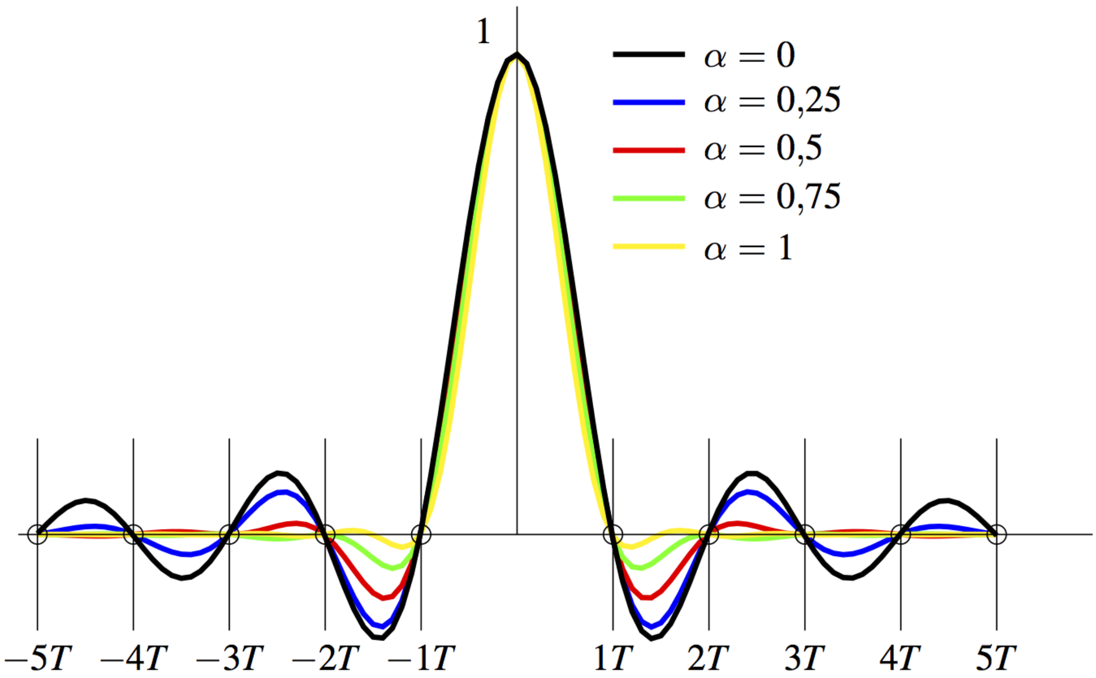
$$H_{RC}^{\alpha,T}(j\omega) = \begin{cases} T & 0 \le |\omega| < (1-\alpha)\frac{\pi}{T} \\ \frac{T}{2}\left[1-\sin\left(\frac{T}{2\alpha}\left(|\omega|-\frac{\pi}{T}\right)\right)\right] & (1-\alpha)\frac{\pi}{T} \le |\omega| \le (1+\alpha)\frac{\pi}{T} \\ 0 & |\omega| > (1+\alpha)\frac{\pi}{T} \end{cases}$$

- Bandwidth for a transmission rate depends on both parameters

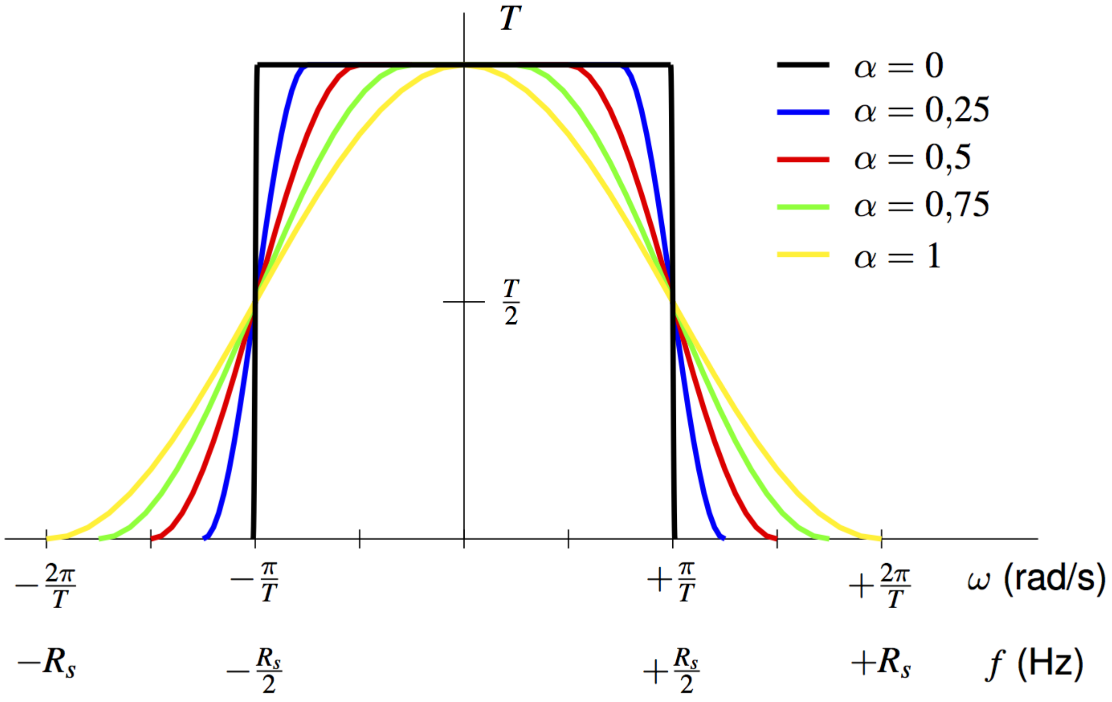
$$W = (1+\alpha)\times\frac{\pi}{T}\ \text{rad/s}, \quad B = (1+\alpha)\times\frac{R_s}{2}\ \text{Hz}$$

Raised cosine pulses: $h_{RC}^{\alpha,T}(t)$

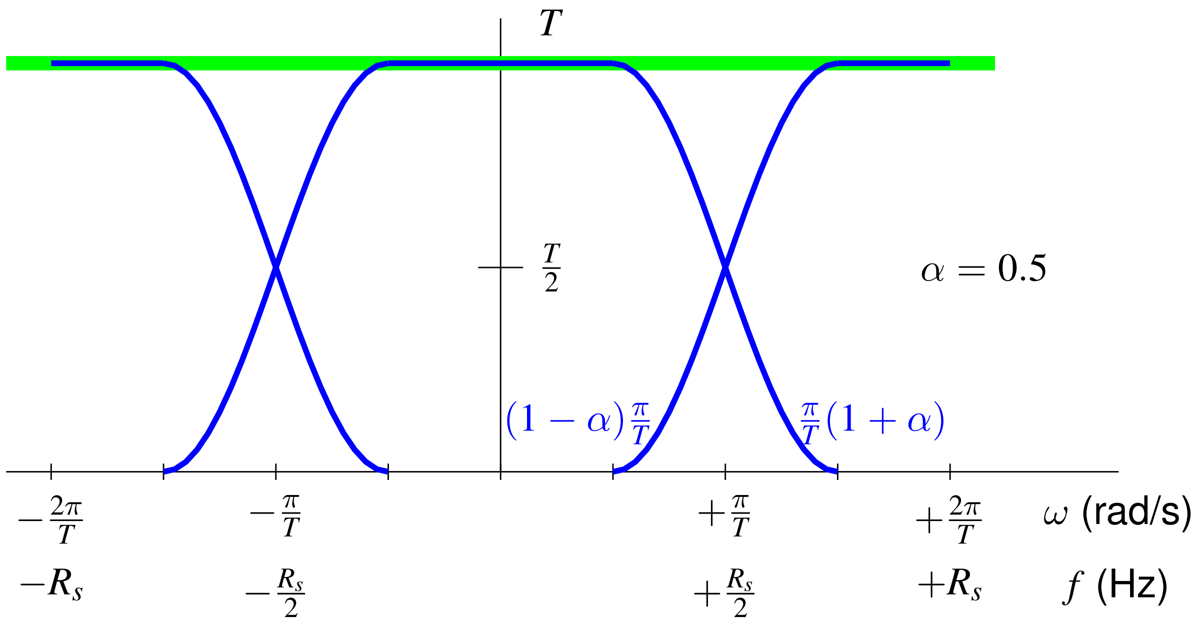
- $h_{RC}^{\alpha,T}(t)$ satisfies the Nyquist criterion at T seconds (or at $R_s = \frac{1}{T}$ bauds)



Raised cosine pulses: Freq. response $H_{RC}^{\alpha,T}(j\omega)$



Raised cosine pulses: Replicas of $H_{RC}^{\alpha,T}(j\omega)$ each $\frac{2\pi}{T}$ rad/s



- $H_{RC}^{\alpha,T}(j\omega)$ satisfies the Nyquist criterion at T seconds (or at $R_s = \frac{1}{T}$ bauds)

Root-raised cosine pulses (Squared-root-raised cosine)

- Filters whose joint response (of two) is a raised cosine

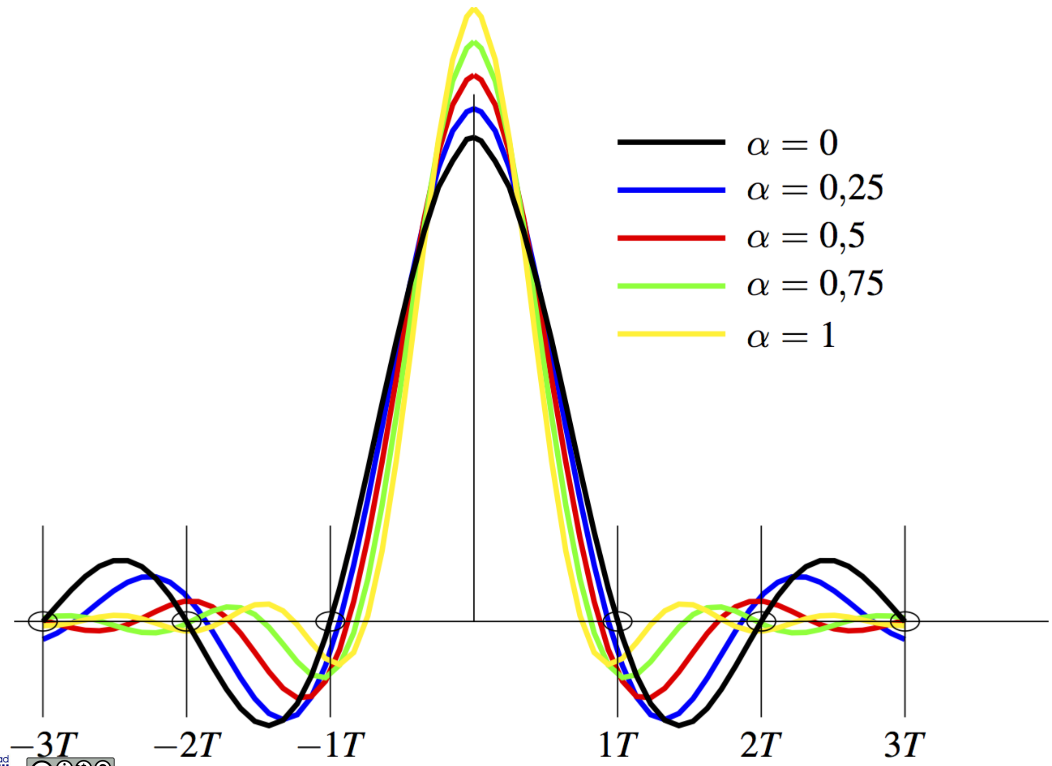
$$h_{RRC}^{\alpha,T}(t) * h_{RRC}^{\alpha,T}(t) = h_{RC}^{\alpha,T}(t) \qquad H_{RRC}^{\alpha,T}(j\omega)\, H_{RRC}^{\alpha,T}(j\omega) = H_{RC}^{\alpha,T}(j\omega)$$

- General procedure to obtain transmission filter $h_{RRC}(t)$
  1. Design in frequency domain from $H_{RC}^{\alpha,T}(j\omega)$
  2. Divide in two contributions: $H_{RRC}^{\alpha,T}(j\omega) = \sqrt{H_{RC}^{\alpha,T}(j\omega)}$
  3. $h_{RRC}^{\alpha,T}(t) = \mathcal{FT}^{-1}\left\{H_{RRC}^{\alpha,T}(j\omega)\right\}$
- Root-raised cosine pulses

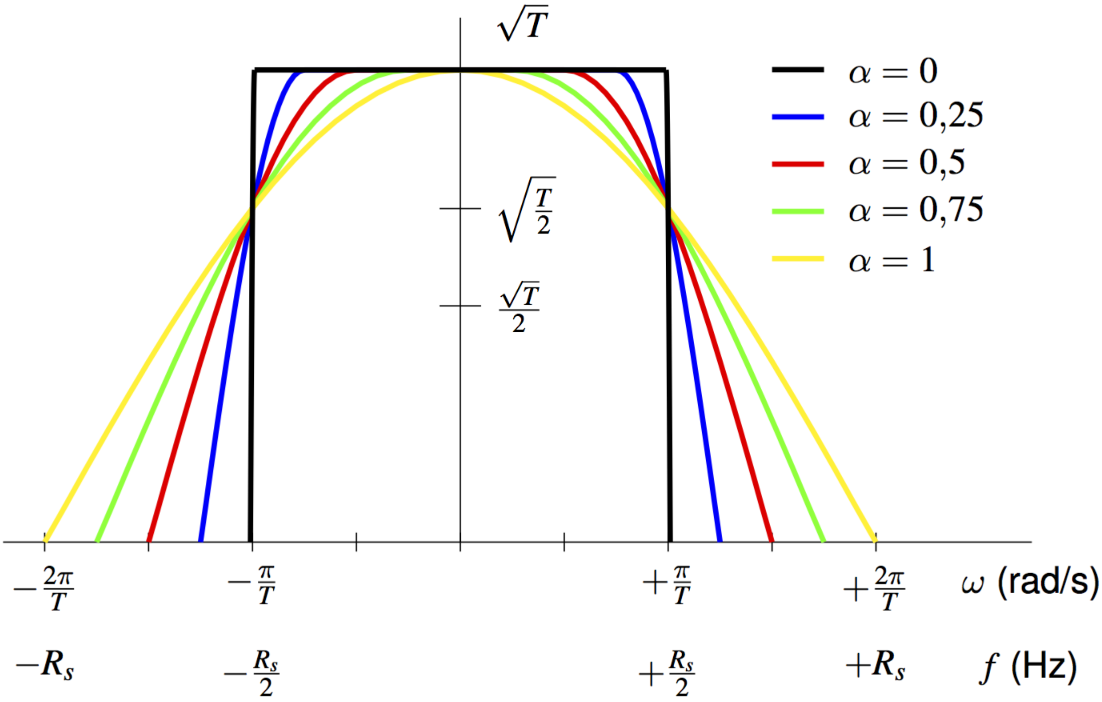
$$h_{RRC}^{\alpha,T}(t) = \frac{4\alpha}{\pi\sqrt{T}} \frac{\cos\left((1+\alpha)\frac{\pi t}{T}\right) + T\frac{\operatorname{sen}\left((1-\alpha)\frac{\pi t}{T}\right)}{4\alpha t}}{1-\left(\frac{4\alpha t}{T}\right)^2}$$

Root-raised cosine pulses: $h_{RRC}^{\alpha,T}(t)$

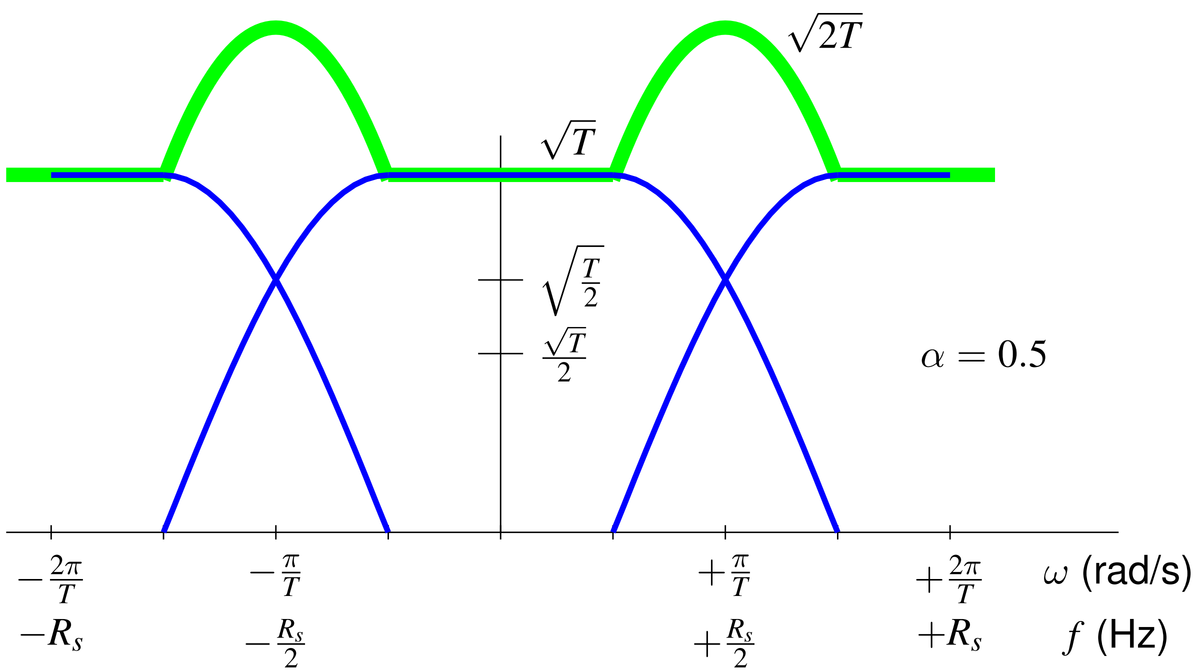
- $h_{RRC}^{\alpha,T}(t)$ does **NOT satisfy** the Nyquist crit. at T s (or at $R_s = \frac{1}{T}$ bauds)
  - Except for $\alpha = 0$, because $h_{RRC}^{0,T}(t) = \frac{1}{\sqrt{T}}\operatorname{sinc}\left(\frac{t}{T}\right)$



Root-raised cosine pulses: $H_{RRC}^{\alpha,T}(j\omega)$

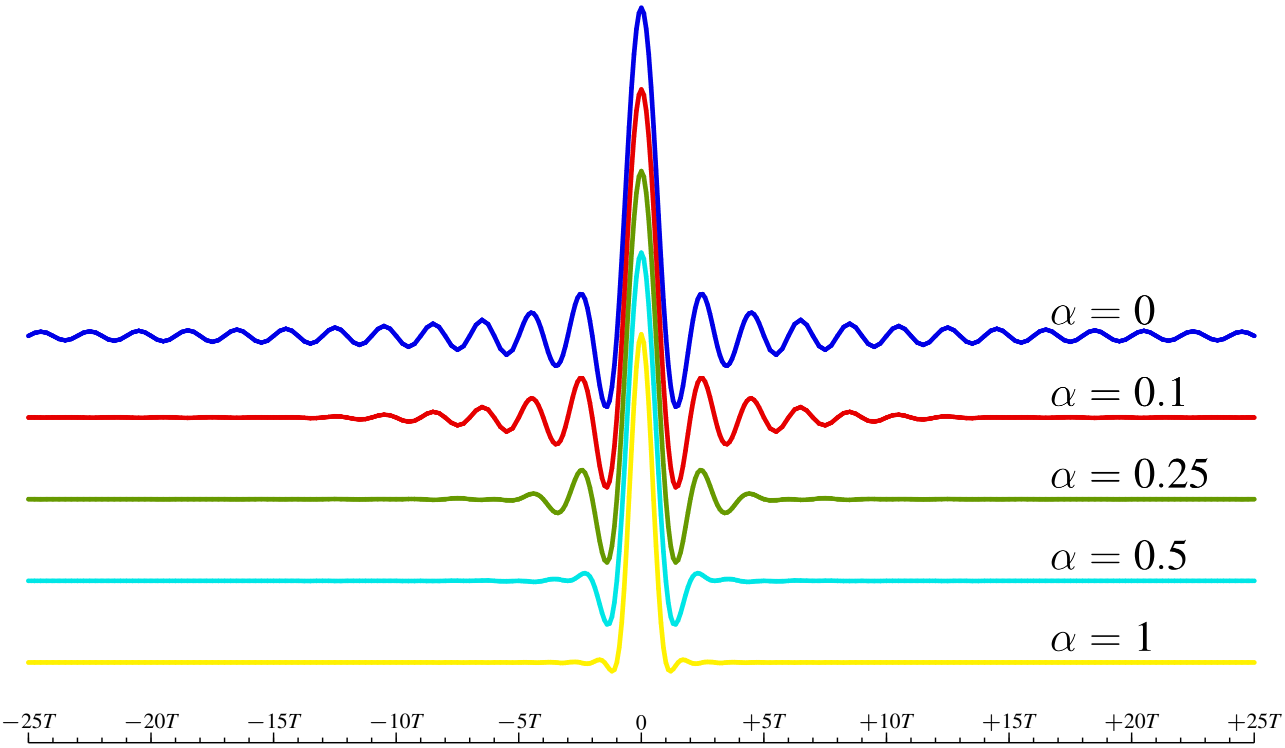


Root-raised cosine pulses: Replicas of $H_{RRC}^{\alpha,T}(j\omega)$ each $\frac{2\pi}{T}$ rad/s



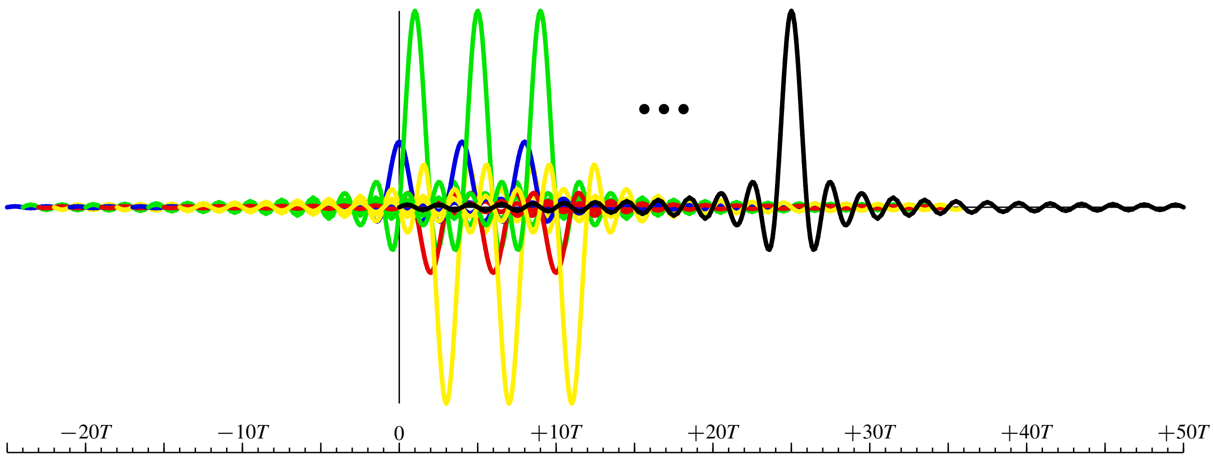
- $H_{RRC}^{\alpha,T}(j\omega)$ does **NOT satisfy** the Nyquist crit. at T s (or at $R_s = \frac{1}{T}$ bauds)
  - Except for $\alpha = 0$, because $H_{RRC}^{0,T}(j\omega) = \sqrt{T}\,\Pi\left(\frac{\omega T}{2\pi}\right)$ is a squared pulse

