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Digital Communications

Grades in English

Chapter 2

Design of digital communications receivers in the presence of intersymbol interference (ISI)

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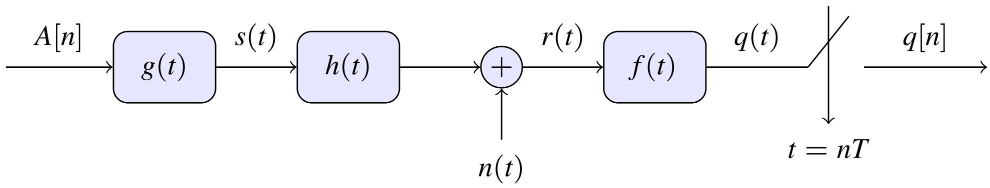
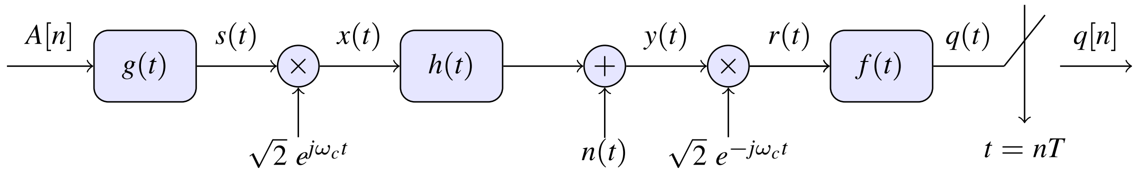
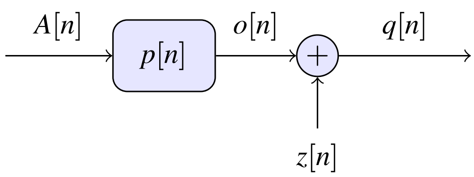
PAM digital communication system



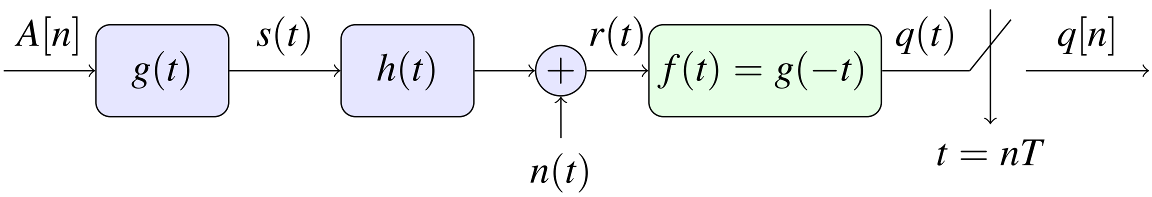


- Discrete-time equivalent model
  - Symbols $A[n]$
  - Equivalent discrete channel $p[n]$
  - Discrete-time noise $z[n]$
    - Real in baseband PAM
    - Complex in bandpass PAM



Typical set up for linear channels

- Receiver uses a matched filter $f(t) = g(-t)$ with $r_f(t) = r_g(t)$ fulfilling Nyquist condition



  - Common choice: root-raised cosine filters

$$g(t) = h_{RRC}^{\alpha,T}(t) \quad \rightarrow \quad f(t) = h_{RRC}^{\alpha,T}(t)$$

$$g(t) * f(t) = r_g(t) = r_f(t) = h_{RC}^{\alpha,T}(t)$$

- Consequences:
  - This ensures discrete filtered noise $z[n]$ is white
  - ISI is present in the system (joint response $p(t)$ then does not meet Nyquist condition)
    - Receivers can be specifically designed to deal with ISI (as it will be seen in Chapter 2)

Detection under ISI - Problem statement

- Receiver: matched filter $f(t) = g(-t)$, normalized, and $r_f(t) = r_g(t)$ satisfies Nyquist (i.e., $r_f[n] = \delta[n]$)
  - $z[n]$ white and Gaussian: $\sigma_z^2 = \begin{cases} N_0/2, & A[n] \in \mathbf{R} \\ N_0, & A[n] \in \mathcal{C} \end{cases}$
- Sequence of symbols $A[n]$: constellation with M symbols
  - Stationnary white sequence with mean energy $E_s = E\left[|A[n]|^2\right]$

$$R_A[k] = E\left[A[n+k]\, A^*[n]\right] = E_s\, \delta[n]$$

$$S_A\left(e^{j\omega}\right) = E_s$$

- Response $p(t)$ is causal and time-limited (T_p seconds)
  - $p[n]$ causal of length $K_p + 1$, $\Rightarrow K_p = \lfloor T_p/T \rfloor$
  - Observation at the output of the demodulator

$$q[n] = A[n] * p[n] + z[n] = o[n] + z[n]$$

    - ★ Noiseless output of the equivalent discrete channel

$$o[n] = A[n] * p[n] = \sum_{k=0}^{K_p} p[k]\, A[n-k]$$

$K_p \equiv$ memory of the equivalent discrete channel

Memory of the equivalent discrete channel (K_p)

$$\mathbf{p} = \left[p[0], \underbrace{p[1], p[2], \cdots, p[K_p]}_{K_p \text{ memory terms}} \right]$$

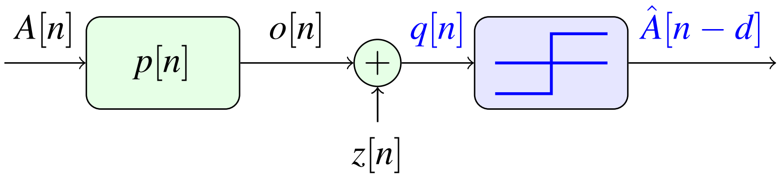
$$o[n] = \underbrace{A[n]\, p[0]}_{\text{current}} + \underbrace{A[n-1]\, p[1] + A[n-2]\, p[2] + \cdots + A[n-K_p]\, p[K_p]}_{K_p \text{ previous}}$$

- Symbols with influence in $o[n]$ at instant n:
  - The symbol at the same instant n: $\boxed{A[n]}$
  - The K_p previous symbols

$$\boxed{A[n-1], A[n-2], \cdots, A[n-K_p]}$$

- $A[n]$ has influence on $o[n]$ (and $q[n]$) at $K_p + 1$ instants
  - At its current instant: n
  - At K_p posterior instants: $n+1, n+2, \cdots, n+K_p$

Memoryles symbol-by-symbol detector



- Detector is based on decision regions
  - 1D constellations : regions are defined by thresholds
  - 2D constellations : regions are defined by 2D boundaries
- Design parameters
  - Delay d associated to the decision
    - ⋆ Decision rule is applied to $q[n]$
    - ⋆ The result is the decision for symbol $A[n-d]$

    Processing of $q[n] \rightarrow \hat{A}[n-d]$ (decision for $A[n-d]$)
  - Decision regions
    - ⋆ A region per symbol: I_j for $\boldsymbol{a}_j$ $\forall j$
    - ⋆ Determine the decision rule

    If $q[n] \in I_j \rightarrow \hat{A}[n-d] = \boldsymbol{a}_j$

Memoryles symbol-by-symbol detection - Delay d

- From observation $q[n]$ a decision is made for $A[n-d]$
  - Detector has a delay d
- Ideal channel with delay d lags:

$$p[n] = C\,\delta[n-d] \quad \rightarrow \quad q[n] = C\,A[n-d] + z[n]$$

- Non ideal channel

$$q[n] = \underbrace{p[d]\,A[n-d]}_{\text{desired term}} + \underbrace{\sum_{k\neq d} p[k]\,A[n-k]}_{\text{ISI}} + \underbrace{z[n]}_{\text{noise}}$$