Raised cosines - side lobe attenuation



Raised cosines - implementation delay

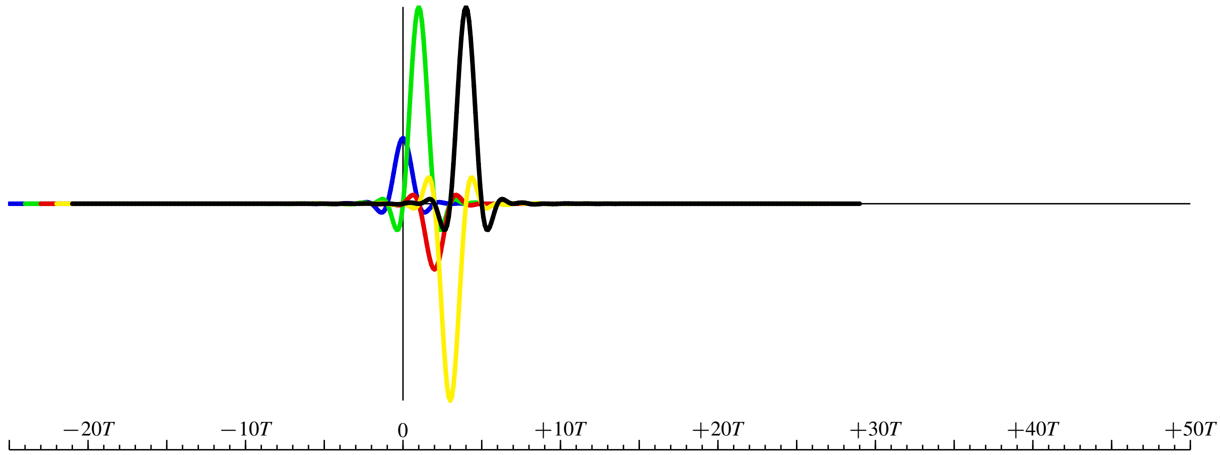
- A raised cosine has a number of "*relevant*" side lobes that is decreasing with roll-off factor
  - Non-relevant lobes could be truncated to make easier the implementation
- For implemententing the modulated waveform, a delay is necessary
  - Delay is related with the number of relevant side lobes that have to be cosidered before truncation
  - Delay is lower for higher values of α (higher bandwidth requirement)
- Example: generation of a 4-PAM waveform with $\alpha = 0$
  - In the example, 25 side lobes are considered relevant (and therefore 25 side lobes are depicted)
  - A delay of $25 \times T$ seconds is necessary to compute the addition
  - Black signal is the last one with relevant contribution at $t = 0$ (related with $A[25]$)



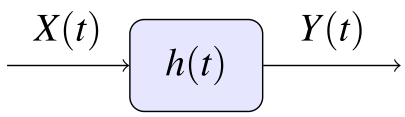
Raised cosines - implementation delay (II)

- Lower delays can be achieved by using higher roll-off factors
  - The price to be paid is a higher required bandwidth
- Example: generation of a 4-PAM waveform with $\alpha = 0.5$
  - In the example, 4 side lobes are considered relevant
  - A delay of $4 \times T$ seconds is necessary to compute the addition
  - Black signal is the last one with relevant contribution at $t = 0$ (related with $A[4]$)
  - Delay is decreased from $25 \times T$ to $4 \times T$ in this example (more than 6 times lower)
  - Required bandwidth is 50 % higher

    NOTE: the number of "*relevant*" lobes depends on required accuracy, this is just a simple example (numbers can not be taken as a precise reference)



Review: random processes and linear systems



Theorem: $X(t)$ is stationary, with mean m_X and autocorrelation function $R_X(\tau)$. The process is the input of a time-invariant linear system with impulse response $h(t)$. In this case, *input and output processes, $X(t)$ and $Y(t)$, are jointly stationary*, being

$$m_Y = m_X \int_{-\infty}^{\infty} h(t)\, dt$$

$$R_Y(\tau) = R_X(\tau) * h(\tau) * h(-\tau)$$

$$R_{XY}(\tau) = R_X(\tau) * h(-\tau)$$

Moreover, it can be seen that

$$R_Y(\tau) = R_{XY}(\tau) * h(\tau)$$

Review: expressions in the frequency domain

- Mean for output process

$$m_Y = m_X \ H(0)$$

- Power spectral density of the output process

$$S_Y(j\omega) = S_X(j\omega) \ |H(j\omega)|^2$$

- Crossed power spectral densities

$$S_{XY}(j\omega) \stackrel{def}{=} \mathcal{FT}\left\{R_{XY}(\tau)\right\}$$

$$S_{XY}(j\omega) = S_X(j\omega)H^*(j\omega)$$

$$S_{YX}(j\omega) = S_{XY}^*(j\omega) = S_X(j\omega)H(j\omega)$$

Review: spectrum of continuous/discrete time signals

- Continuous signal $x(t)$ and discretized $x[n]$ sampled at T seg.

$$x[n] = x(t)\big|_{t=nT} = x(nT)$$

- Usual notation
  - $X(j\omega)$: spectrum (Fourier transform) of $x(t)$
  - $X\left(e^{j\omega}\right)$: spectrum of $x[n]$
- Relationship between both spectral responses
  - To obtain discrete from continuous

$$X\left(e^{j\omega}\right) = \frac{1}{T}\sum_k X\left(j\frac{\omega}{T} - j\frac{2\pi}{T}k\right)$$

  - To obtain continuous from discrete

$$X(j\omega) = T\,X\left(e^{j\omega T}\right), \ |\omega| \leq \frac{\pi}{T}$$

Review: properties of the continuous time autocorrelation function (time ambiguity function)

- Definition for deterministic finite energy function $x(t)$

$$r_x(t) = x(t) * x^*(-t)$$

Informally: measure of similarity between a function and itself with a delay t

- Expression in the frequency domain

$$R_x(j\omega) = \mathcal{FT}\{r_x(t)\} = \mathcal{FT}\{x(t)\} \times \mathcal{FT}\{x^*(-t)\}$$
$$= X(j\omega) \times X^*(j\omega) = |X(j\omega)|^2$$

- Maximum value is at $t = 0$: $|r_x(0)| \geq |r_x(t)|$
- Energy of the signal

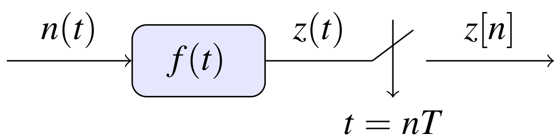
$$\text{Parseval: } \mathcal{E}\{x(t)\} = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega$$

Using the continuous autocorrelation function (temporal ambiguity func.)

$$\mathcal{E}\{x(t)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_x(j\omega) \, d\omega \quad \rightarrow \quad \mathcal{E}\{x(t)\} = r_x(0)$$

Properties of the noise at the receiver

- White noise $n(t)$ ($S_n(j\omega) = N_0/2$) is filtered by receiver filter $f(t)$



- Analysis in the frequency domain
  - PSD of filtered noise $z(t)$

$$S_z(j\omega) = S_n(j\omega) \, |F(j\omega)|^2 = \frac{N_0}{2} \, |F(j\omega)|^2$$

    - ★ Non-flat PSD: Coloured (non-white) noise

      REMARK: unless $|F(j\omega)| = C$, i.e., an all-pass filter (amplifies/attenuates)

  - PSD of sampled noise $z[n]$

$$S_z(e^{j\omega}) = \frac{N_0}{2} \, \frac{1}{T} \sum_k \underbrace{\left| F\left(j\frac{\omega}{T} - j\frac{2\pi}{T}k\right) \right|^2}_{R_f\left(j\frac{\omega}{T} - j\frac{2\pi}{T}k\right)}$$

    - ★ Sampled noise can be white !!!!

$$\text{Condition: } \frac{1}{T} \sum_k R_f\left(j\frac{\omega}{T} - j\frac{2\pi}{T}k\right) = \text{constant}$$

Conditions for sampled noise $z[n]$ being white

- Sampled noise $z[n]$ is white if

$$\frac{1}{T}\sum_k R_f\left(j\frac{\omega}{T}-j\frac{2\pi k}{T}\right)=C \text{ , which is equivalent to } R_f(e^{j\omega})=C$$

  - Equivalent condition in the time domain

$$r_f[n]=r_f(t)\big|_{t=nT}=C\;\delta[n], \text{ which implies } C=r_f(0)=\mathcal{E}\{f(t)\}$$

- Equivalent statement for $z[n]$ being white
  - $z[n]$ is white if the continuous autocorrelation function of receiver filter $r_f(t)$ (or $R_f(j\omega)$) fulfills the same conditions that $p(t)$ has to satisfy for zero ISI (Nyquist conditions)
- <u>REMARK</u>
  - Conditions for $z[n]$ being white only depend on the shape of receiver filter $f(t)$!!!
- Power spectral density for $z[n]$ when it is white

$$S_z(e^{j\omega})=\frac{N_0}{2}\times\mathcal{E}\{f(t)\}\rightarrow \text{if } f(t) \text{ is normalized } S_z(e^{j\omega})=\frac{N_0}{2}$$

Noise power and signal to noise ratio (SNR)

- If Nyquist ISI criterion is meet (ISI=0), the received observation is

$$q[n]=A[n]+z[n]$$

- In this case, signal to noise ratio at $q[n]$ is

$$\left(\frac{S}{N}\right)_q=\frac{E\left[|A[n]|^2\right]}{E\left[|z[n]|^2\right]}=\frac{E_s}{\sigma_z^2}$$

- σ_z^2 is the power (variance) of noise sequence $z[n]$

$$\sigma_z^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}S_z(e^{j\omega})\;d\omega$$

  - If noise $z[n]$ is white, with PSD $S_z(e^{j\omega})=\frac{N_0}{2}\;\mathcal{E}\{f(t)\}$

$$\sigma_z^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{N_0}{2}\times\mathcal{E}\{f(t)\}\;d\omega=\frac{N_0}{2}\times\mathcal{E}\{f(t)\}=\frac{N_0}{2}\times r_f(0)$$

    - ★ For a normalized receiver filter: $\sigma_z^2=\frac{N_0}{2}$

Consequences of Nyquist criterion for Gaussian channels

- A matched filter is assumed at the receiver

$$f(t) = g^*(-t) = g(-t) \text{ since } g(t) \text{ is a real function}$$

- Condition to avoid ISI
  - Joint response $p(t) = g(t) * f(t)$ meets Nyquist criterion
    - ★ Using matched filters $\boxed{p(t) = r_g(t)}$
- Condition for $z[n]$ being white
  - Continuous autocorrelation of the receiver filter, $r_f(t)$, satisfies conditions of the Nyquist criterion
    - ★ Using matched filters $\boxed{r_f(t) = r_g(t)}$
- Conclusion: both conditions are equivalent $\boxed{p(t) = r_f(t) = r_g(t)}$
  - Transmitting through a Gaussian channel using matched filters, if ISI is avoided, sampled noise $z[n]$ is white

Avoidance of ISI in linear channels using matched filters

- Nyquist ISI criterion must be fulfilled for $p[n]$ (or $P(j\omega)$)
  - Definition of $p(t)$ includes now the effect of linear channel $h(t)$
- Design of $p(t)|P(j\omega)$ to fulfill Nyquist at symbol period T
- Design using matched filters at the receiver
  Response of transmitter filter in the frequency domain
  - $P(j\omega) = H(j\omega)\ |G(j\omega)|^2$
  - Therefore

$$G(j\omega) = \begin{cases} \sqrt{\frac{P(j\omega)}{H(j\omega)}}, & \text{if } H(j\omega) \neq 0 \\ 0, & \text{in other case} \end{cases}$$

  If the receiver filter is matched to the transmitter filter, this choice for the transmitter filter eliminates ISI
  - $P(j\omega)$ is a design option
    - ★ Tipically, a raised-cosine response is selected

$$\boxed{P(j\omega) = H_{RC}^{\alpha,T}(j\omega)}$$

Drawbacks of this design option

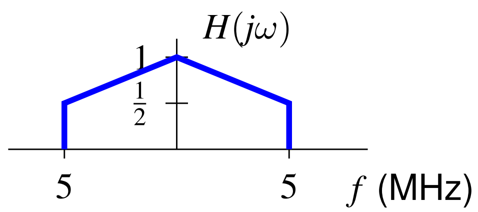
- Channel response, $H(j\omega)$, must be known
  - It can be difficult to know it
  - Channel can be time variant in practice
- Discrete noise sequence, $z[n]$, is not white

$$S_z\left(e^{j\omega}\right) = \frac{N_0}{2}\,\frac{1}{T}\sum_k \underbrace{\left|F\left(j\frac{\omega}{T} - j\frac{2\pi}{T}k\right)\right|^2}_{\left|G\left(j\frac{\omega}{T} - j\frac{2\pi}{T}k\right)\right|^2} = \frac{N_0}{2}\,\frac{1}{T}\sum_k \left|\frac{P\left(j\frac{\omega}{T} - j\frac{2\pi}{T}k\right)}{H\left(j\frac{\omega}{T} - j\frac{2\pi}{T}k\right)}\right|$$

REMARK: For matched filters $F(j\omega) = G^*(j\omega)$, which means $|F(j\omega)| = |G(j\omega)|$

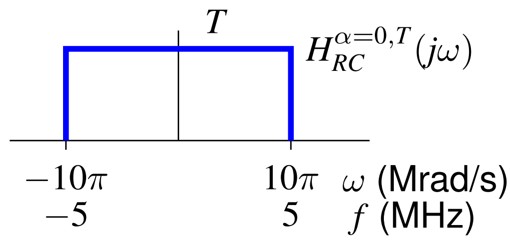
  - Memoryless symbol by symbol detector is not optimal
  - All sequence $q[n]$ has to be used to estimate the symbol at a given discrete instant n_0, $A[n_0]$
  - Noise can be amplified
    - ⋆ Channels with deep attenuation at some frequencies in the band
  - Conclusion:
    - ⋆ Using matched filters, in general is not possible to simultaneously avoiding ISI and having white noise

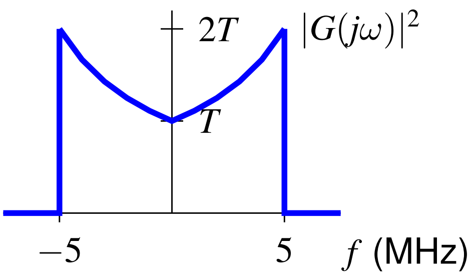
Example: ISI=0 with matched filters

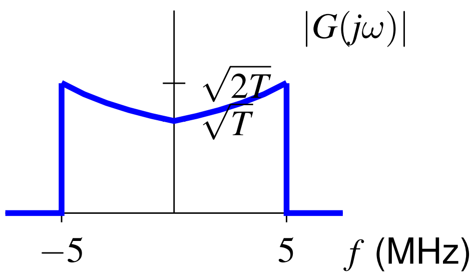


$$P(j\omega) = G(j\omega)\ F(j\omega)\ H(j\omega)$$
$$P(j\omega) = |G(j\omega)|^2\ H(j\omega)$$
$$P(j\omega) = H_{RC}^{\alpha,T}(j\omega)$$
$$G(j\omega) = \sqrt{\frac{P(j\omega)=H_{RC}^{\alpha=0,T}(j\omega)}{H(j\omega)}}$$

Max. Rate: $\alpha = 0$





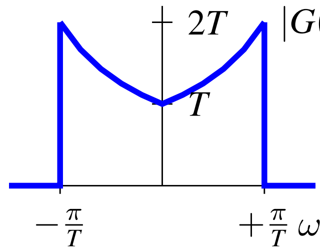


Example: ISI=0 with matched filters (II)

- Power spectral density of noise $z[n]$

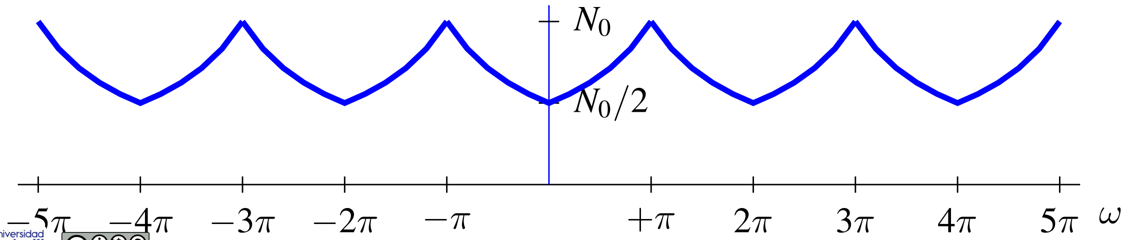
$$S_z\left(e^{j\omega}\right) = \frac{N_0}{2} \times \frac{1}{T} \sum_k R_f\left(j\frac{\omega}{T} - j\frac{2\pi}{T}k\right) = \frac{N_0}{2} \times \frac{1}{T} \sum_k \left| F\left(j\frac{\omega}{T} - j\frac{2\pi}{T}k\right)\right|^2$$

Matched filters: $|F(j\omega)|^2 = |G(j\omega)|^2$



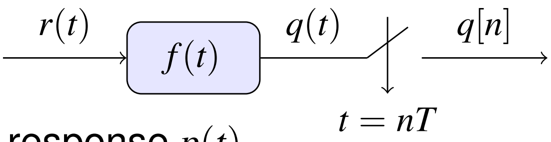
$$S_z\left(e^{j\omega}\right) = \frac{N_0}{2} \times \frac{1}{T} \sum_k \left| G\left(j\frac{\omega}{T} - j\frac{2\pi}{T}k\right)\right|^2$$

$S_z\left(e^{j\omega}\right) \neq C$, therefore $z[n]$ is **NOT** white!!!



Using a generic receiver filter

- Generic receiver, not necesarily a matched filter



- Definition of joint response $p(t)$

$$p(t) = g(t) * h(t) * f(t),\ P(j\omega) = G(j\omega)\ H(j\omega)\ F(j\omega)$$

- Equivalent discrete channel at symbol rate $p[n]$

$$p[n] = p(nT) = (g(t) * h(t) * f(t))\ \big|_{t=nT}$$

- Filtered noise

$$z(t) = n(t) * f(t),\ z[n] = z(nT)$$

  - Power spectral density for discrete noise $z[n]$

$$S_z\left(e^{j\omega}\right) = \frac{N_0}{2} \times \frac{1}{T} \sum_k \left| F\left(j\frac{\omega}{T} - j\frac{2\pi}{T}k\right)\right|^2$$

Design of $g(t)$ and $f(t)$

- Simultaneous avoidance of ISI and white noise (precoding)
  - Selection of $P(j\omega)$ fulfilling Nyquist
  - Selection of $F(j\omega)$ with $R_f(j\omega) = |F(j\omega)|^2$ fulfilling Nyquist
  - Then, transmitter filter is given by

$$G(j\omega) = \frac{P(j\omega)}{H(j\omega)\, F(j\omega)}$$

    - Usually presents serious implementation problems
- The usual design
  - Joint response: $P(j\omega) = H_{RC}^{\alpha,T}(j\omega)$
    - Transmission without ISI
  - Receiver filter: $F(j\omega) = H_{RRC}^{\alpha,T}(j\omega)$
    - Noise $z[n]$ is white (because $R_f(j\omega) = H_{RC}^{\alpha,T}(j\omega)$)
  - Transmitter filter

$$G(j\omega) = \frac{H_{RC}^{\alpha,T}(j\omega)}{H(j\omega)\, H_{RRC}^{\alpha,T}(j\omega)} = \frac{H_{RRC}^{\alpha,T}(j\omega)}{H(j\omega)}$$

Other design criteria

- Filter matched to the joint transmitter-channel response

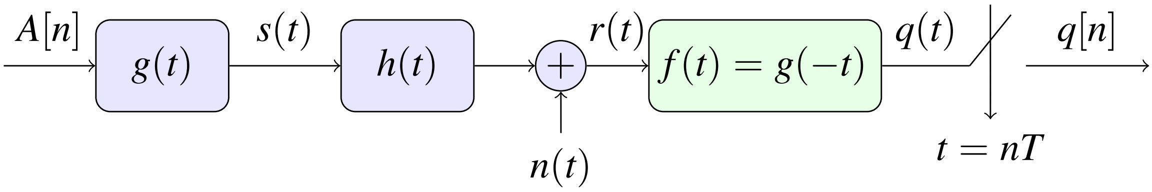
$$f(t) = g_h(-t), \text{ with } g_h(t) = g(t) * h(t)$$

  - Maximizes the signal to noise ratio
  - Does not provides zero ISI and noise $z[n]$ is not white
- Minimum mean squared error criterion: to maximize

$$\frac{E\left[|A[n]\ p[0]|^2\right]}{E\left[\left|\sum_{\substack{k\\k\neq n}} A[k]\ p[n-k] + z[n]\right|^2\right]}$$

Typical set up for linear channels

- Receiver uses a matched filter $f(t) = g(-t)$ with $r_f(t) = r_g(t)$ fulfilling Nyquist condition



  - Common choice: root-raised cosine filters

$$g(t) = h_{RRC}^{\alpha,T}(t) \quad \rightarrow \quad f(t) = h_{RRC}^{\alpha,T}(t)$$
$$g(t) * f(t) = r_g(t) = r_f(t) = h_{RC}^{\alpha,T}(t)$$

- Consequences:
  - This ensures discrete filtered noise $z[n]$ is white
  - ISI is present in the system (joint response $p(t)$ then does not meet Nyquist condition)
    - ⋆ Receivers can be specifically designed to deal with ISI (as it will be seen in Chapter 2)

Review - Evaluation of Probability of Symbol Error (P_e)

- Definition

$$P_e = P(\hat{A}[n] \neq A[n])$$

- Evaluation - Averaging of probability of symbol error for each symbol in the constellation

$$P_e = \sum_{i=0}^{M-1} p_A(\boldsymbol{a}_i)\, P_{e|\boldsymbol{a}_i}$$

- Calculation of conditional probabilities of symbol error (conditional probabilities of error)

$$P_{e|\boldsymbol{a}_i} = \int_{\boldsymbol{q} \notin I_i} f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}|\boldsymbol{a}_i)\, d\boldsymbol{q}$$

Conditional distribution of observations conditioned to transmission of the symbol $\boldsymbol{a}_i$ is integrated out of its decision region I_i

Review - Calculation of Bit Error Rate (BER)

- Conditional BER for each symbol $\boldsymbol{a}_i$ are averaged

$$BER = \sum_{i=0}^{M-1} p_A(\boldsymbol{a}_i)\ BER_{\boldsymbol{a}_i}$$

- Calculation of conditional BER for $\boldsymbol{a}_i$

$$BER_{\boldsymbol{a}_i} = \sum_{\substack{j=0\\ j\neq i}}^{M-1} P_{e|\boldsymbol{a}_i\to\boldsymbol{a}_j} \frac{m_{e|\boldsymbol{a}_i\to\boldsymbol{a}_j}}{m}$$

  - $P_{e|\boldsymbol{a}_i\to\boldsymbol{a}_j}$: probability of deciding $\hat{A} = \boldsymbol{a}_j$ when $\boldsymbol{A} = \boldsymbol{a}_i$ was transmitted

$$P_{e|\boldsymbol{a}_i\to\boldsymbol{a}_j} = \int_{\boldsymbol{q}_0\in I_j} f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}_0|\boldsymbol{a}_i)\ d\boldsymbol{q}_0$$