- Optimal choice for delay d
  - Looking for the symbol with the greatest contribution on $q[n]$
  - Alternative interpretation:
    - ⋆ Normalization of observation to compensate gain $p[d]$

    $$q_n[n] = \frac{q[n]}{p[d]} = A[n-d] + \sum_{k\neq d} \frac{p[k]}{p[d]}\,A[n-k] + \frac{z[n]}{p[d]}$$
    - ⋆ Once d is selected, term $p[d]$ divides ISI and noise
  - Optimal choice: select d such that $|p[d]| \geq |p[n]|$ for all n
    - ⋆ Minimizes the joint effect of ISI and noise

Choice for optimal delay - Example

- Transmission of 2-PAM through $p[n] = \frac{1}{2}\delta[n] + \delta[n-1] + \frac{1}{4}\delta[n-2]$
- $o[n] = A[n] * p[n] = \frac{1}{2}A[n] + A[n-1] + \frac{1}{4}A[n-2]$

Delay $d = 0$

$A[n]$	$A[n-1]$	$A[n-2]$	$o[n]$
$+1$	$+1$	$+1$	$+\frac{7}{4}$
$+1$	$+1$	-1	$+\frac{5}{4}$
$+1$	-1	$+1$	$-\frac{1}{4}$
$+1$	-1	-1	$-\frac{3}{4}$
-1	$+1$	$+1$	$+\frac{3}{4}$
-1	$+1$	-1	$+\frac{1}{4}$
-1	-1	$+1$	$-\frac{5}{4}$
-1	-1	-1	$-\frac{7}{4}$

Delay $d = 1$ (optimal)

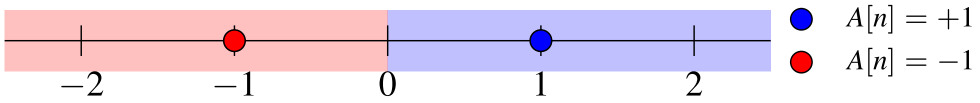
$A[n]$	$A[n-1]$	$A[n-2]$	$o[n]$
$+1$	$+1$	$+1$	$+\frac{7}{4}$
$+1$	$+1$	-1	$+\frac{5}{4}$
$+1$	-1	$+1$	$-\frac{1}{4}$
$+1$	-1	-1	$-\frac{3}{4}$
-1	$+1$	$+1$	$+\frac{3}{4}$
-1	$+1$	-1	$+\frac{1}{4}$
-1	-1	$+1$	$-\frac{5}{4}$
-1	-1	-1	$-\frac{7}{4}$

Blue : $A[n-d] = +1$ Red : $A[n-d] = -1$

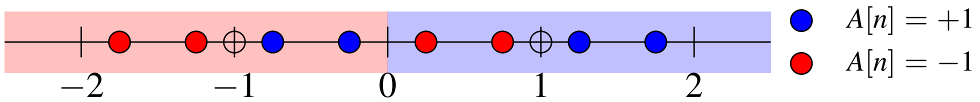
- Symbol associated to highest $|p[n]|$ has the highest contribution in $o[n]$
  - Sign of $o[n]$ in this case depends on $A[n-1]$ ($d = 1$)

Choice for optimal delay - Example (II)

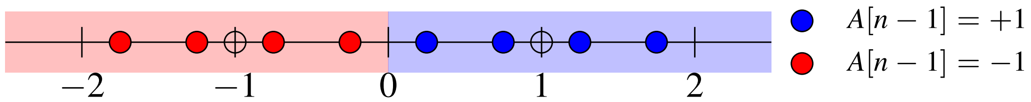
- Without ISI ($p[n] = \delta[n]$)



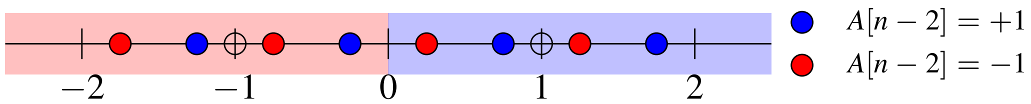
- With ISI ($p[n] = \frac{1}{2}\delta[n] + \delta[n-1] + \frac{1}{4}\delta[n-2]$): delay $d = 0$