  - $m_{e|\boldsymbol{a}_i\to\boldsymbol{a}_j}$: number of bit errors associated to that decision
  - m: number of bits per symbol in the constellation

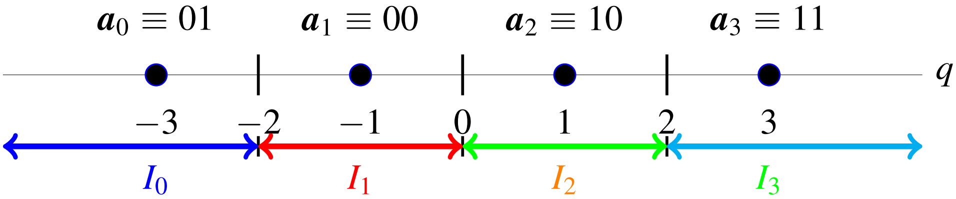
Example - 1-D M-ary constellation

- Example:
  - $M = 4$, equiprobable symbols $p_A(\boldsymbol{a}_i) = \frac{1}{4}$
  - Constellation: $\boldsymbol{a}_0 = -3,\ \boldsymbol{a}_1 = -1,\ \boldsymbol{a}_2 = +1,\ \boldsymbol{a}_3 = +3$
  - Decision regions: thresholds $q_{u1} = -2,\ q_{u2} = 0,\ q_{u3} = +2$

$$I_0 = (-\infty, -2],\ I_1 = (-2, 0],\ I_2 = (0, +2],\ I_3 = (+2, +\infty)$$

  - Binary assignment

$$\boldsymbol{a}_0 \equiv 01,\ \boldsymbol{a}_1 \equiv 00,\ \boldsymbol{a}_2 \equiv 10,\ \boldsymbol{a}_3 \equiv 11$$



- No ISI ($p[n] = \delta[n]$) and white noise with variance $N_0/2$

$$q[n] = A[n] + z[n]$$

Case that was studied in "Communications Theory"

Example - 1-D M-ary constellation (II)

- Probability of error

$$P_e = \frac{1}{4}\sum_{i=0}^{3} P_{e|\boldsymbol{a}_i} = \frac{3}{2} Q\left(\frac{1}{\sqrt{N_0/2}}\right)$$

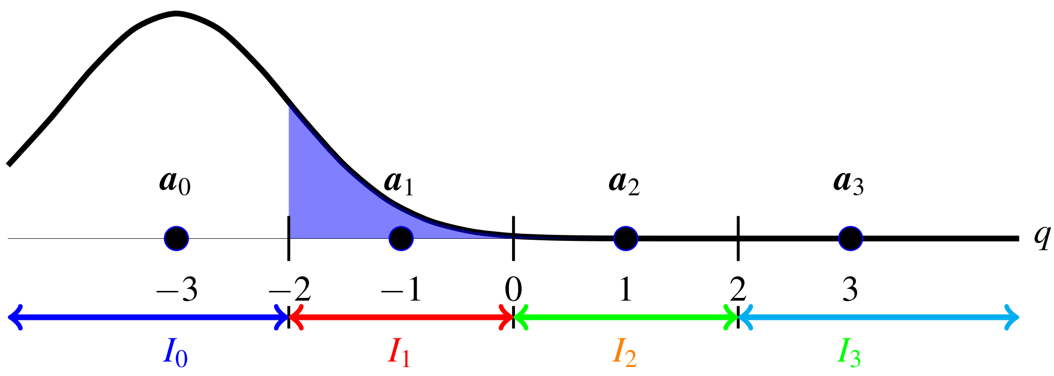
- Bit error rate (BER)

$$BER = \frac{1}{4}\sum_{i=0}^{3} BER_{\boldsymbol{a}_i}$$

$$= \frac{3}{4} Q\left(\frac{1}{\sqrt{N_o/2}}\right) + \frac{1}{2} Q\left(\frac{3}{\sqrt{N_o/2}}\right) - \frac{1}{4} Q\left(\frac{5}{\sqrt{N_o/2}}\right)$$

- Analytic developments are detailed in next slides

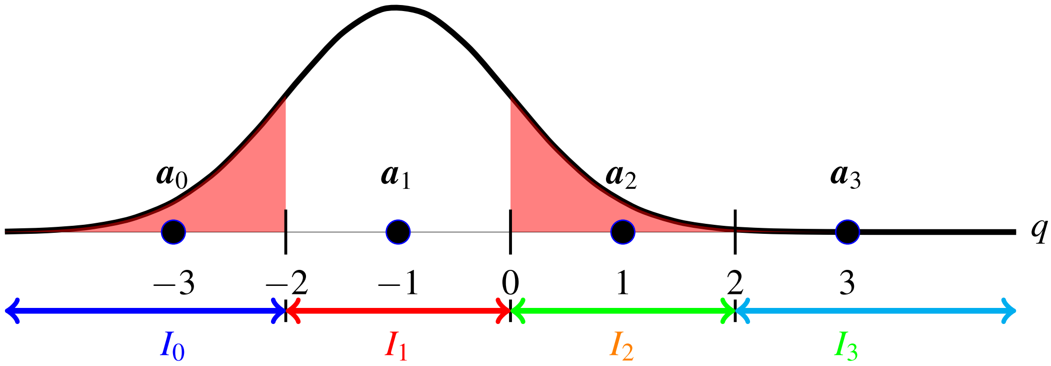
Calculation of $P_{e|\boldsymbol{a}_0}$



- Distribution $f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}|\boldsymbol{a}_0)$
  - Gaussian with mean $\boldsymbol{a}_0 = -3$ and variance $N_0/2$
- Conditional probability of error
  - Integration of $f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}|\boldsymbol{a}_0)$ out of I_0

$$P_{e|\boldsymbol{a}_0} = \int_{q \notin I_0} f_{\boldsymbol{q}|\boldsymbol{A}}(q|\boldsymbol{a}_0)\, dq = Q\left(\frac{1}{\sqrt{N_0/2}}\right)$$

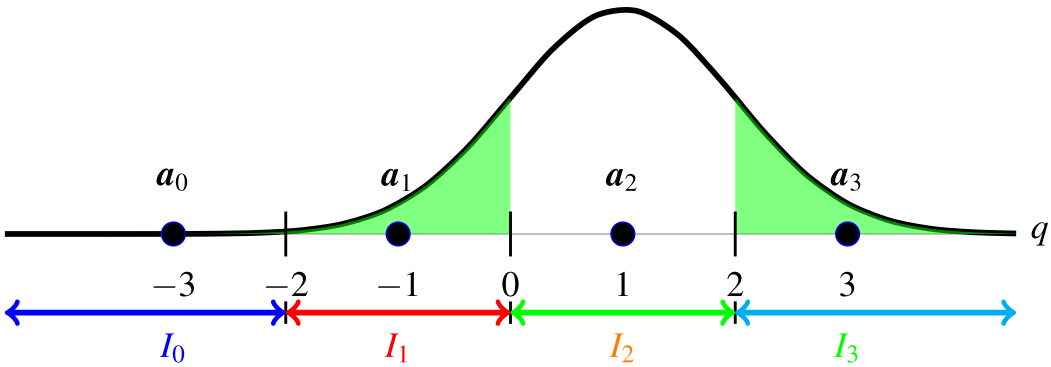
Calculation of $P_{e|\boldsymbol{a}_1}$



- Distribution $f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}|\boldsymbol{a}_1)$
  - Gaussian with mean $\boldsymbol{a}_1 = -1$ and variance $N_0/2$
- Conditional probability of error
  - Integration of $f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}|\boldsymbol{a}_1)$ out of I_1

$$P_{e|\boldsymbol{a}_1} = \int_{q \notin I_1} f_{\boldsymbol{q}|\boldsymbol{A}}(q|\boldsymbol{a}_1)\, dq = 2Q\left(\frac{1}{\sqrt{N_0/2}}\right)$$

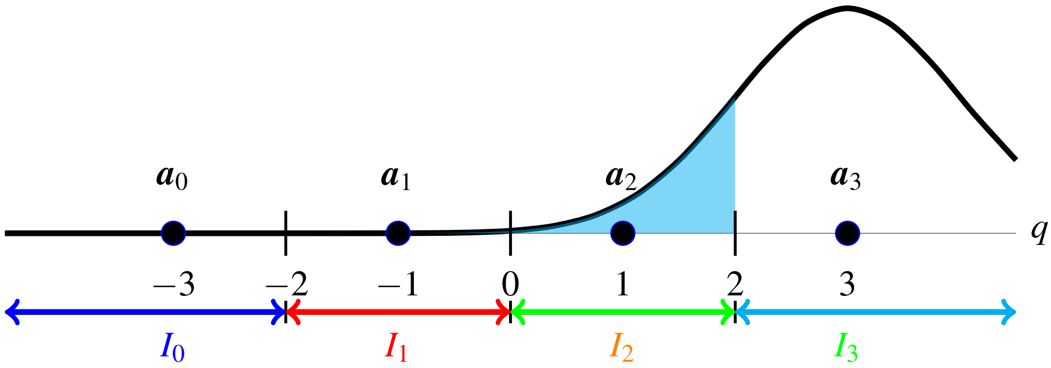
Calculation of $P_{e|\boldsymbol{a}_2}$



- Distribution $f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}|\boldsymbol{a}_2)$
  - Gaussian with mean $\boldsymbol{a}_2 = +1$ and variance $N_0/2$
- Probability of error
  - Integration of $f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}|\boldsymbol{a}_2)$ out of I_2

$$P_{e|\boldsymbol{a}_2} = \int_{q \notin I_2} f_{\boldsymbol{q}|\boldsymbol{A}}(q|\boldsymbol{a}_2)\, dq = 2Q\left(\frac{1}{\sqrt{N_0/2}}\right)$$

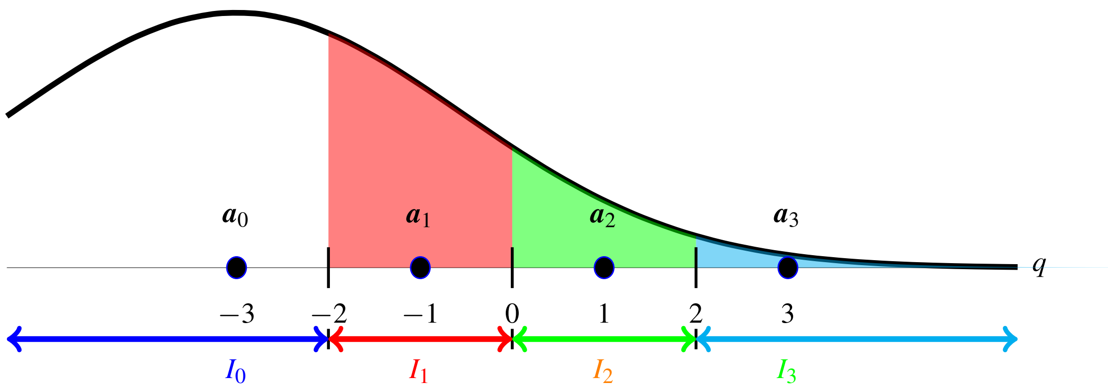
Calculation of $P_{e|a_3}$



- Distribution $f_{q|A}(q|a_3)$
  - Gaussian with mean $a_3 = -3$ and variance $N_0/2$
- Probability of error
  - Integration of $f_{q|A}(q|a_3)$ out of I_3

$$P_{e|a_3} = \int_{q \notin I_3} f_{q|A}(q|a_3)\, dq = Q\left(\frac{1}{\sqrt{N_0/2}}\right)$$

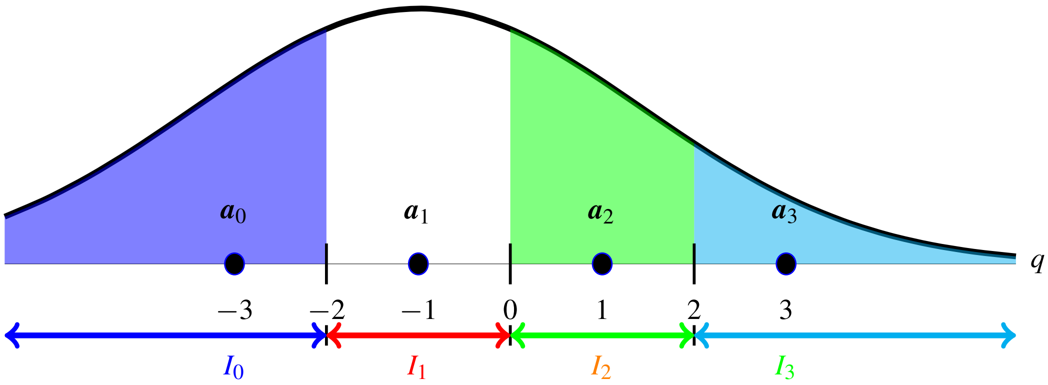
Calculation of BER_{a_0}



- Binary assignment: $a_0 \equiv 01,\ a_1 \equiv 00,\ a_2 \equiv 10,\ a_3 \equiv 11$
- Distribution $f_{q|A}(q|a_0)$: Gaussian with mean a_0 and variance $N_0/2$

$$BER_{a_0} = \underbrace{\left[Q\left(\frac{1}{\sqrt{N_0/2}}\right) - Q\left(\frac{3}{\sqrt{N_0/2}}\right)\right]}_{P_{e|a_0 \to a_1}} \times \underbrace{\frac{1}{2}}_{\frac{m_{e|a_0 \to a_1}}{m}}$$
$$+ \underbrace{\left[Q\left(\frac{3}{\sqrt{N_0/2}}\right) - Q\left(\frac{5}{\sqrt{N_0/2}}\right)\right]}_{P_{e|a_0 \to a_2}} \times \underbrace{\frac{2}{2}}_{\frac{m_{e|a_0 \to a_2}}{m}} + \underbrace{\left[Q\left(\frac{5}{\sqrt{N_0/2}}\right)\right]}_{P_{e|a_0 \to a_3}} \times \underbrace{\frac{1}{2}}_{\frac{m_{e|a_0 \to a_3}}{m}}$$

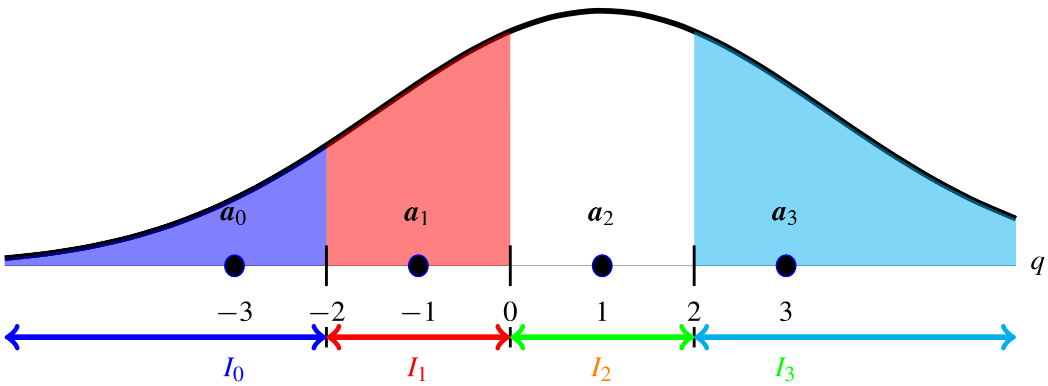
Calculation of BER_{a_1}



- Binary assignment: $a_0 \equiv 01,\ a_1 \equiv 00,\ a_2 \equiv 10,\ a_3 \equiv 11$
- Distribution $f_{q|A}(q|a_1)$: Gaussian with mean a_1 and variance $N_0/2$

$$BER_{a_1} = \underbrace{\left[Q\left(\frac{1}{\sqrt{N_0/2}}\right)\right]}_{P_{e|a_1 \to a_0}} \times \underbrace{\frac{1}{2}}_{\frac{m_{e|a_1 \to a_0}}{m}} + \underbrace{\left[Q\left(\frac{1}{\sqrt{N_0/2}}\right) - Q\left(\frac{3}{\sqrt{N_0/2}}\right)\right]}_{P_{e|a_1 \to a_2}} \times \underbrace{\frac{1}{2}}_{\frac{m_{e|a_1 \to a_2}}{m}}$$
$$+ \underbrace{\left[Q\left(\frac{3}{\sqrt{N_0/2}}\right)\right]}_{P_{e|a_1 \to a_3}} \times \underbrace{\frac{2}{2}}_{\frac{m_{e|a_1 \to a_3}}{m}}$$

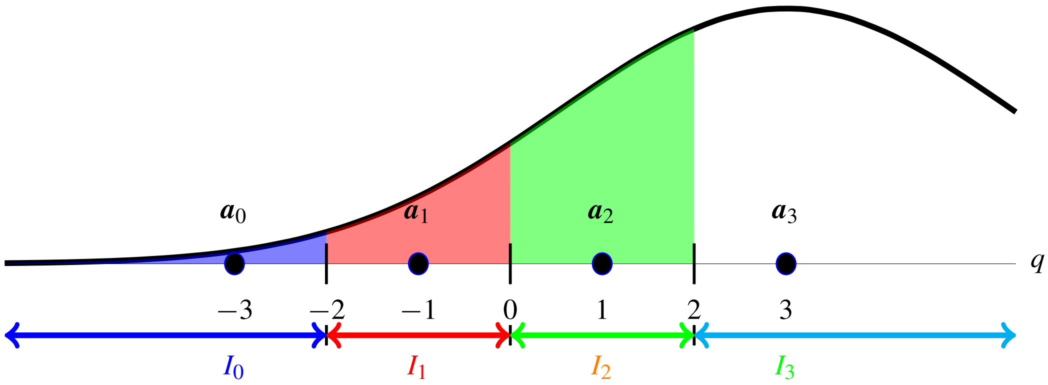
Calculation of BER_{a_2}



- Binary assignment: $a_0 \equiv 01,\ a_1 \equiv 00,\ a_2 \equiv 10,\ a_3 \equiv 11$
- Distribution $f_{q|A}(q|a_2)$: Gaussian with mean a_2 and variance $N_0/2$

$$BER_{a_2} = \underbrace{\left[Q\left(\frac{3}{\sqrt{N_0/2}}\right)\right]}_{P_{e|a_2 \to a_0}} \times \underbrace{\frac{2}{2}}_{\frac{m_{e|a_2 \to a_0}}{m}} + \underbrace{\left[Q\left(\frac{1}{\sqrt{N_0/2}}\right) - Q\left(\frac{3}{\sqrt{N_0/2}}\right)\right]}_{P_{e|a_2 \to a_1}} \times \underbrace{\frac{1}{2}}_{\frac{m_{e|a_2 \to a_1}}{m}}$$
$$+ \underbrace{\left[Q\left(\frac{1}{\sqrt{N_0/2}}\right)\right]}_{P_{e|a_2 \to a_3}} \times \underbrace{\frac{1}{2}}_{\frac{m_{e|a_2 \to a_3}}{m}}$$

Calculation of BER_{a_3}

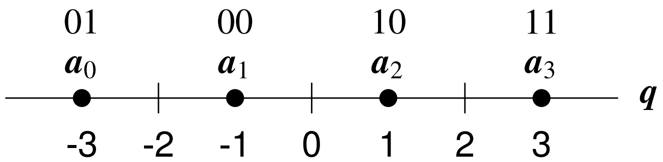


- Binary assignment: $\boldsymbol{a}_0 \equiv 01,\ \boldsymbol{a}_1 \equiv 00,\ \boldsymbol{a}_2 \equiv 10,\ \boldsymbol{a}_3 \equiv 11$
- Distribution $f_{\boldsymbol{q}|A}(\boldsymbol{q}|\boldsymbol{a}_3)$: Gaussian with mean $\boldsymbol{a}_3$ and variance $N_0/2$

$$BER_{a_3} = \underbrace{\left[Q\left(\frac{5}{\sqrt{N_0/2}}\right)\right]}_{P_{e|\boldsymbol{a}_3\to\boldsymbol{a}_0}} \times \underbrace{\frac{1}{2}}_{\frac{m_{e|\boldsymbol{a}_3\to\boldsymbol{a}_0}}{m}} + \underbrace{\left[Q\left(\frac{3}{\sqrt{N_0/2}}\right) - Q\left(\frac{5}{\sqrt{N_0/2}}\right)\right]}_{P_{e|\boldsymbol{a}_3\to\boldsymbol{a}_1}} \times \underbrace{\frac{2}{2}}_{\frac{m_{e|\boldsymbol{a}_3\to\boldsymbol{a}_1}}{m}}$$
$$+ \underbrace{\left[Q\left(\frac{1}{\sqrt{N_0/2}}\right) - Q\left(\frac{3}{\sqrt{N_0/2}}\right)\right]}_{P_{e|\boldsymbol{a}_3\to\boldsymbol{a}_2}} \times \underbrace{\frac{1}{2}}_{\frac{m_{e|\boldsymbol{a}_3\to\boldsymbol{a}_2}}{m}}$$

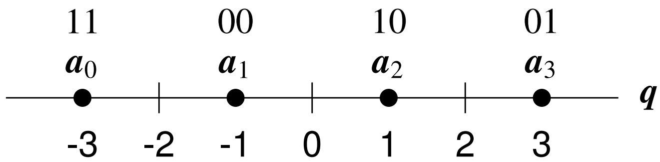
Modification of the binary assignment

- Final result for previous binary assignment



$$BER = \frac{3}{4}Q\left(\frac{1}{\sqrt{N_0/2}}\right) + \frac{1}{2}Q\left(\frac{3}{\sqrt{N_0/2}}\right) - \frac{1}{4}Q\left(\frac{5}{\sqrt{N_0/2}}\right)$$

- If binary assignment is modified

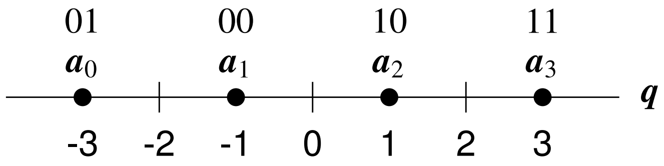


  - Terms $P_{e|\boldsymbol{a}_i\to\boldsymbol{a}_j}$ do not vary
  - Terms $m_{e|\boldsymbol{a}_i\to\boldsymbol{a}_j}$ do vary $\Rightarrow$ BER is modified !!!

$$BER = \frac{5}{4}Q\left(\frac{1}{\sqrt{N_0/2}}\right) - \frac{1}{4}Q\left(\frac{3}{\sqrt{N_0/2}}\right)$$

Gray Coding

- Blocks of m bits assigned to symbols at minimum distance differ in only a single bit



  - This assignment minimizes BER for a given constellation
- Terms $P_{e|a_i \to a_j}$ depend on the constellation
  - Values depend on distance between a_i and a_j
  - Highest values for symbols at minimum distance
- Terms $\frac{m_{e|a_i \to a_j}}{m}$ depend on bit assignment
  - These terms weight the contribution of $P_{e|a_i \to a_j}$
    - Gray coding: minimizes impact of highest values of $P_{e|a_i \to a_j}$
    - For high values of signal to noise ratio (SNR), in most cases, a symbol error produces a single erroneous bit

$$BER \approx \frac{1}{m} P_e$$

Probability of error with and without ISI

- Example: 2-PAM modulation: $A[n] \in \{\pm 1\}$ at $R_s = \frac{1}{T}$ bauds
- Receiver: normalized root-raised cosine with roll-off α