- With ISI ($p[n] = \frac{1}{2}\delta[n] + \delta[n-1] + \frac{1}{4}\delta[n-2]$): delay $d = 1$

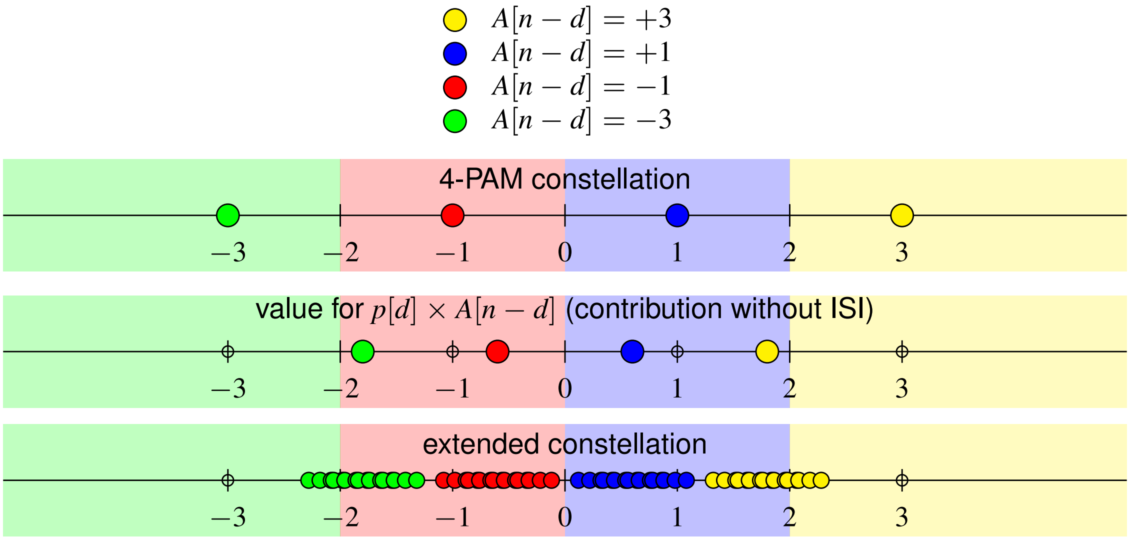


- With ISI ($p[n] = \frac{1}{2}\delta[n] + \delta[n-1] + \frac{1}{4}\delta[n-2]$): delay $d = 2$



Effect of the cursor's value $p[d]$: scaling

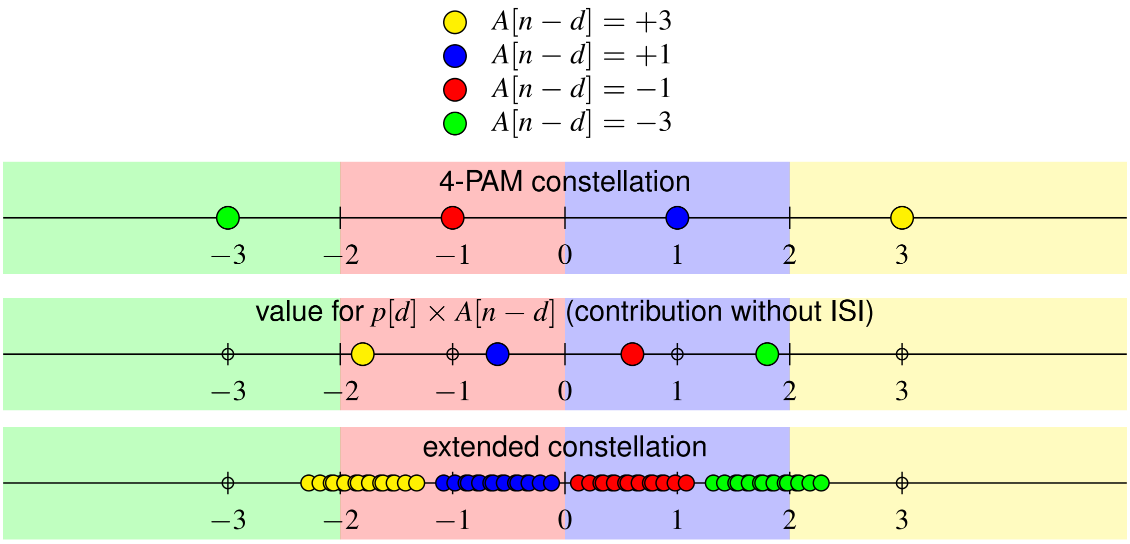
- Example: 4-PAM constellation : $A[n] \in \{\pm 1, \pm 3\}$
- Equivalent discrete channel: $p[n] = 0.1\ \delta[n] + 0.6\ \delta[n-1] - 0.05\ \delta[n-2]$
  - Optimal delay for decision: $d = 1$ ($p[d] = 0.6$)