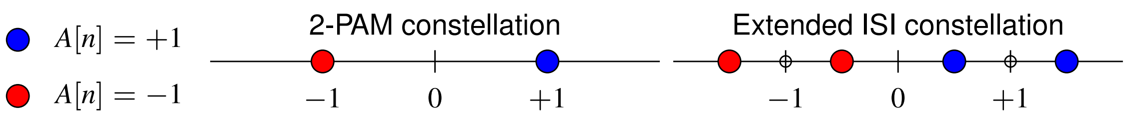
$$f(t) = h_{RRC}^{\alpha,T}(t) \to r_f(t) = f(t) * f(-t) = h_{RC}^{\alpha,T}(t)$$

$$z[n] \text{ is white with } \sigma_z^2 = \frac{N_0}{2}$$

- Equivalent discrete channel: $p[n] = \delta[n] + \frac{1}{2}\delta[n-1]$
- ISI produces an extended constellation at the receiver side

$$o[n] = A[n] * p[n] = A[n] + \frac{1}{2} A[n-1]$$

$A[n]$	$A[n-1]$	$o[n]$
$+1$	$+1$	$+\frac{3}{2}$
$+1$	-1	$+\frac{1}{2}$
-1	$+1$	$-\frac{1}{2}$
-1	-1	$-\frac{3}{2}$

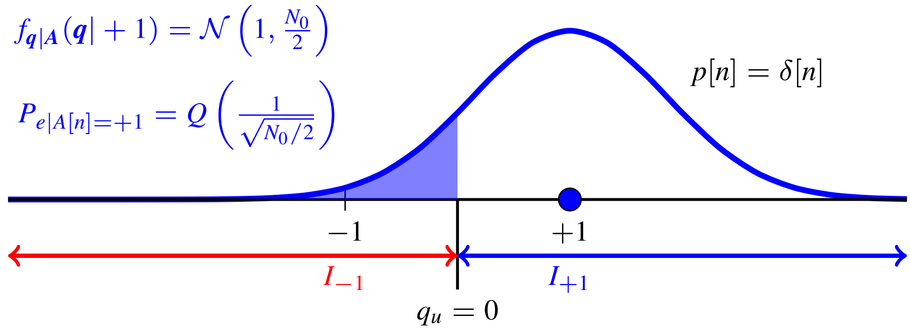


Probability of error with and without ISI (II)

Conditional probability of error for $A[n] = +1$, i.e., $P_{e|A[n]=+1}$

$$f_{\boldsymbol{q}|A}(\boldsymbol{q}|+1) = \mathcal{N}\left(1, \frac{N_0}{2}\right)$$

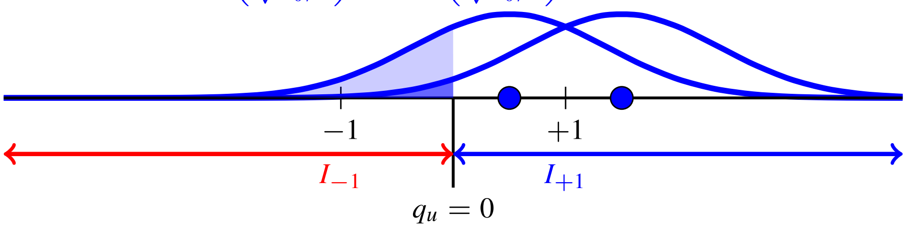
$$P_{e|A[n]=+1} = Q\left(\frac{1}{\sqrt{N_0/2}}\right)$$



$$f_{\boldsymbol{q}|A}(\boldsymbol{q}|+1) = \frac{1}{2}\mathcal{N}\left(\frac{1}{2}, \frac{N_0}{2}\right) + \frac{1}{2}\mathcal{N}\left(\frac{3}{2}, \frac{N_0}{2}\right)$$

$$P_{e|A[n]=+1} = \frac{1}{2}Q\left(\frac{1/2}{\sqrt{N_0/2}}\right) + \frac{1}{2}Q\left(\frac{3/2}{\sqrt{N_0/2}}\right)$$

$p[n] = \delta[n] + \frac{1}{2}\delta[n-1]$

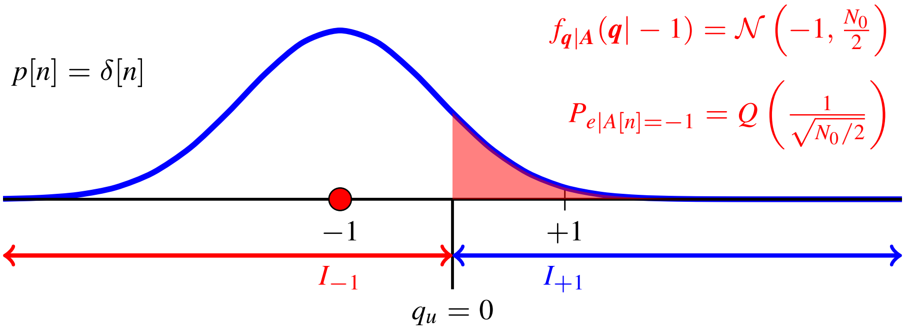


Probability of error with and without ISI (III)

Conditional probability of error for $A[n] = -1$, i.e., $P_{e|A[n]=-1}$

$$f_{\boldsymbol{q}|A}(\boldsymbol{q}|-1) = \mathcal{N}\left(-1, \frac{N_0}{2}\right)$$

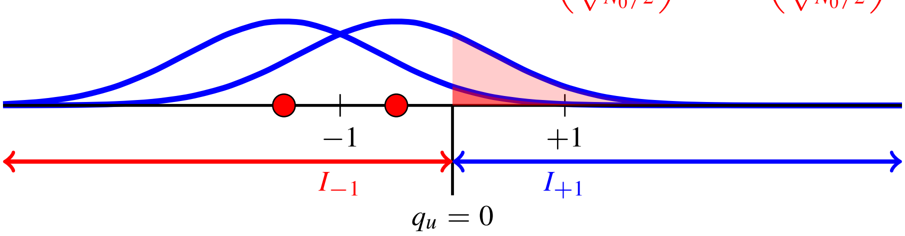
$$P_{e|A[n]=-1} = Q\left(\frac{1}{\sqrt{N_0/2}}\right)$$



$$f_{\boldsymbol{q}|A}(\boldsymbol{q}|-1) = \frac{1}{2}\mathcal{N}\left(-\frac{1}{2}, \frac{N_0}{2}\right) + \frac{1}{2}\mathcal{N}\left(-\frac{3}{2}, \frac{N_0}{2}\right)$$

$p[n] = \delta[n] + \frac{1}{2}\delta[n-1]$

$$P_{e|A[n]=-1} = \frac{1}{2}Q\left(\frac{1/2}{\sqrt{N_0/2}}\right) + \frac{1}{2}Q\left(\frac{3/2}{\sqrt{N_0/2}}\right)$$



Probability of error with and without ISI (IV)

- Probability of error without ISI

$$P_e = \frac{1}{2}P_{e|A[n]=+1} + \frac{1}{2}P_{e|A[n]=-1} = Q\left(\frac{1}{\sqrt{N_0/2}}\right)$$

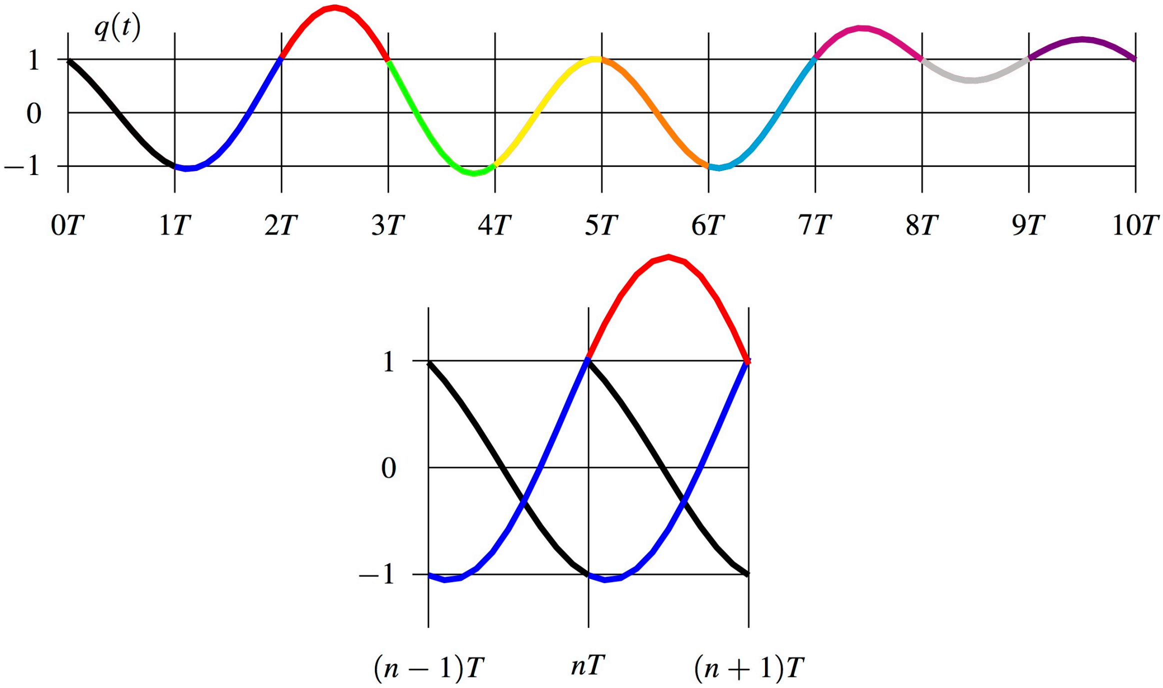
- Probability of error with ISI

$$P_e = \frac{1}{2}P_{e|A[n]=+1} + \frac{1}{2}P_{e|A[n]=-1} = \frac{1}{2}Q\left(\frac{1/2}{\sqrt{N_0/2}}\right) + \frac{1}{2}Q\left(\frac{3/2}{\sqrt{N_0/2}}\right)$$

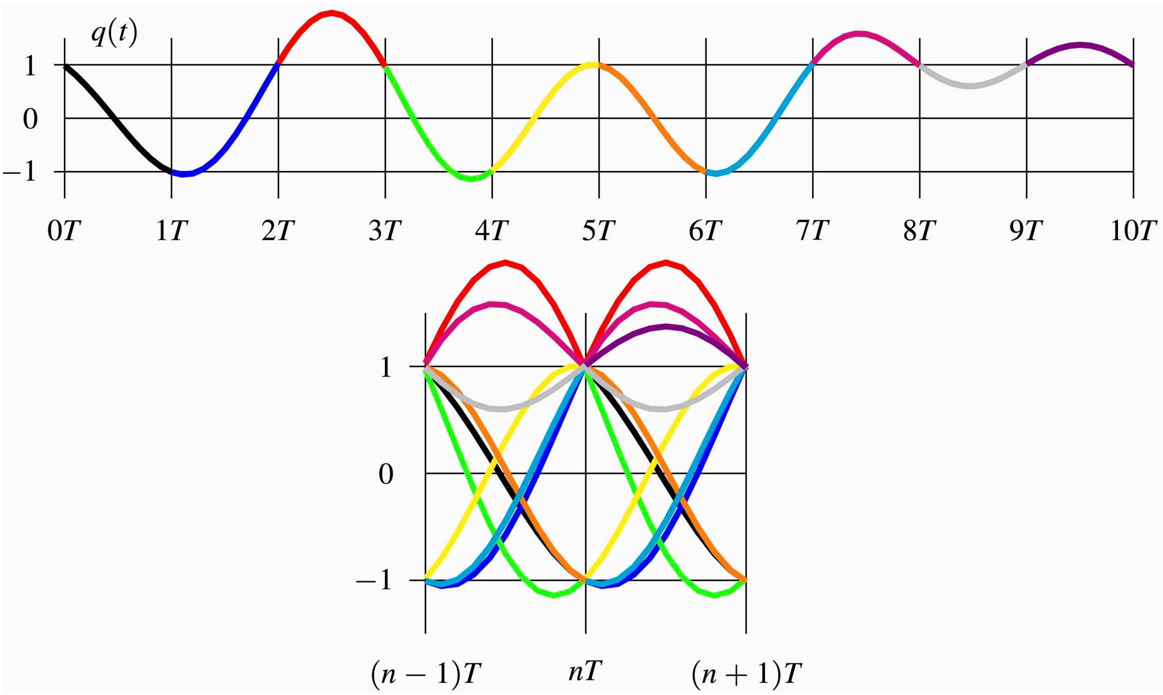
Eye diagram (eye pattern)

- Monitoring tool for a digital communication system
  - Superposition of waveform pieces around a sampling point
  - Duration of each piece: $2T$
  - Obtained using an osciloscope
    - ★ Trigger: governed by sampling signal
    - ★ Timebase: to cover $2T$
- Main features
  - In the middle and in both sides (horizontaly), there are sampling instants
    - ★ Traces should have to go through values of the constellation
  - Diversity of transition between sampling instants depend on the shape of transmitter and receiver filters
- It allows to detect several problems:
  - Problems/sensitivity to synchronism
  - Level of noise
  - Presence (and level) of ISI

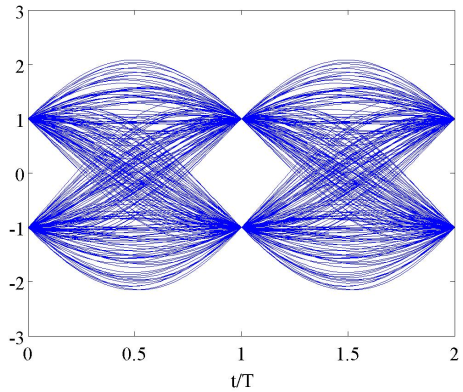
Eye Diagram - $\alpha = 0$



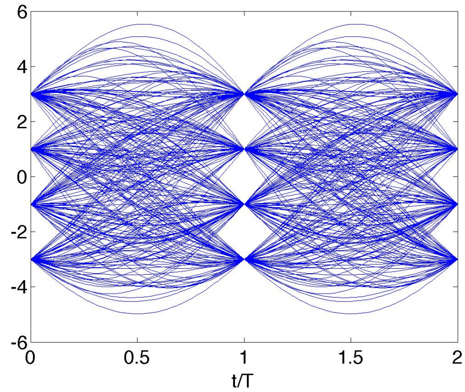
Eye Diagram - $\alpha = 0$



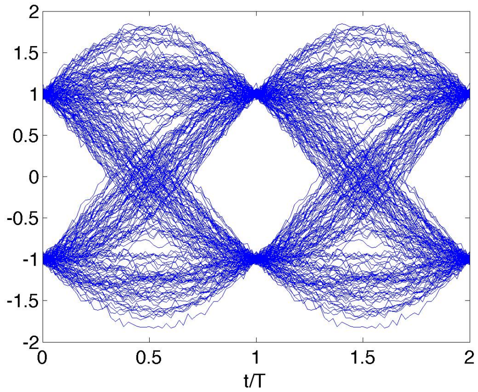
Eye diagram - Examples



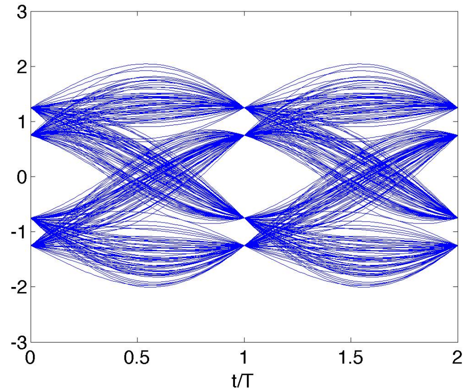
2-PAM $\alpha = 0$



4-PAM $\alpha = 0$

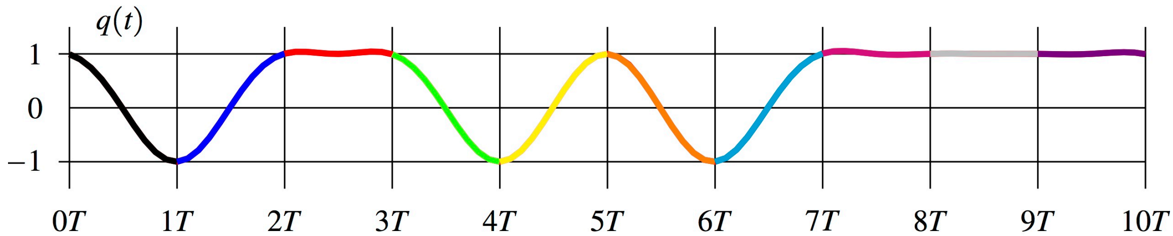


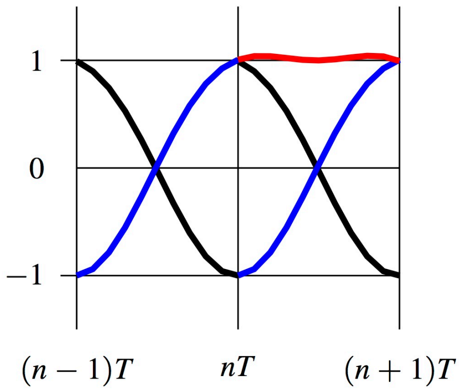
2-PAM $\alpha = 0$ Noisy



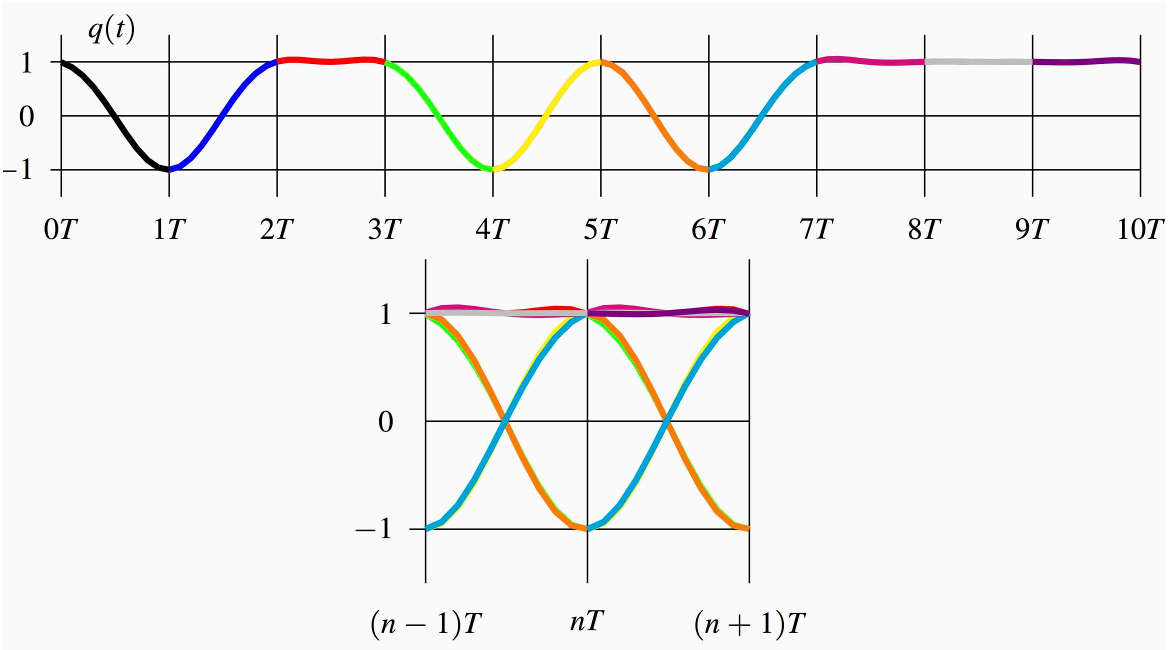
2-PAM $\alpha = 0$, ISI

Eye Diagram - $\alpha = 1$



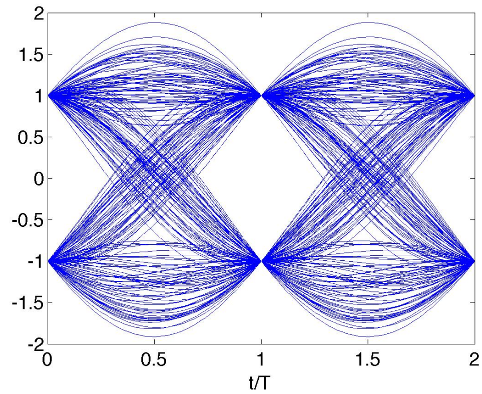


Eye Diagram - $\alpha = 1$

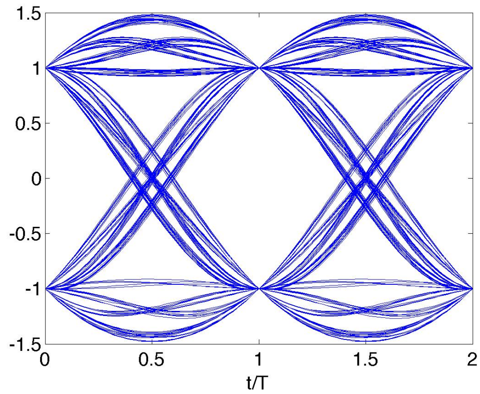


- Eye width is higher as α increases
  - Lower sensitivity to sampling synchronization and jitter effects

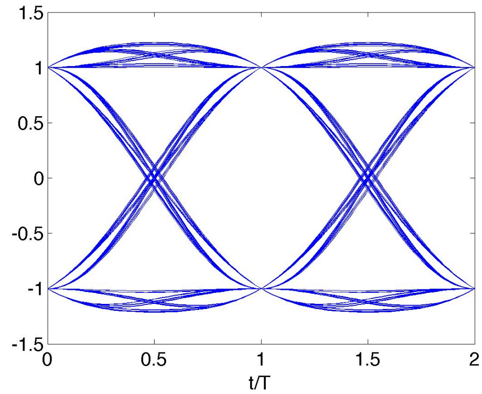
Eye diagram - Examples (II)



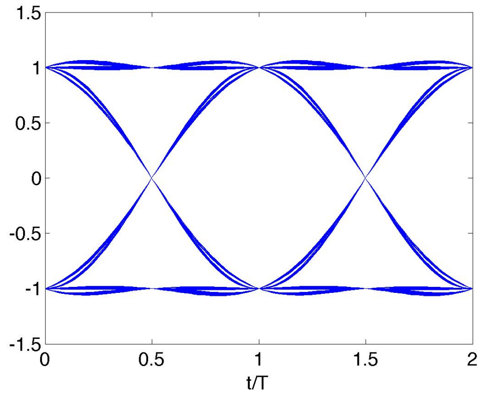
2-PAM $\alpha = 0.25$



2-PAM $\alpha = 0.5$

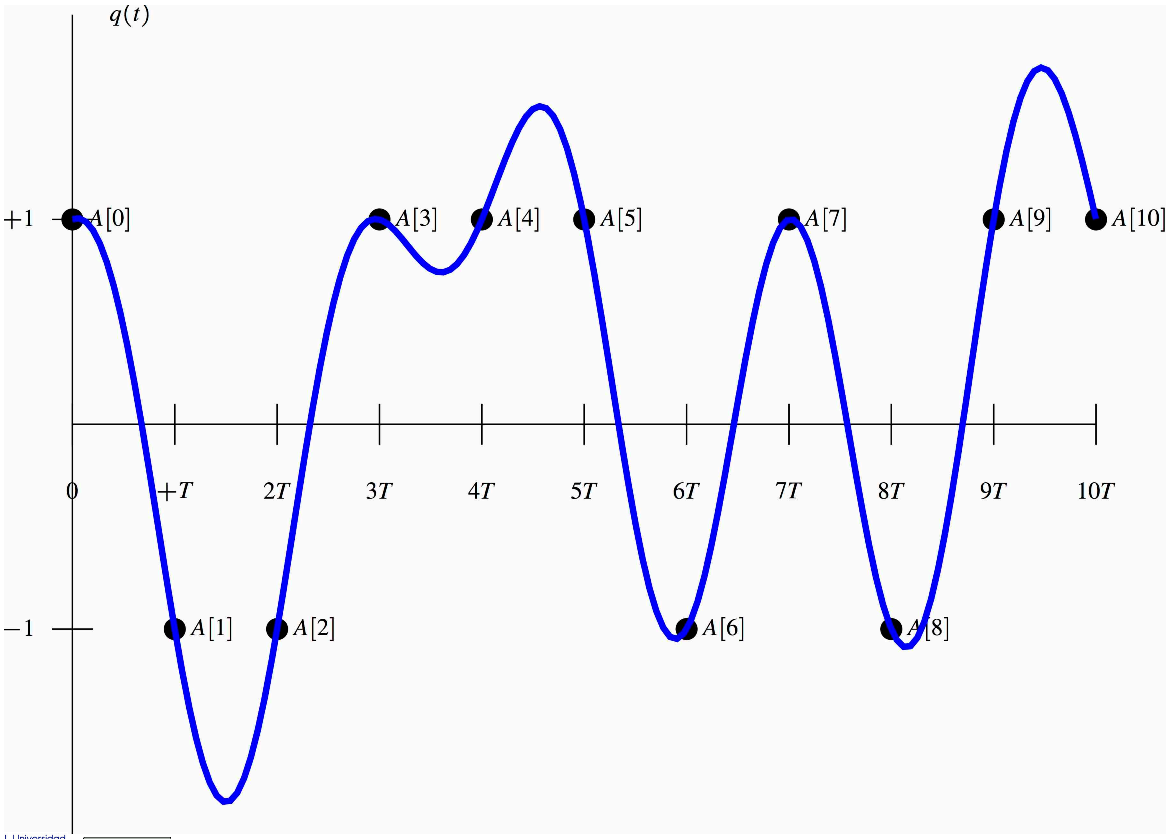


2-PAM $\alpha = 0.75$

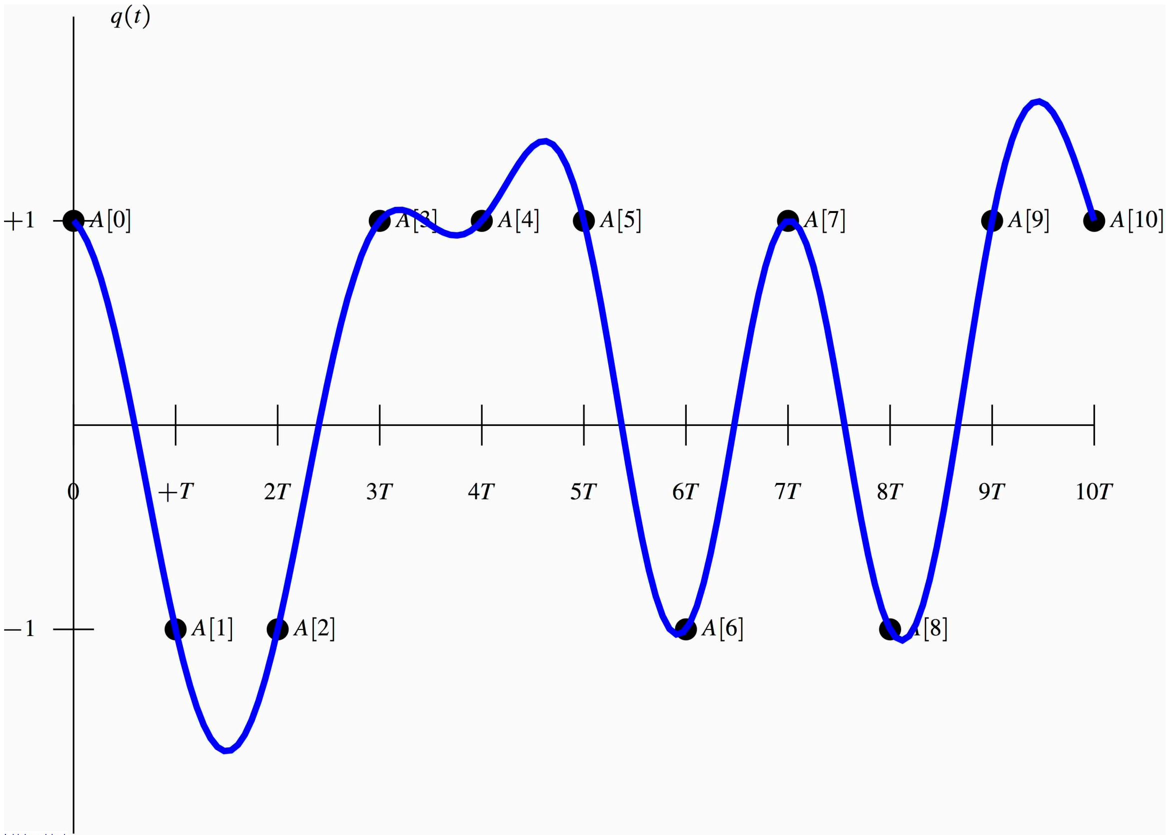


2-PAM $\alpha = 1$

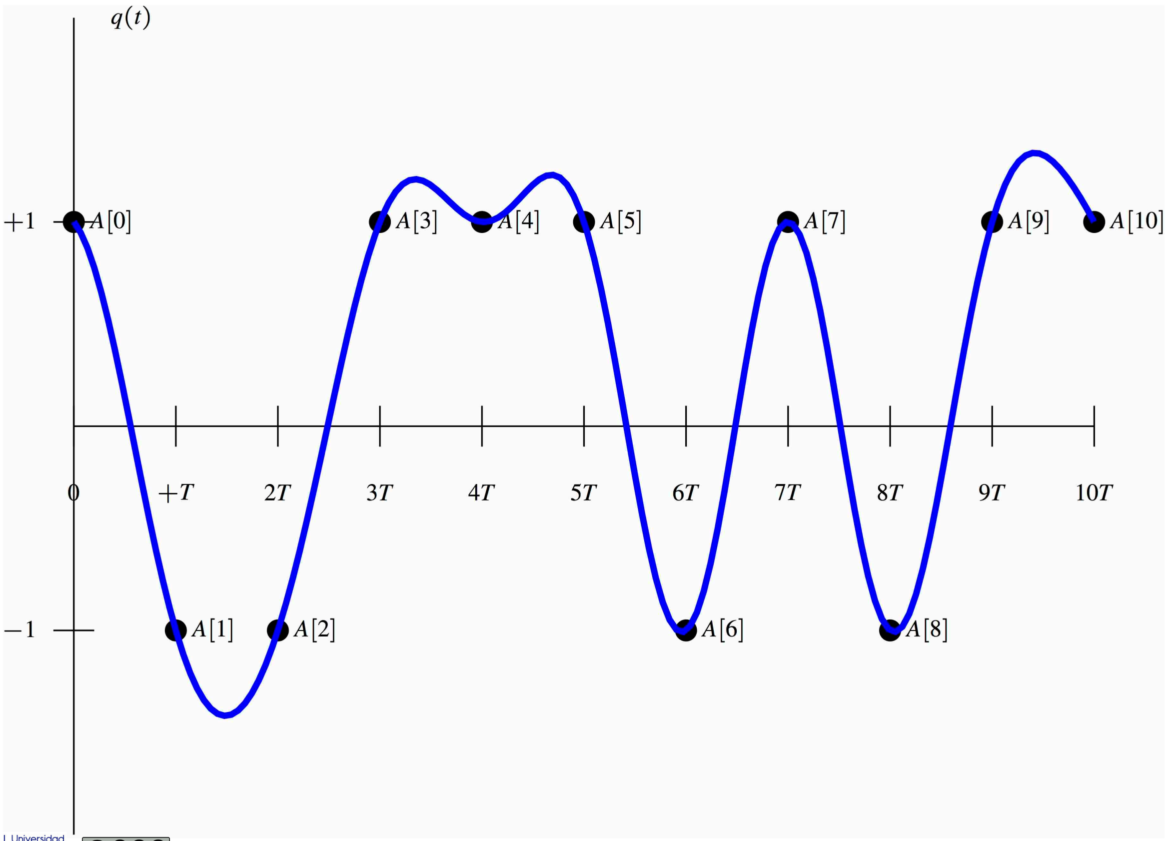
Signals with raised cosines (ideal) - $\alpha = 0$



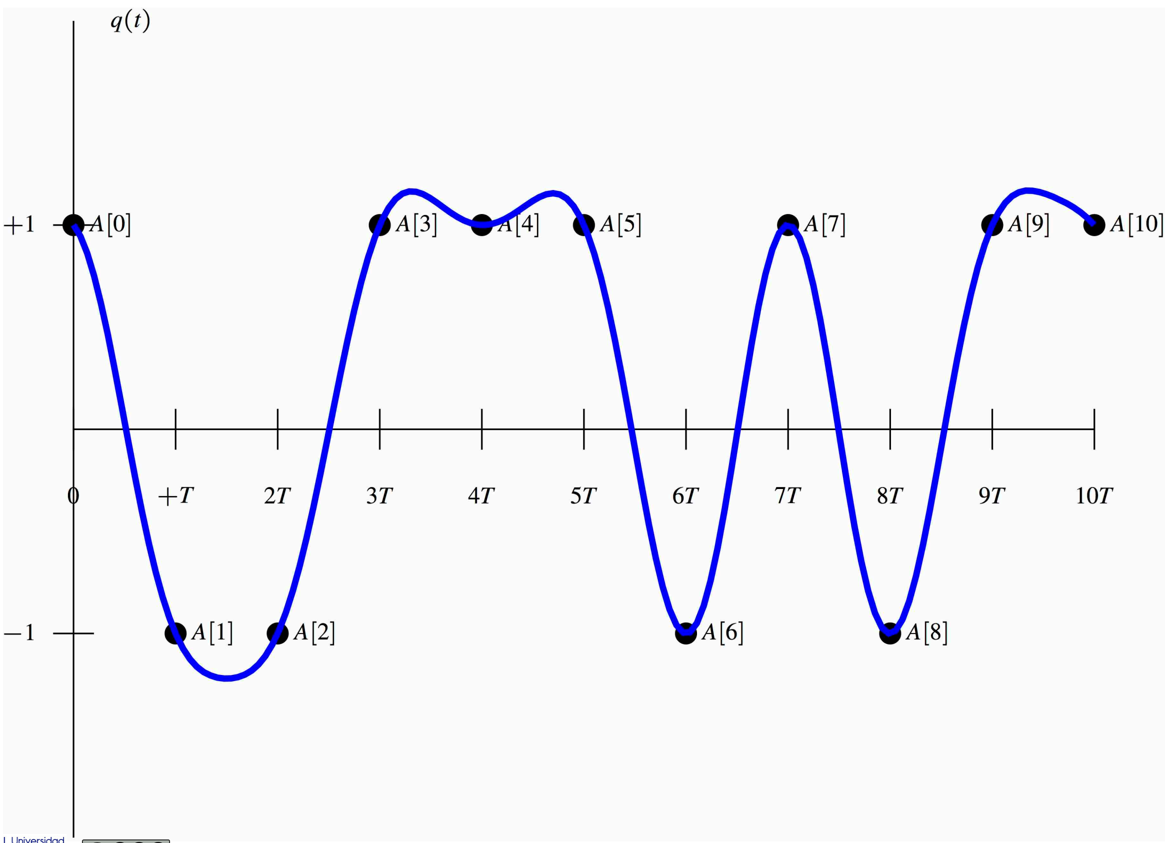
Signals with raised cosines (ideal) - $\alpha = 0.25$



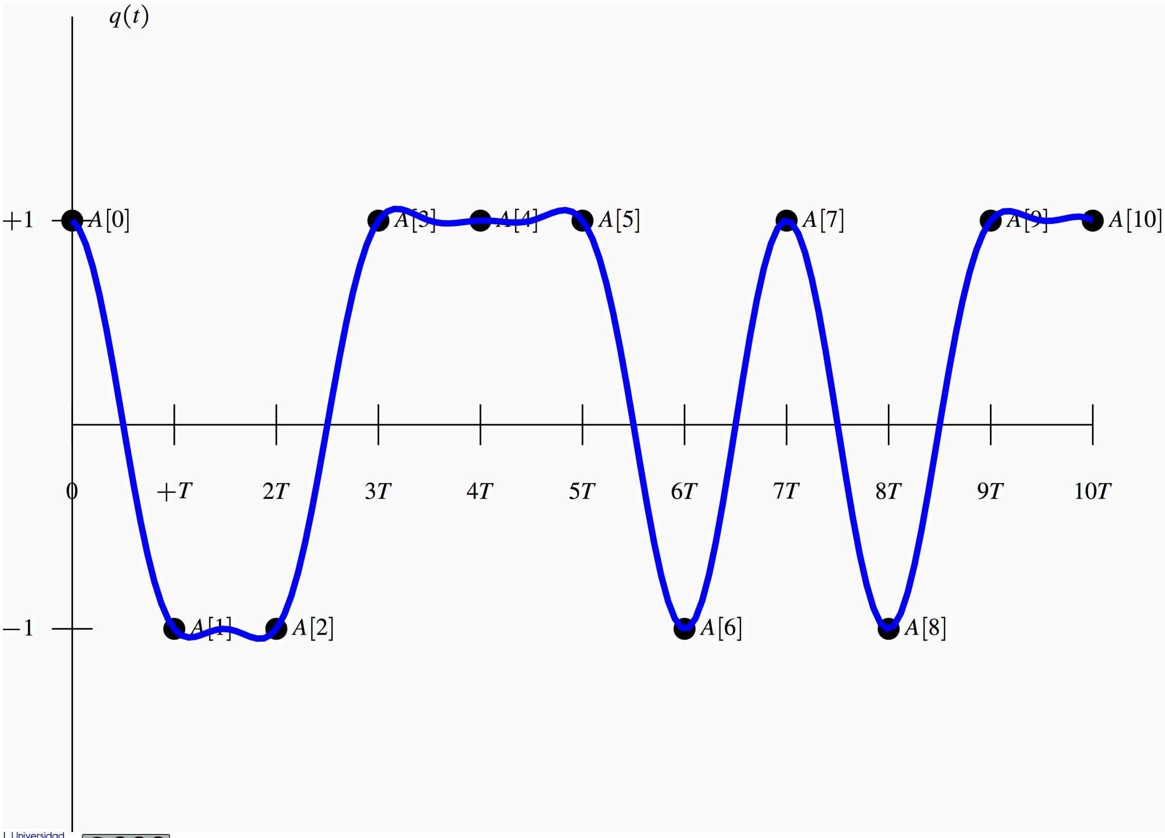
Signals with raised cosines (ideal) - $\alpha = 0.5$



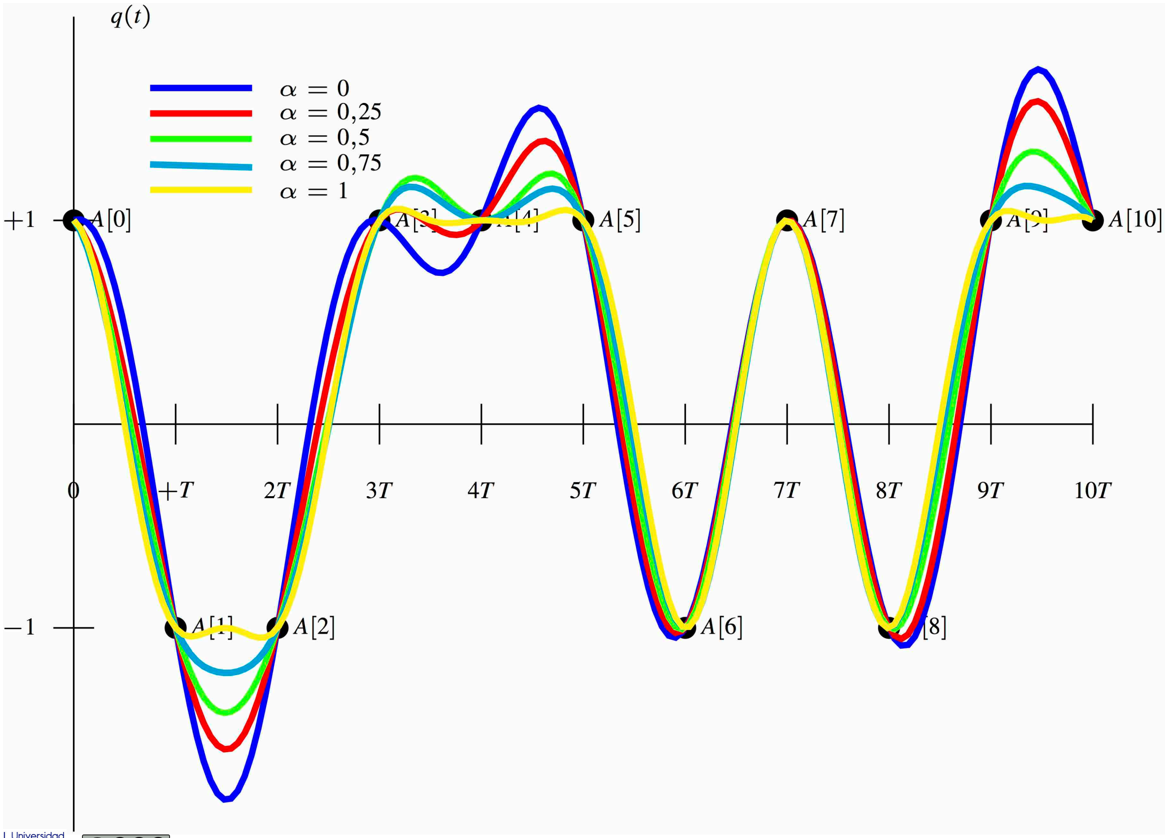
Signals with raised cosines (ideal) - $\alpha = 0.75$



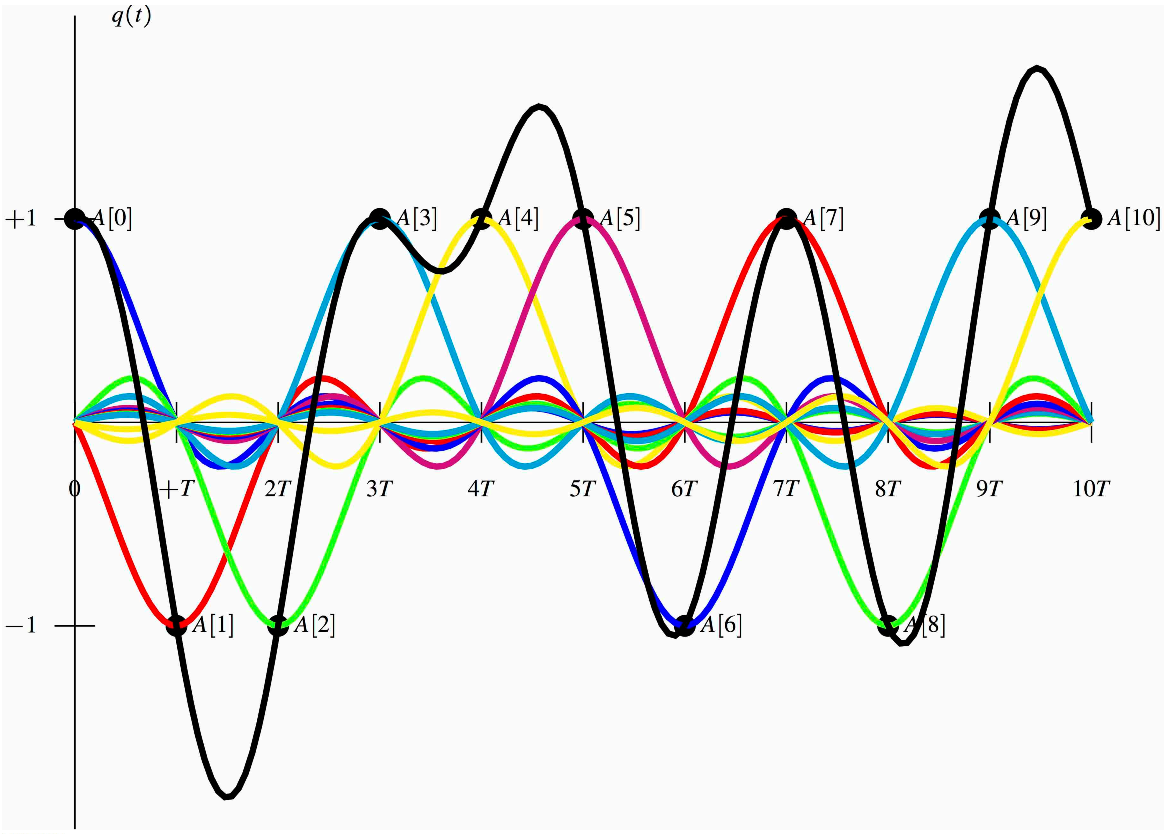
Signals with raised cosines (ideal) - $\alpha = 1$



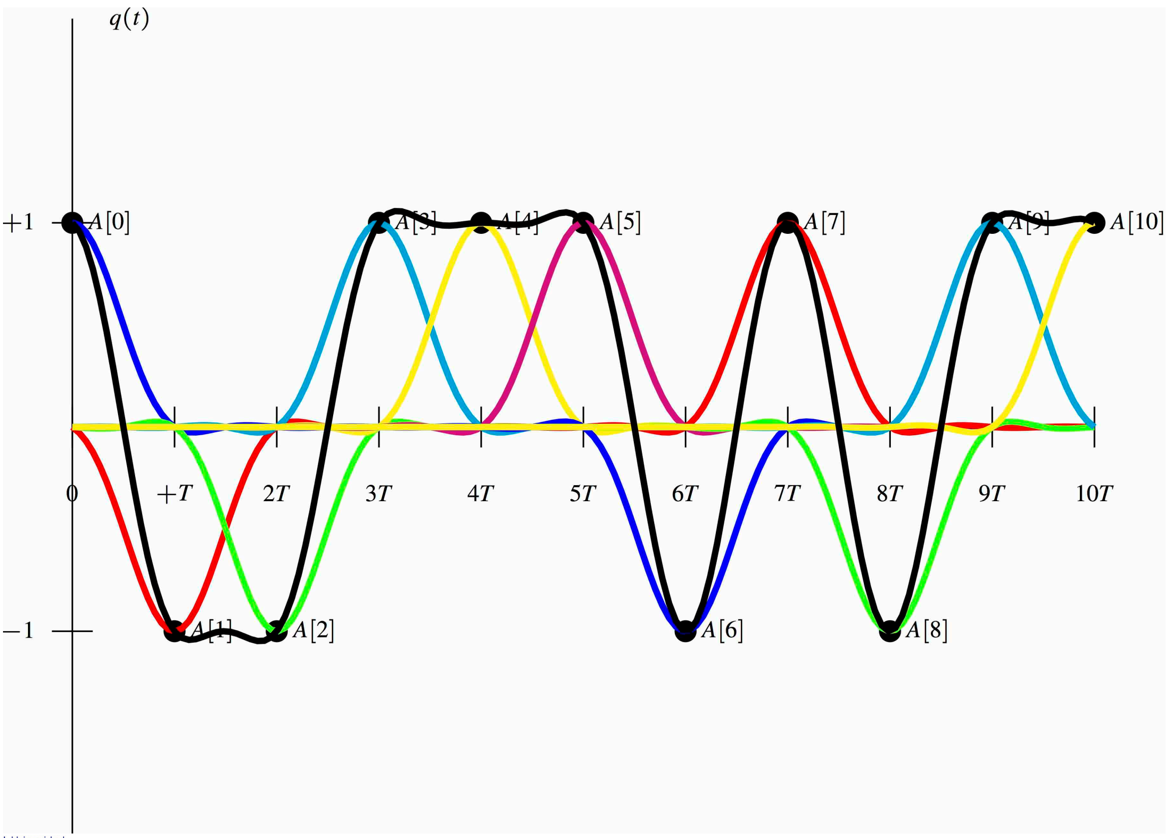
Signals with raised cosines (ideal) - Comparison