- Value of the cursor determines a general scaling of the received constellation

Effect of the cursor's value $p[d]$: sign of the cursor

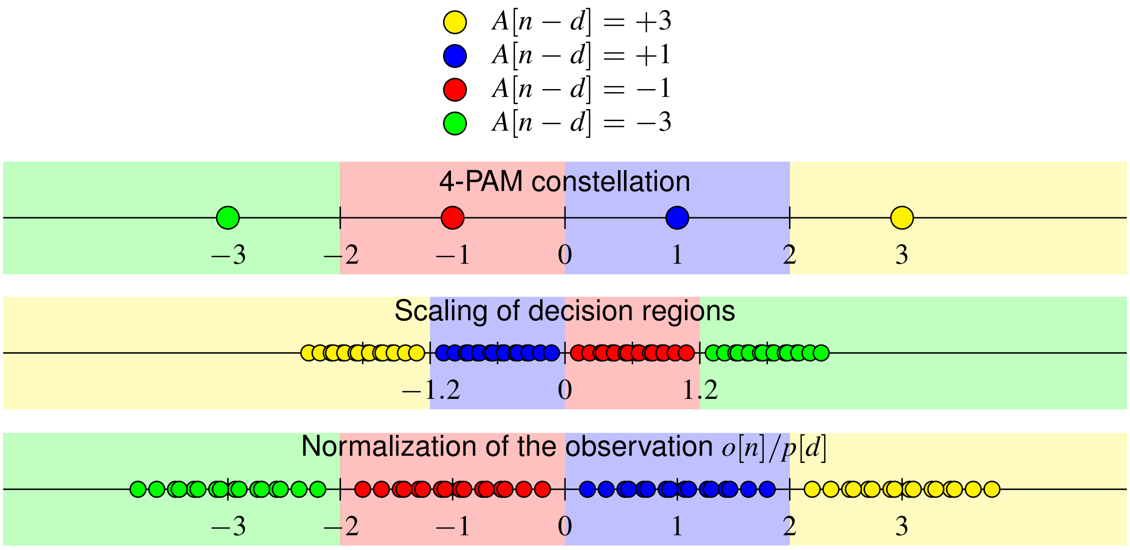
- Example: 4-PAM constellation : $A[n] \in \{\pm 1, \pm 3\}$
- Equivalent discrete channel: $p[n] = 0.1\ \delta[n] - 0.6\ \delta[n-1] - 0.05\ \delta[n-2]$
  - Optimal delay for decision: $d = 1$ ($p[d] = -0.6$)



- A negative sign for cursor "*inverts*" the constellation

Effect of the cursor's value $p[d]$: options

- Example: 4-PAM constellation : $A[n] \in \{\pm 1, \pm 3\}$
- Equivalent discrete channel: $p[n] = 0.1\ \delta[n] - 0.6\ \delta[n-1] - 0.05\ \delta[n-2]$
  - Optimal delay for decision: $d = 1$ ($p[d] = -0.6$)
- There are two options to take into account the cursor
  - To scale decision regions according to the cursor (considering sign)
  - To normalize the observation before decisions: dividing by cursor