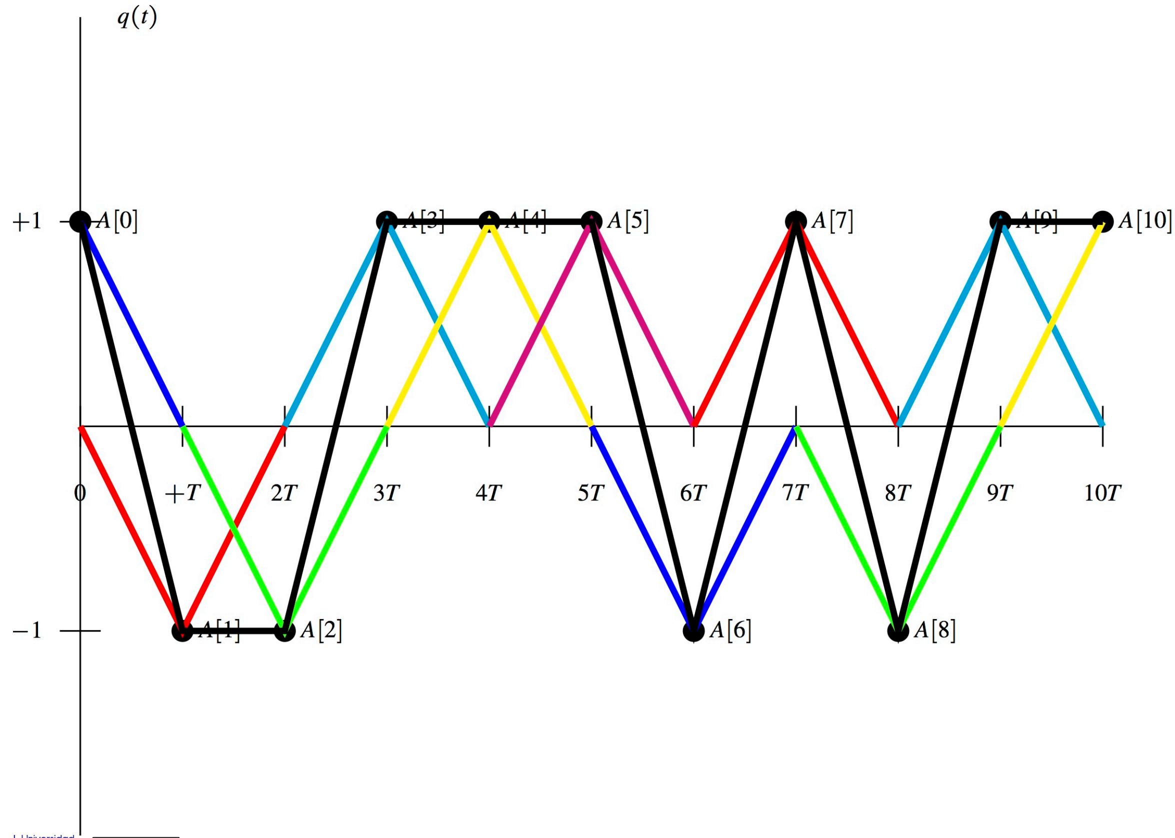
Constituents - $\alpha = 0$



Constituents - $\alpha = 1$

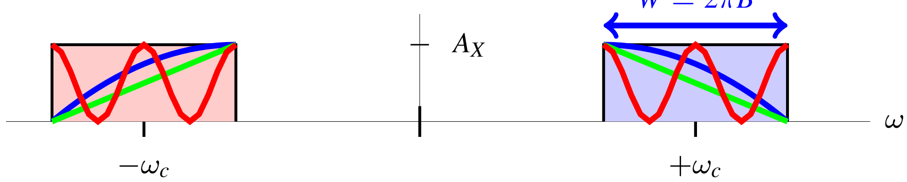


Constituents - $p(t)$: triangle



Bandpass PAM modulations

- Goal of a bandpass PAM modulation
  - To generate bandlimited modulated signals whose frequency response is bandpass
    - Central frequency ω_c rad/s (or $f_c = \frac{\omega_c}{2\pi}$ Hz)
    - Limited bandwidth W rad/s (or $B = \frac{W}{2\pi}$ Hz)
  - Appropriate signals to be transmitted through a bandpass channel



Bandpass (central frequency ω_c rad/s)

Bandpass PAM - Generation by AM modulation

- Simplest approach
- A baseband PAM is initially generated

$$s(t) = \sum_n A[n]\, g(t - nT)$$

- Then, this baseband PAM signal is modulated with an amplitude modulation. Several options are available
  - Conventional AM (double sided band with carrier)
  - Double sided band PAM (DSB-PAM)
  - Single sided band PAM (SSB-PAM)
    - ⋆ Lower sided band
    - ⋆ Upper sided band
  - Vestigial sided band PAM (VSB-PAM)
    - ⋆ Lower sided band
    - ⋆ Upper sided band

Drawbacks of using a AM modulation

- Conventional AM and double side band AM (DSB-AM)
  - Spectral efficiency is reduced to the half (bandwidth is doubled)
- Single side band AM (SSB-AM)
  - Ideal analog side band filters are required
    - ⋆ Real filters introduce a distortion
- Vestigial side band AM (VSB-AM)
  - Analog vestigial band filters are required
    - ⋆ Strong implementation constraints
  - Spectral efficiency is reduced (slightly)
    - ⋆ The bandwidth is increased by the size of the vestige

Modulation by using quadrature carriers

- Two sequences of symbols (not necessarily independent) are simultaneously transmitted (rate $R_s = \frac{1}{T}$ in both cases)

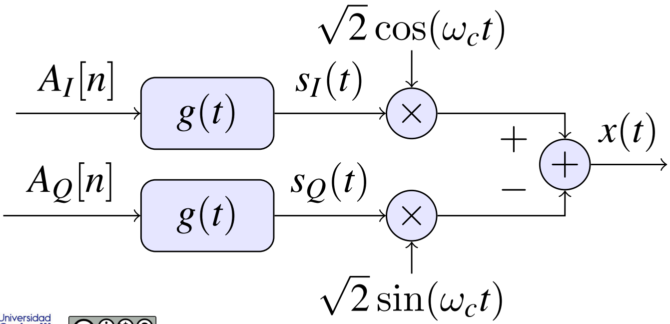
$$A_I[n] \qquad A_Q[n]$$

- Two baseband PAM signals are generated using $g(t)$

$$s_I(t) = \sum_n A_I[n]\ g(t - nT) \qquad s_Q(t) = \sum_n A_Q[n]\ g(t - nT)$$

$s_I(t)$: in-phase component, $s_Q(t)$: quadrature component

- Generation of the bandpass signal, $x(t)$, from $s_I(t)$ and $s_Q(t)$



$$x(t) = \sqrt{2}\ s_I(t)\ \cos(\omega_c t) - \sqrt{2}\ s_Q(t)\ \sin(\omega_c t)$$

Complex notation for bandpass PAM

- Complex sequence of symbols

$$A[n] = A_I[n] + jA_Q[n]$$

  - $A_I[n] = \mathcal{R}e\{A[n]\}, \quad A_Q[n] = \mathcal{I}m\{A[n]\}$

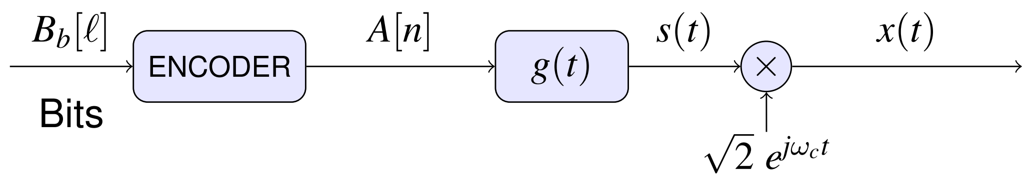
- Complex baseband signal, $s(t)$:

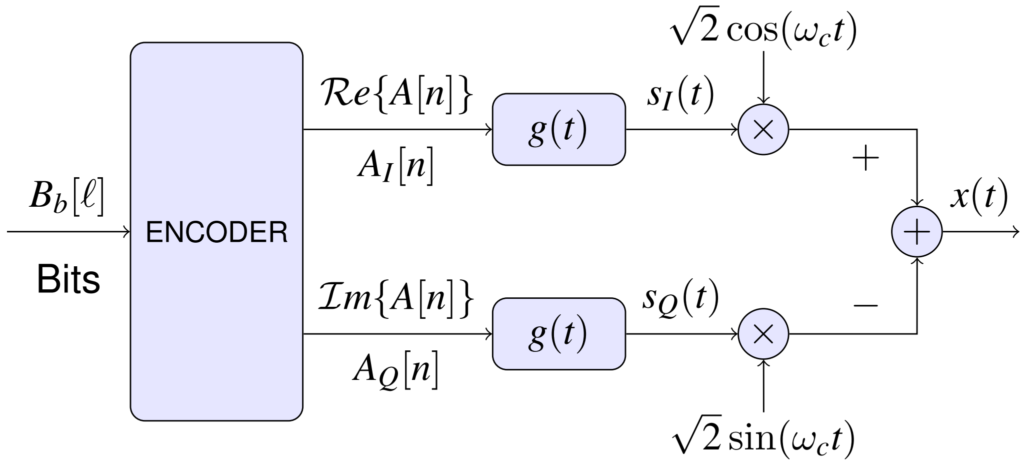
$$s(t) = s_I(t) + js_Q(t) = \sum_n A[n]\ g(t - nT)$$

- The bandpass PAM signal can be written as follows

$$x(t) = \sqrt{2}\ \mathcal{R}e\left\{s(t)\ e^{j\omega_c t}\right\} = \sqrt{2}\ \mathcal{R}e\left\{\sum_n A[n]\ g(t - nT)\ e^{j\omega_c t}\right\}$$

Bandpass PAM modulator





Relationship with a 2D signal space

- Signal in a 2D signal space can be written as

$$x(t) = \sum_n A_0[n]\ \phi_0(t-nT) + \sum_n A_1[n]\ \phi_1(t-nT)$$

  - $\phi_0(t)$ and $\phi_1(t)$ are orthonormal signals
- In this case, this only happens if

$$\omega_c = \frac{2\pi}{T} \times k, \quad \text{with } k \in \mathbb{Z}$$

In this case

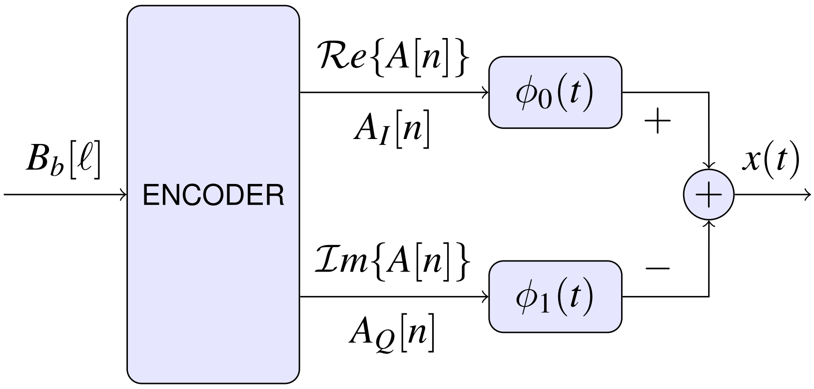
$$A_0[n] = A_I[n],\ A_Q[n] = A_1[n]$$

$$\phi_0(t) = g(t)\ \cos(\omega_c t), \qquad \phi_1(t) = -g(t)\ \sin(\omega_c t)$$

$$\phi_0(t-nT) = g(t-nT)\ \cos(\omega_c(t-nT)) = g(t-nT)\ \cos(\omega_c t)$$

$$\phi_1(t-nT) = -g(t-nT)\ \sin(\omega_c(t-nT)) = -g(t-nT)\ \sin(\omega_c t)$$

Modulator 2D signal space



Bandpass PAM constellations

- 2D plotting of possible combinations for $A_I[n]$ and $A_Q[n]$
- Typical constellations
  - QAM (Quadrature Amplitude Modulation) constellations
    - ⋆ $M = 2^m$ symbols, with m even
    - ⋆ Symbols arranged in a full squared lattice ($2^{m/2} \times 2^{m/2}$ levels)
      - Both $A_I[n]$ and $A_Q[n]$ use baseband PAM constellations
      - Independent symbol mapping, bit assignment, and definition of decision regions are possible

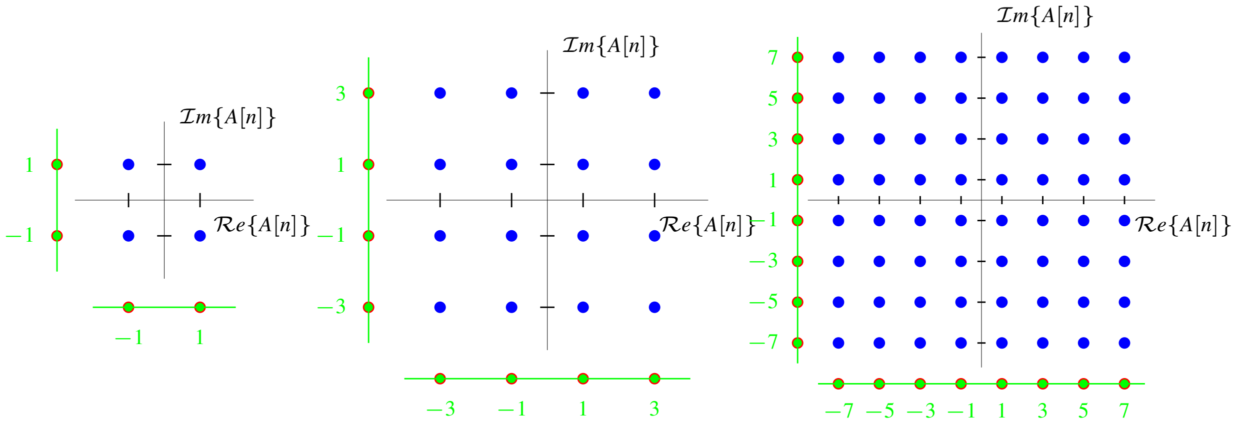
$$E_s = \frac{2(M-1)}{3} \text{ J}$$

  - Crossed QAM constellations
    - ⋆ $M = 2^m$ symbols, with m odd
    - ⋆ Symbols arranged in a non-full squared lattice
      - Independent symbol mapping, bit assignment, and definition of decision regions are not possible

$$E_s = \frac{2}{3}\left(\frac{31}{32}M - 1\right) \text{ J}$$

  - PSK (Phase Shift Keying) constellations
    - ⋆ Symbols are drawn as points in a circle (radius $\sqrt{E_s}$)
      - Constant energy for all symbols $E_s = |A[n]|^2$

QAM constellations

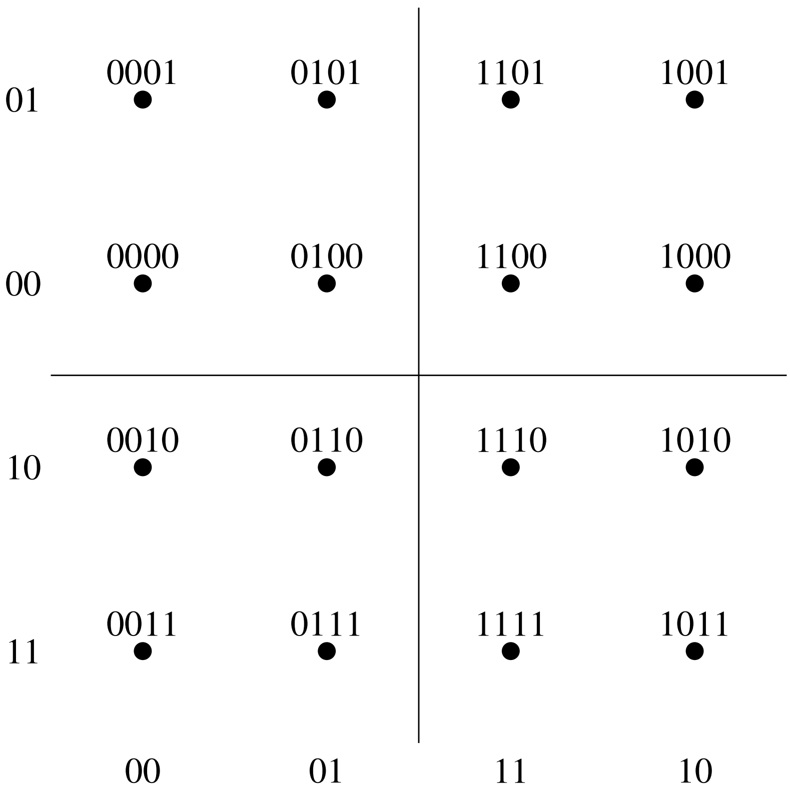


4-QAM (2-PAM)×(2-PAM)

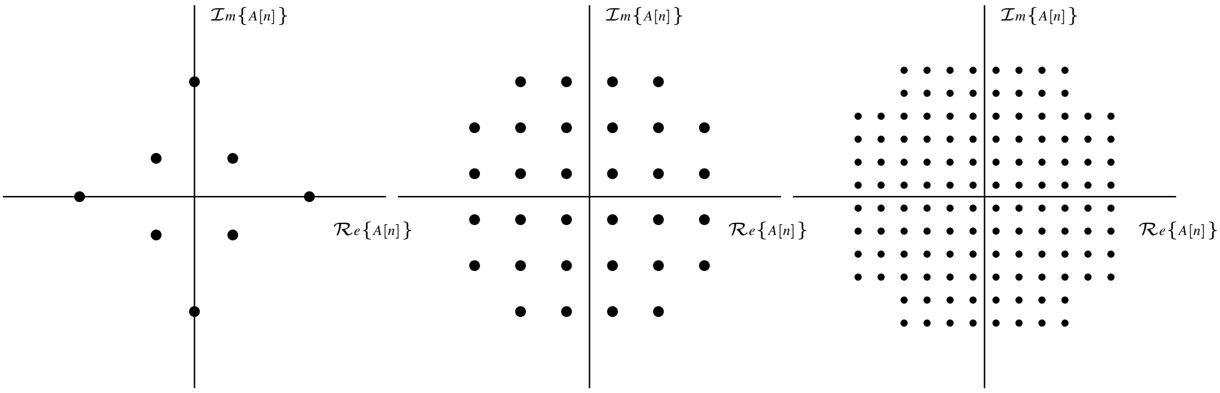
16-QAM (4-PAM)×(4-PAM)

64-QAM (8-PAM)×(8-PAM)

Gray coding for QAM



Crossed QAM constellations



Constellations: 8-QAM, 32-QAM y 128-QAM

Phase shift keying (PSK) modulation

- PSK constellation

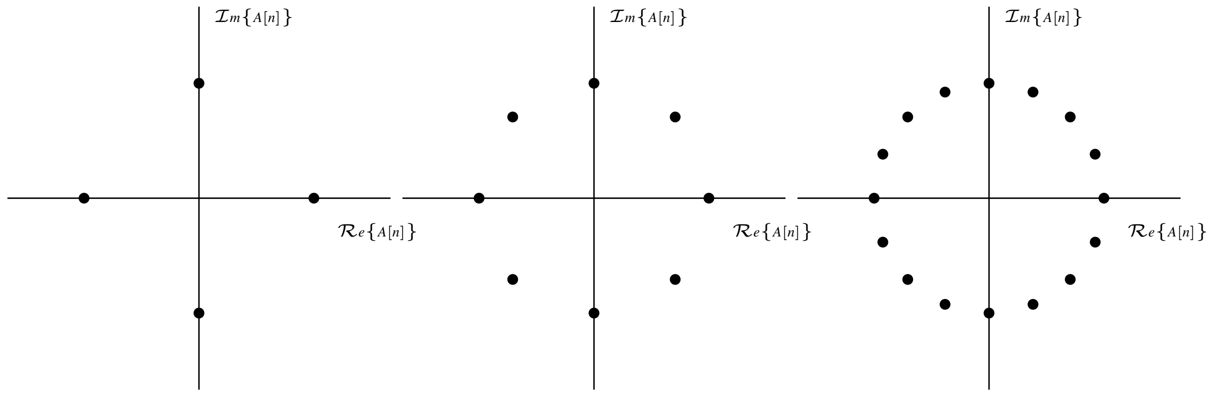
$$A[n] = \sqrt{E_s}\, e^{j\varphi[n]}$$

  - Constant modulus
  - Information is conveyed in the symbol phase
- Waveform for PSK modulations

$$x(t) = \sqrt{2E_s} \mathcal{R}e\left\{ \sum_n g(t - nT)\, e^{j(\omega_c t + \varphi[n])} \right\}$$
$$= \sqrt{2E_s} \sum_n g(t - nT) \cos(\omega_c t + \varphi[n])$$

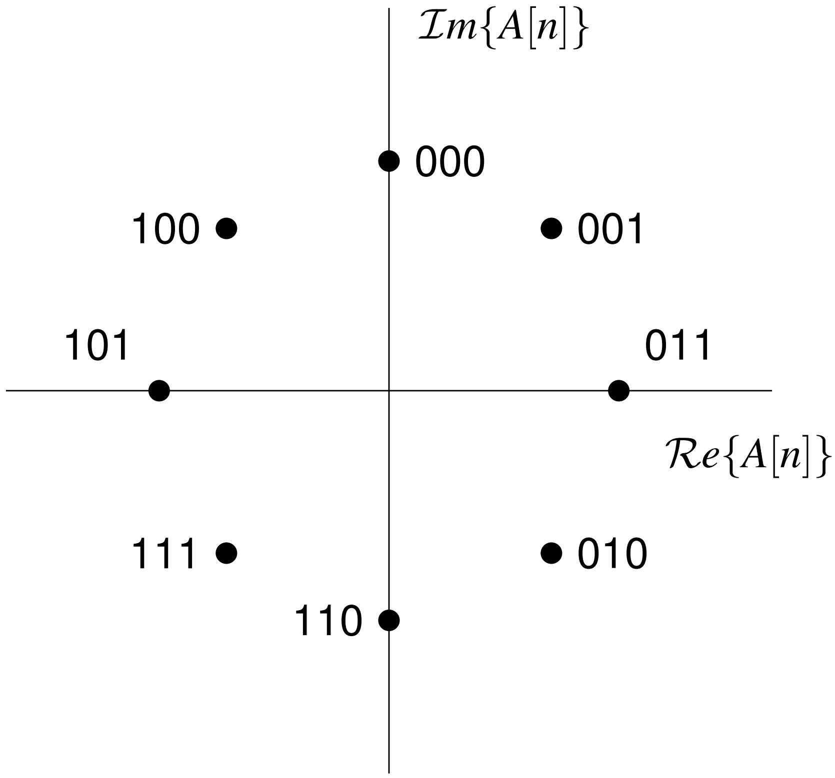
  - Phase shifts in transitions from symbol to symbol

PSK constellations

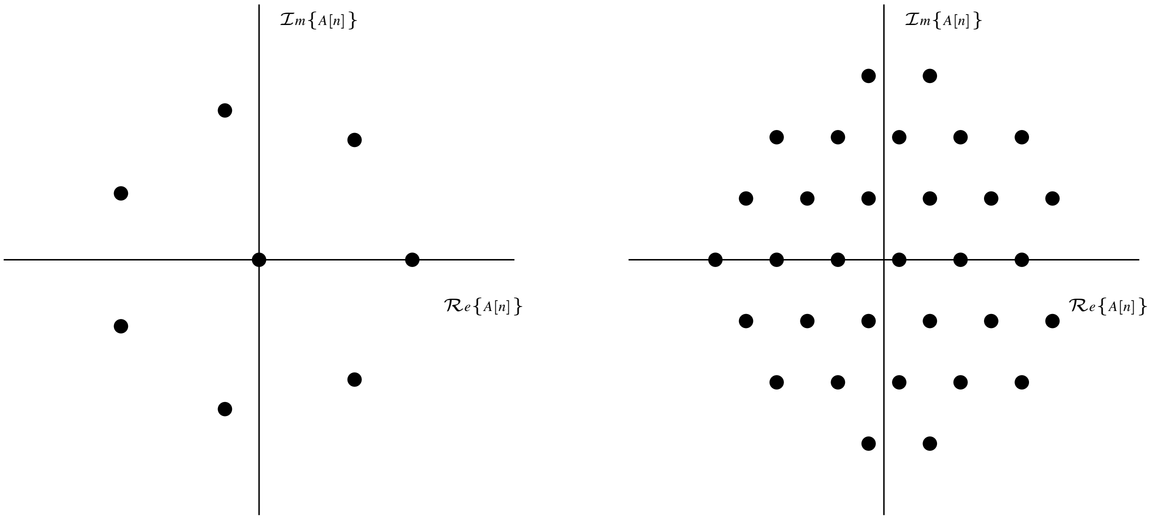


Constellations: 4-PSK (QPSK), 8-PSK y 16-PSK

Gray coding for PSK



Other constellations



Constellations 1-7-AM-PM y 32-hexagonal

Spectrum of a bandpass PAM

- Condition for cyclostationarity of signal $x(t)$:

$$E\left[A[k+m]\,A[k]\right] = 0, \text{ for all } k, m, m \neq 0$$

  - Conditions for QAM constellations
    - Symbol sequences $A_I[n]$ and $A_Q[n]$ are mutually independent
    - Autocorrelation functions of $A_I[n]$ and $A_Q[n]$ are identical
  - Conditions for PSK constellations
    - Samples of $\varphi[n]$ are independent

- Under cyclostationarity the power spectral density function is

$$S_X(j\omega) = \frac{1}{2}\left[S_S(j\omega - j\omega_c) + S_S^*(-(j\omega + j\omega_c))\right]$$

$$S_S(j\omega) = \frac{1}{T}\,S_A\left(e^{j\omega T}\right)\,|G(j\omega)|^2$$

REMARK: $A[n]$ is a complex sequence in bandpass PAM

Spectrum of a bandpass PAM (II)

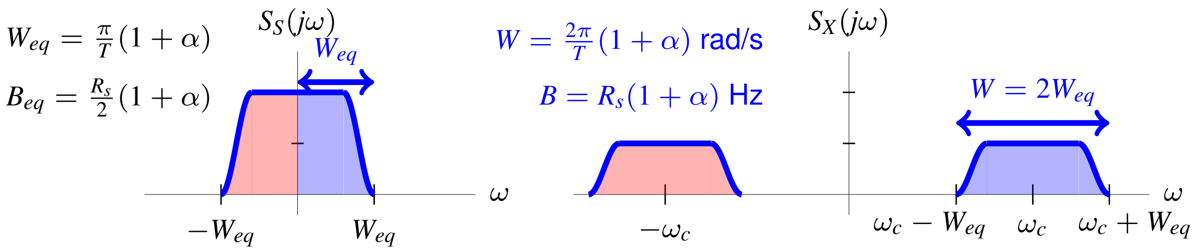
- For white sequences of symbols: $S_A\left(e^{j\omega}\right) = E_s$

$$S_S(j\omega) = \frac{E_s}{T}\,|G(j\omega)|^2$$

  - $G(j\omega)$ is responsible of the shape of the spectrum
    - ★ $S_S(j\omega)$ is real and symmetric

$$S_X(j\omega) = \frac{1}{2}\frac{E_s}{T}\left[|G(j\omega - j\omega_c)|^2 + |G(j\omega + j\omega_c)|^2\right]$$

- Example using pulses of raised cosine family



★ Bandpass bandwidth W is double of equivalent baseband bandwidth W_{eq}

★Spectral efficiency is the same: now two sequences are simultaneously transmitted

Transmitted power

- The mean transmitted power is

$$P_X = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(j\omega)\,d\omega$$

- If symbol sequence $A[n]$ is white

$$S_A\left(e^{j\omega}\right) = E_s, \qquad S_S(j\omega) = \frac{E_s}{T}\,|G(j\omega)|^2$$

  - Power for a white symbol sequence

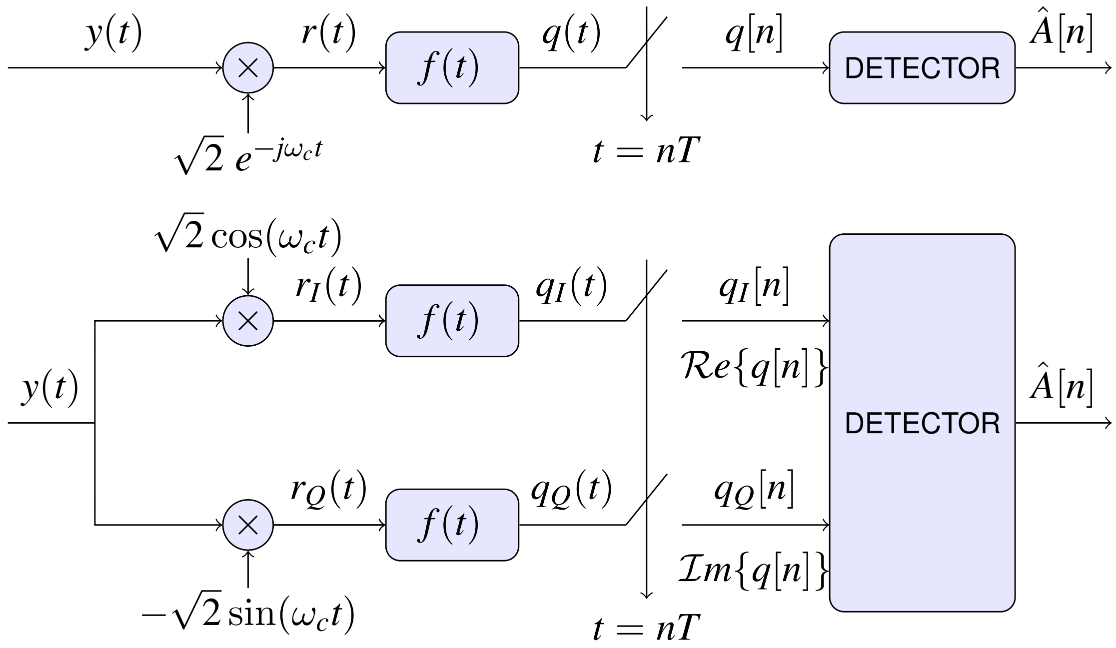
$$P_X = \frac{E_s}{T}\frac{1}{2\pi}\int_{-\infty}^{\infty}|G(j\omega)|^2\,d\omega = \frac{E_s}{T}\times\mathcal{E}\{g(t)\}$$

    - ★ For normalized pulses (with unitary energy)

$$P_X = \frac{E_s}{T} = E_s \times R_s \text{ Watts}$$

Demodulator for bandpass PAM

- Demodulation and a baseband filter structure can be used
  - Complex notation and implementation by components can be seen in the following pictures



Equivalent alternative demodulator

- Signal at the input of the sampler (using complex notation)

$$q(t) = \left(y(t)\, e^{-j\omega_c t}\right) * \left(\sqrt{2}\, f(t)\right)$$

- Expression for the convolution

$$q(t) = \sqrt{2} \int_{-\infty}^{\infty} f(\tau)\, y(t-\tau)\, e^{j\omega_c \tau}\, e^{-j\omega_c t}\, d\tau$$

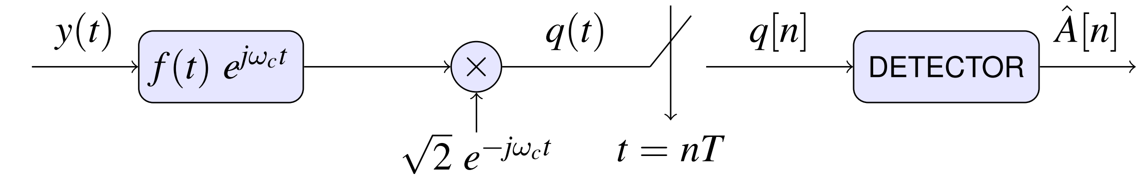
- Rearranging terms, an equivalent demodulation scheme is obtained

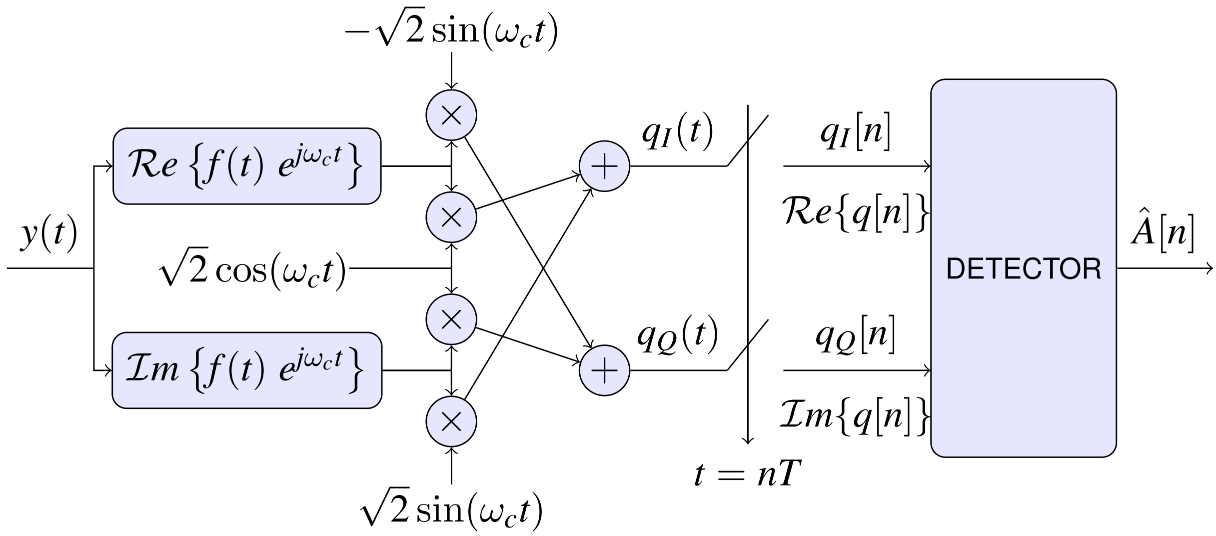
$$q(t) = e^{-j\omega_c t} \int_{-\infty}^{\infty} \sqrt{2}\, f(\tau)\, e^{j\omega_c \tau}\, y(t-\tau)\, d\tau$$

$$q(t) = e^{-j\omega_c t} \left(y(t) * \left(\sqrt{2}\, f(t)\, e^{j\omega_c t}\right)\right)$$

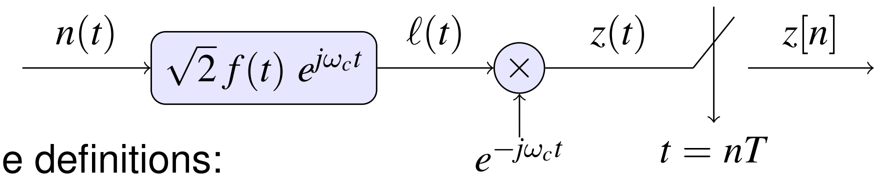
Bandpass filtering and then demodulation

Equivalent alternative demodulator (II)





Noise characteristics at the receiver



- Some definitions:

$$f_c(t) = \sqrt{2}\,f(t)\ e^{j\omega_c t},\ \ F_c(j\omega) = \sqrt{2}\ F(j\omega - j\omega_c)$$

- Properties:
  1. $z(t)$ is strict sense stationary only if $\ell(t)$ es circularly symmetric
     NOTE: A complex process $X(t)$ is circularly symmetric if real and imaginary parts, $X_r(t)$ and $X_i(t)$, are jointly stationary, and their correlations satisfy
     $$R_{X_r}(\tau) = R_{X_i}(\tau),\ \ R_{X_r,X_i}(\tau) = -R_{X_i,X_r}(\tau)$$
  2. $\ell(t)$ is circularly symmetric if ω_c is higher than bandwidth of filter $f_c(t)$ (narrow band system)
     $$S_\ell(j\omega) = 2\ S_n(j\omega)\ \ |F(j\omega - j\omega_c)|^2$$

Noise signal $z(t)$ at the receiver

$$z(t) = z_I(t) + j\, z_Q(t)$$

- $z(t)$ is circularly symmetric and its power spectral density is

$$S_z(j\omega) = 2\ S_n(j\omega + j\omega_c)\ \left|F(j\omega)\right|^2$$

  - If the process is symmetric, its real and imaginary parts, $z_I(t)$ and $z_Q(t)$, have the same variance and are independent for any time instant t
  - In general, $z_I(t_1)$ and $z_Q(t_2)$, for $t_1 \neq t_2$ are not independent
  - If spectrum is hermitic, $S_z(j\omega) = S_z^*(-j\omega)$, $z_I(t_1)$ and $z_Q(t_2)$, for $t_1 \neq t_2$ are also independent
    - ★ If $n(t)$ is white, this is fulfilled when $f(t)$ is real

Discrete noise sequence $z[n]$ at the receiver

$$z[n] = z_I[n] + j\, z_Q[n]$$

- $z[n]$ is circularly symmetric

$$S_z\left(e^{j\omega}\right) = \frac{2}{T}\sum_k S_n\left(j\frac{\omega}{T} + j\frac{\omega_c}{T} - j\frac{2\pi k}{T}\right)\ \left|F\left(j\frac{\omega}{T} - j\frac{2\pi k}{T}\right)\right|^2$$

For white noise $n(t)$

$$S_n(j\omega) = \frac{N_0}{2} \Rightarrow S_z\left(e^{j\omega}\right) = N_0\ \frac{1}{T}\sum_k \left|F\left(j\frac{\omega}{T} - j\frac{2\pi k}{T}\right)\right|^2$$

Sampled noise $z[n]$ can be white

  - This happens if the ambiguity function of $f(t)$, $r_f(t) = f(t) * f^*(-t)$, satisfies the conditions of Nyquist ISI criterion at symbol rate

$$S_z(j\omega) = N_0 \times \mathcal{E}\left\{f(t)\right\}$$

    - ★ $z_I[n]$ and $z_Q[n]$ are independent for any instant n
    - ★ $z_I[n_1]$ and $z_Q[n_2]$, for $n_1 \neq n_2$, are independent

$$S_{z_I}(j\omega) = S_{z_Q}(j\omega) = \frac{N_0}{2} \times \mathcal{E}\left\{f(t)\right\}$$

Variance and distribution of $z[n]$

- The variance of complex discrete noise is

$$\sigma_z^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_z\left(e^{j\omega}\right)\, d\omega$$

- In noise $n(t)$ is white, with $S_n(j\omega) = N_0/2$ W/Hz, and if $r_f(t)$ is normalized and satisfies the Nyquist ISI criterion at R_s (T)

$$\boxed{\sigma_z^2 = N_0 \qquad \left(\sigma_{z_I}^2 = \sigma_{z_Q}^2 = \frac{N_0}{2}\right)}$$

REMARK: remember that $z[n] = z_I[n] + j\, z_Q[n]$

- If noise is circularly symmetric
  - Real and imaginary parts ($z_I[n]$ and $z_Q[n]$) are independent and both have variance $N_0/2$
  - Probability density function of noise level is

$$f_Z(z) = \frac{1}{\pi N_0} e^{-\frac{|z|^2}{N_0}}$$

NOTE: If receiver filter is not normalized, noise variance is multiplied by $\mathcal{E}\{f(t)\}$

Equivalent discrete channel

- Sampled signal at the output of the matched filter

$$q[n] = q(t)\big|_{t=nT} = q(nT), \text{ with } q(t) = \sum_n A[n]\, p(t-nT) + z(t)$$

- Bandpass equivalent discrete channel:

$$\boxed{p[n] = p(t)\big|_{t=nT} = p(nT)} \quad \boxed{q[n] = A[n] * p[n] + z[n]}$$

- Definition of the complex equivalent baseband channel, $h_{eq}(t)$

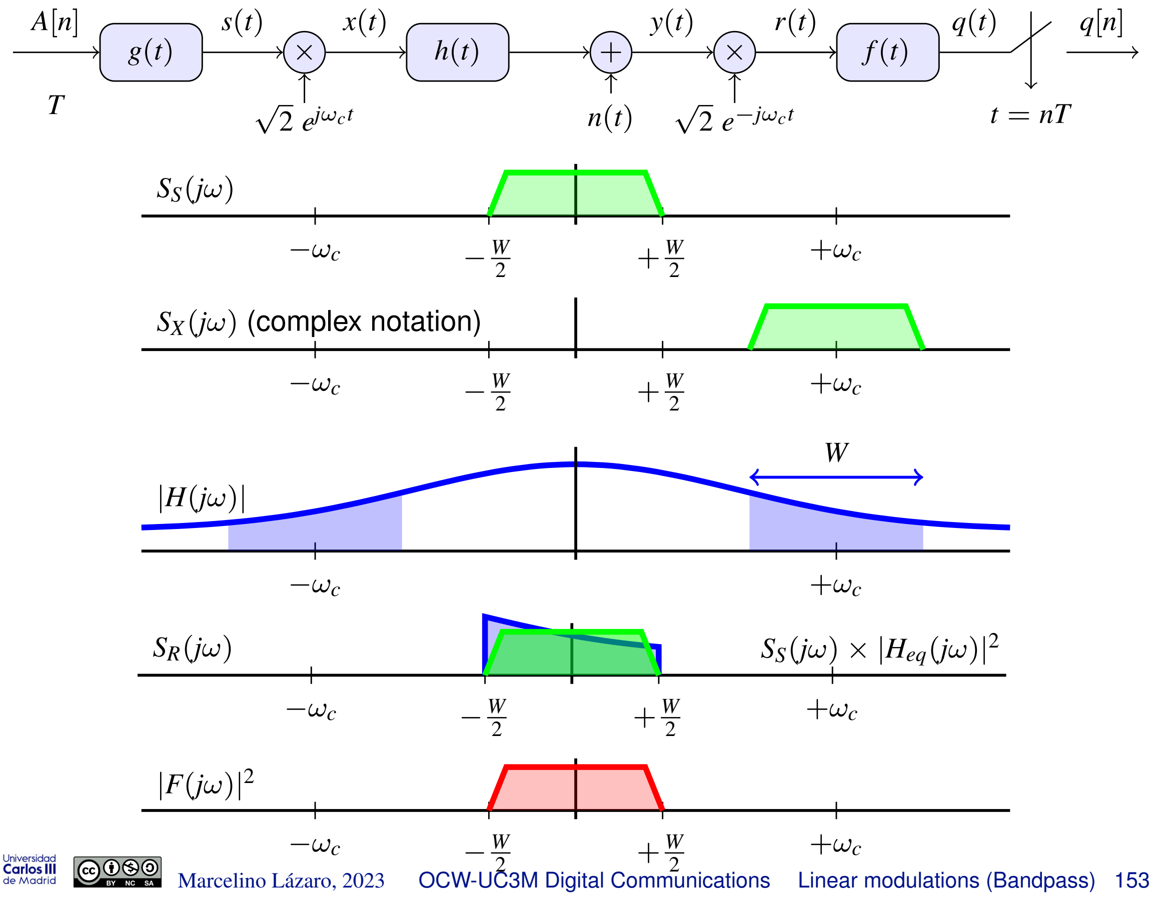
$$\boxed{h_{eq}(t) = e^{-j\omega_c t}\, h(t) \quad \overset{\mathcal{FT}}{\leftrightarrow} \quad H_{eq}(j\omega) = H(j\omega + j\omega_c)}$$

The behavior of the channel around central frequency ω_c is shifted down to baseband

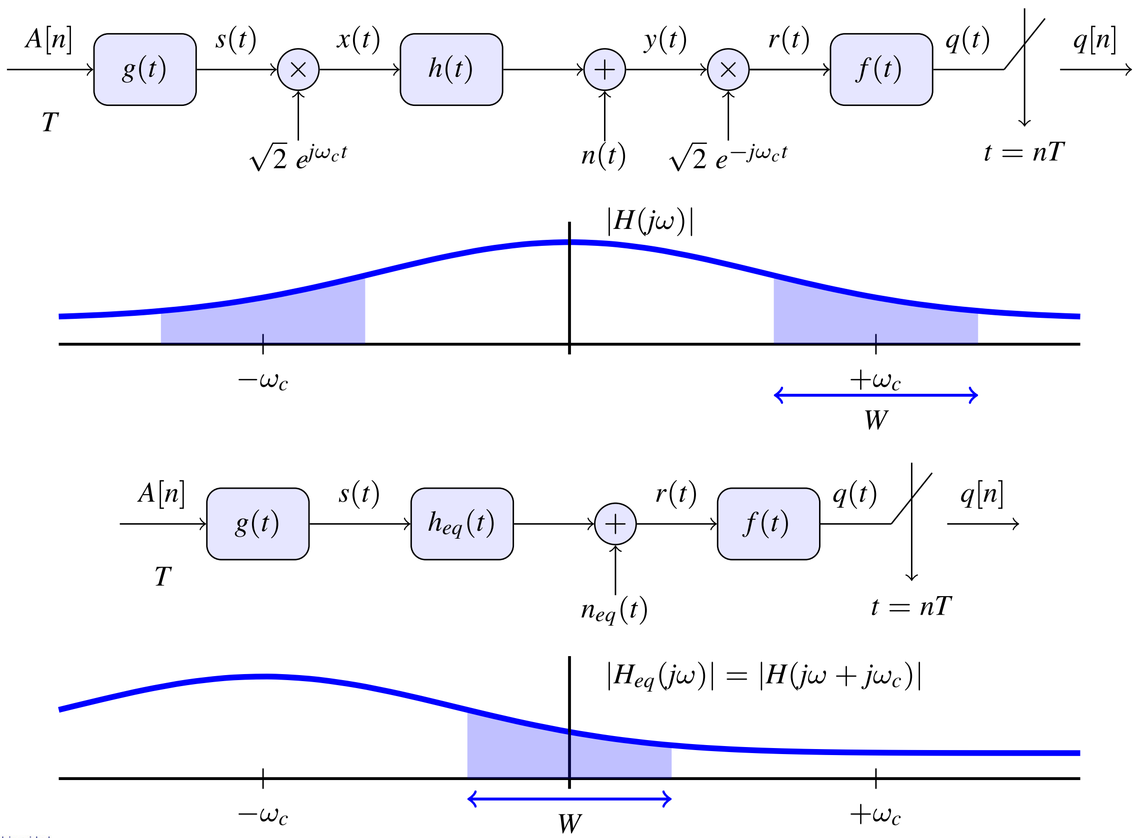
- Joint transmiter-channel-receiver response

$$\boxed{p(t) = g(t) * h_{eq}(t) * f(t) \quad \overset{\mathcal{FT}}{\leftrightarrow} \quad P(j\omega) = G(j\omega)\, H_{eq}(j\omega)\, F(j\omega)}$$

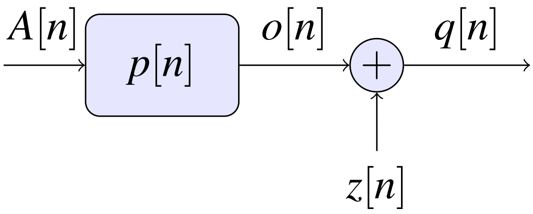
Complex equivalent baseband channel



Complex equivalent baseband channel (II)

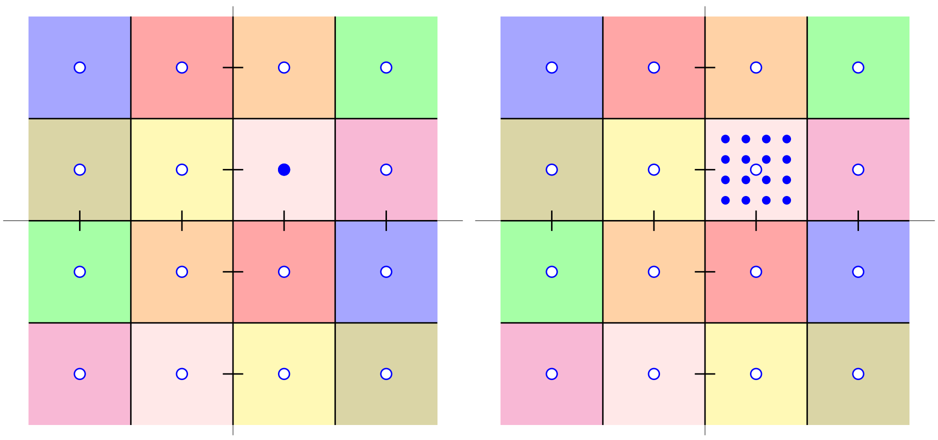


Equivalent discrete channels - baseband and bandpass PAM



- Identification of baseband and bandpass PAM
  - Symbols $A[n]$
  - Equivalent discrete channel $p[n]$
  - Discrete noise $z[n]$
    - Are real in baseband PAM
    - Are complex in bandpass PAM

ISI: Extended constellation

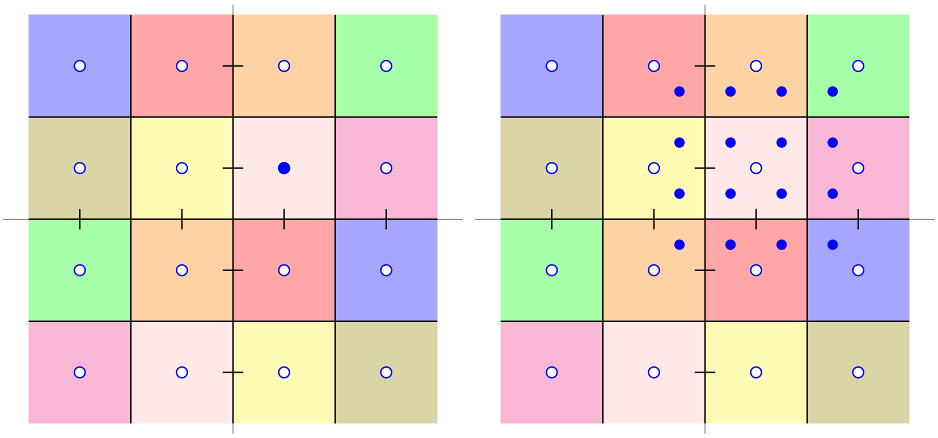


- Example of ISI (memory fo $p[n]$, $K_p = 1$)

$$p[n] = \delta[n] + a\,\delta[n-1], \qquad o[n] = A[n] + a\,A[n-1]$$

  - Transmission of a symbol at $A[n]$
    - At the receiver an extended constellation is seen around this symbol : the point in each instant will depend on the value of the previous symbol (M posibilities))
    - Noise will also be introducing additional distortion

ISI: Extended constellation (II)



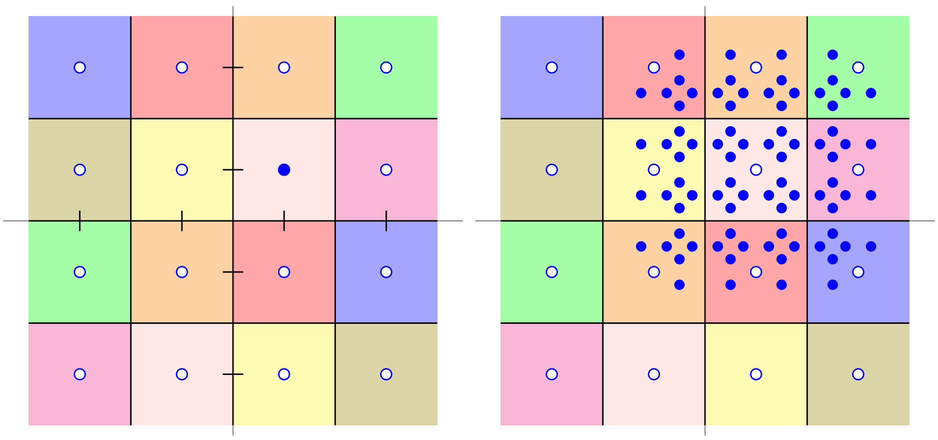
- Example of ISI (memory of $p[n]$, $K_p = 1$)

$$p[n] = \delta[n] + a\,\delta[n-1], \qquad o[n] = A[n] + a\,A[n-1]$$

  - If a increases the points of the extended constellation will separate more from it
  - If memory of $p[n]$ increases, the size of the constellation increases exponentially

M^{K_p} possible values for each symbol

ISI: Extended constellation (III)



- Example of ISI (memory of $p[n]$, $K_p = 2$)

$$p[n] = \delta[n] + a\,\delta[n-1] + b\,\delta[n-2]$$

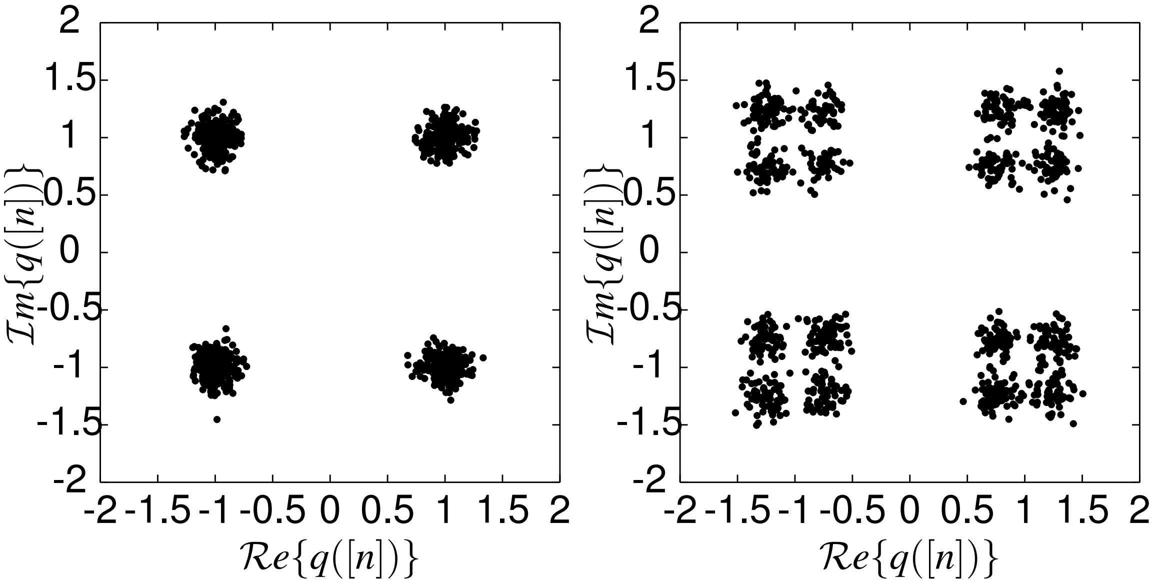
$$o[n] = A[n] + a\,A[n-1] + b\,A[n-2]$$

  - If memory of $p[n]$ increases, the size of the constellation increases exponentially

M^{K_p} posibles valores por cada símbolo

Scattering diagram

- Monitoring tool for bandpass system
  - Plotting of $\mathcal{R}e(q[n])$ versus $\mathcal{I}m(q[n])$
  - Ideally: the transmitted constellation must be plotted
  - Allows to monitor noise level, ISI level, synchronism errors



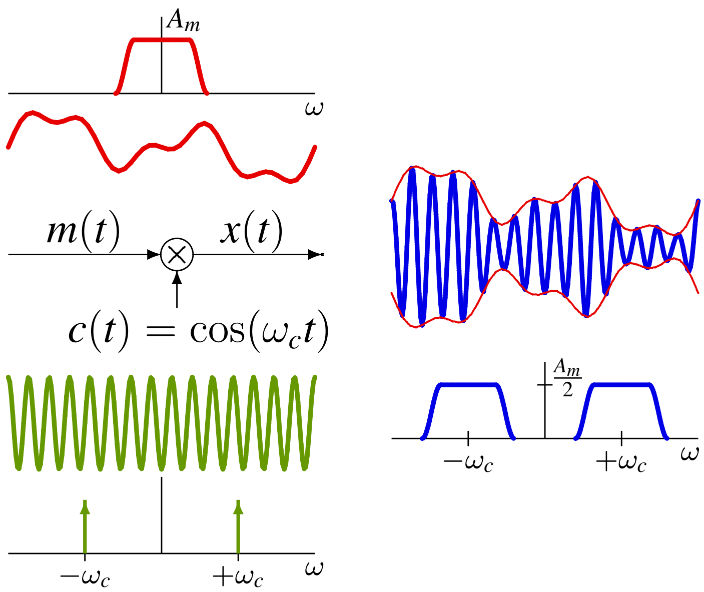
$p[n] = \delta[n]$ $\qquad$ $p[n] = \delta[n] - 0.25\delta[n-1]$

Review - Product with a sinusoid

- To multiply with a sinusoid of frequency ω_c generates, spectraly, two replicas of the signal spectrum, shifted $\pm\omega_c$

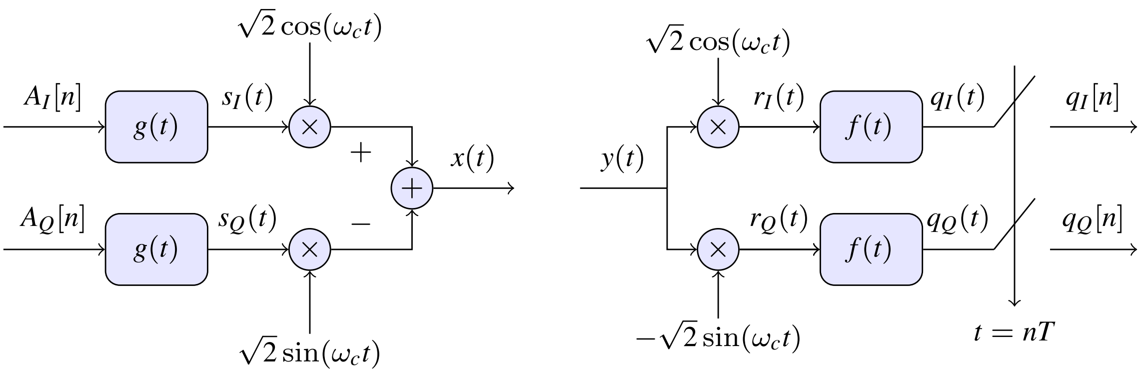
$$x(t) = m(t) \times \cos(\omega_c t) \quad \stackrel{\mathcal{FT}}{\leftrightarrow} \quad X(j\omega) = \frac{1}{2}M(j\omega - j\omega_c) + \frac{1}{2}M(j\omega + j\omega_c)$$

Power spectral density: $S_X(j\omega) = \frac{1}{4}S_M(j\omega - j\omega_c) + \frac{1}{4}S_M(j\omega + j\omega_c)$



Analysis of modulation / demodulation

- Block diagram for transmitter and receiver



- Transmitter multiplies two baseband signals by two orthogonal carriers
- Receiver demodulates each component and then filters with $f(t)$
  - Receiver filter $f(t)$ has a baseband characteristic
  - Typical set-up: root-raised cosine filter

Analysis of modulation / demodulation (II)

- Undistorted received signal (modulated signal) has the shape

$$y(t) = A\ \cos(\omega_c t) + B\ \sin(\omega_c t)$$

- At the receiver, signal processing is splitted in two components

$$q_I(t) \equiv \text{filter}\ [A\ \cos(\omega_c t) + B\ \sin(\omega_c t)] \times \cos(\omega_c t)$$

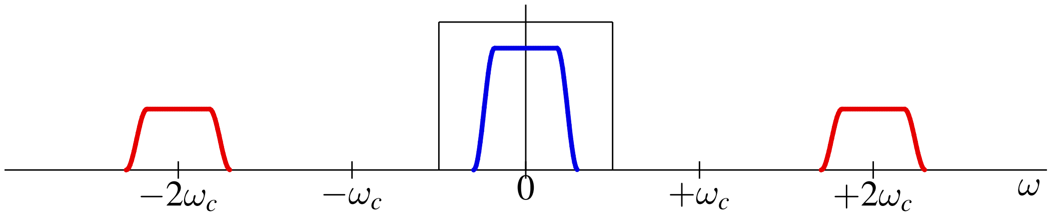
$$q_Q(t) \equiv \text{filter}\ [A\ \cos(\omega_c t) + B\ \sin(\omega_c t)] \times \sin(\omega_c t)$$

- Trigonometric identities and removing (filtering) of bandpass terms

$$X\ \cos(\omega_c t)\ \cos(\omega_c t) = \underbrace{\frac{X}{2}}_{\textit{Desired}} + \underbrace{\frac{X}{2}\ \cos(2\omega_c t)}_{\textit{Bandpass at } 2\omega_c}$$

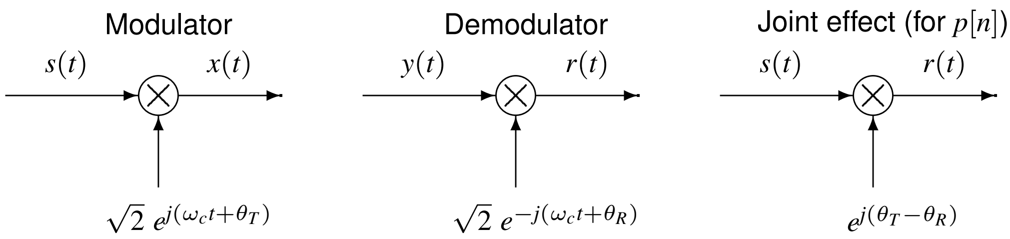
$$X\ \sin(\omega_c t)\ \cos(\omega_c t) = \underbrace{\frac{X}{2}\ \sin(2\omega_c t)}_{\textit{Bandpass at } 2\omega_c}$$

$$X\ \sin(\omega_c t)\ \sin(\omega_c t) = \underbrace{\frac{X}{2}}_{\textit{Desired}} - \underbrace{\frac{X}{2}\ \cos(2\omega_c t)}_{\textit{Bandpass at } 2\omega_c}$$



Analysis of modulation / demodulation (III)

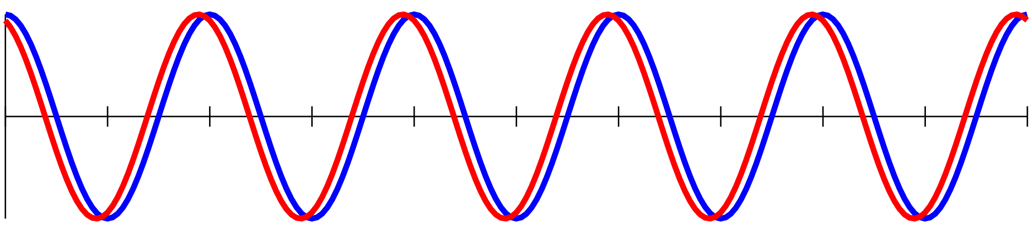
- The product of two carriers allows to recover the transmitted baseband signals
  - Products $\cos(\omega_c t) \times \cos(\omega_c t)$ or $\sin(\omega_c t) \times \sin(\omega_c t)$ introduce a $\frac{1}{2}$ factor
    - ★ Factors $\sqrt{2}$ are introduced at transmiter and receiver to compensate it
  - Complex notation fails to represent this scaling
    - ★ Mathematically: $\sqrt{2}\ e^{j\omega_c} \times \sqrt{2}\ e^{-j\omega_c} = 2$
      - 2 times the amplitude of the product of cosines or sines
    - ★ This has to be taken into account



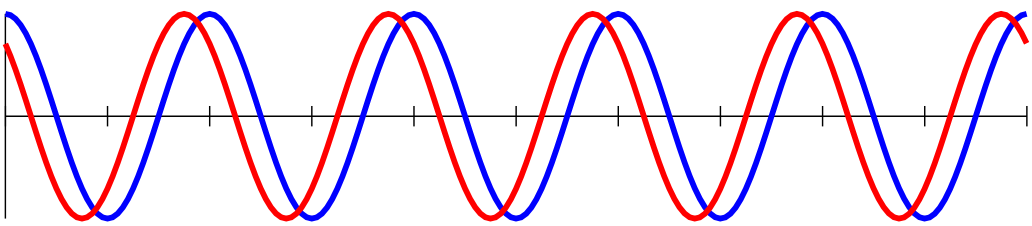
- Non-coherent receivers
  - Receiver whose demodulator has a phase that is different than phase at modulator
  - Produces a rotation in the received constellation
  - A coherent receiver needs to recover the phase of received signal (with a PLL)
    - ★ Additional cost for PLL (*Phase Locked Loop*)

Sinusoids with different phases or frequencies

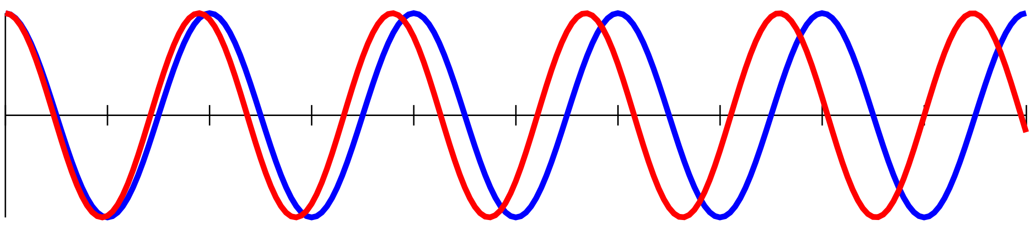
Carriers with same frequency and phase shift of 20º



Carriers with same frequency and phase shift of 45º



Carriers with different frequencies: “variable” phase shift



Binary transmission rate (R_b bits/s)

- Binary transmission rate is obtained as $R_b = m \times R_s$ bits/s
  - Symbol rate: R_s bauds (symbols/s)
  - Number of bits per symbol in the constellation: m

$$m = \log_2(M)$$

  M: number of symbols of the constellation
- Limitation in the achievable binary rate
  - Limitation in R_s: available bandwidth (B Hz)
    Using filters of the raised cosine family

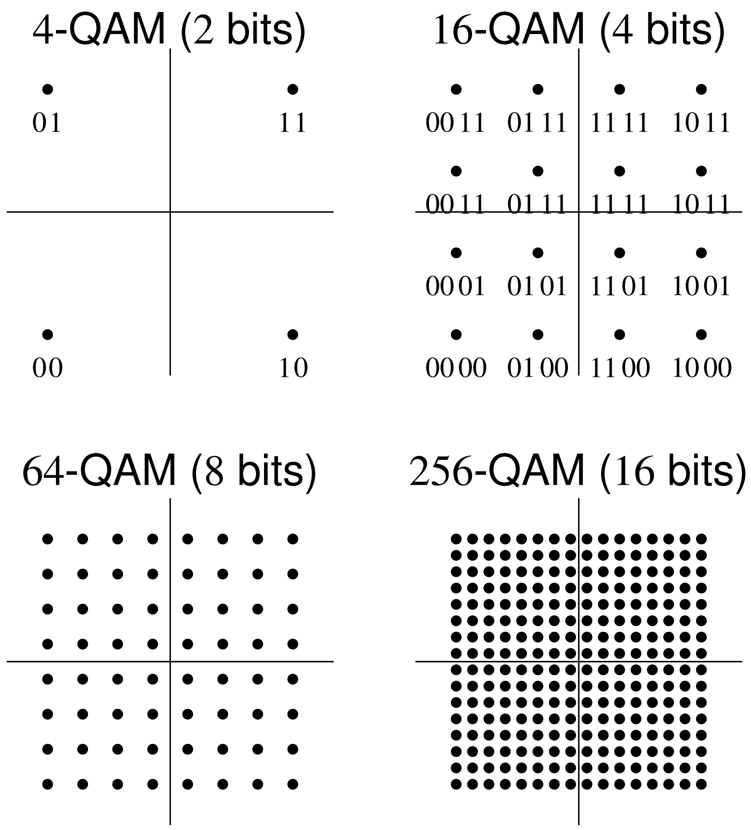
BASEBAND	BANDPASS
$R_{s\|max} = \frac{2B}{1+\alpha}$	$R_{s\|max} = \frac{B}{1+\alpha}$

  - Limitation on the number of symbols M (and therefore in m)
    - ★ Power limitation limits mean energy per symbol $E_s = E\left[|A[n]|^2\right]$
      - This limits the maximum modulus of the constellation
    - ★ Performance requirements limit the minimum distance between symbols

$$P_e \approx k\, Q\left(\frac{d_{min}}{2\sqrt{N_0/2}}\right)$$

    - ★ E_s and P_e determine a maximum constellation density

Constellation density - Example - QAM



Constellation density - Example - QAM

- Increasing constellation size (M symbols):
  - Binary rate is increased
    - Number of bits per symbols is increased $m = \log_2 M$
  - Lower performance for a given E_s
    - Distance between points of the constellation is reduced

Example for M-QAM constellations

M (symbols)	m (bits/symbol)	E_s with normalized levels ($d_{min} = 2$)	d_{min} with $E_s = 2$
4	2	2	2
16	4	10	0.8944
64	8	42	0.4364
256	16	170	0.2169