Normalization of the observation: effect over noise

- Normalization of the observation

$$q_n[n] = \frac{q[n]}{p[d]} = o_n[n] + z_n[n]$$

$$o_n[n] = \frac{o[n]}{p[d]} \qquad z_n[n] = \frac{z[n]}{p[d]}$$

- Effect over noise: power (variance) is modified

$$\sigma_{z_n}^2 = \sigma_z^2 \times \frac{1}{|p[d]|^2}$$

  - Important to evaluate probabilities of error

ISI level γ_{ISI}

- Quantifies the ISI distortion introduced by a channel

$$\gamma_{ISI} = \frac{D_{peak}}{\eta} \geq 0$$

  - D_{peak}: peak distortion for a delay d

$$D_{peak} = \sum_{k \neq d} \frac{|p[k]|}{|p[d]|} \geq 0$$

    Depends on the equivalent discrete channel and selected delay for decision (d)
  - η: constellation efficiency

$$\eta = \frac{(d_{min}/2)}{|A|_{max}} \geq 0$$

    Depends on the constellation used to transmit data
    - $|A|_{max}$: maximum modulus of a symbol in the constellation

$$|A|_{max} = \text{máx}\{|A[n]|\}$$

    - d_{min}: minimum distace between two symbols in the constellation

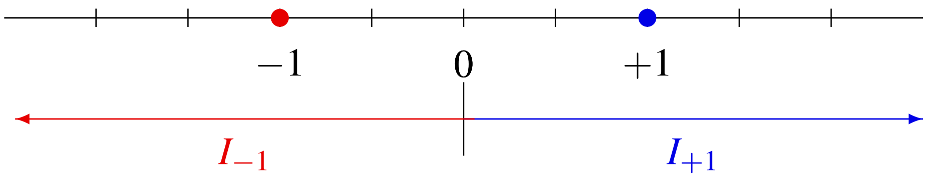
$$d_{min} = \min_{A[n] \neq A[k]} |A[n] - A[k]|$$

γ_{ISI}: Measuring effect of ISI on the decision regions

- ISI level measures the effect of ISI in terms of how it affects to the received constellation (extended constellation generated by ISI)
- Value $\gamma_{ISI} = 1$ indicates the point where the extended constellation achieves the limits of the original decision regions
  - $\gamma_{ISI} < 1$: ISI does not move symbols out of its decision region
    - Without noise unmodified symbol by symbol detector does not make erroneous decisions
  - $\gamma_{ISI} > 1$: ISI does move symbols out of its decision region
    - Unmodified symbol by symbol detector makes erroneous decisions even without noise
    - In this case, a re-definition of the decision regions, taking into account the underlying ISI, is necessary to guarantee a maximum performance using memoryless symbol by symbol detectors

Example

- ISI level will be presented for the following case
  - Transmitted constellation: 2-PAM ($A[n] \in \{\pm 1\}$)
    - ★ Efficiency $\eta = 1$
    - ★ Constellation and original decision regions ($I_{+1},\ I_{-1}$)



  - Equivalent discrete channel

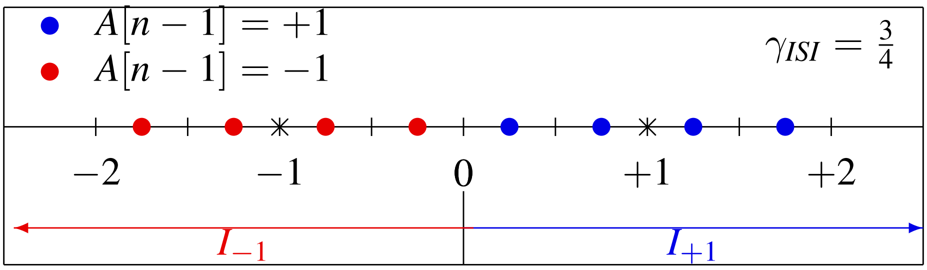
$$p[n] = \frac{1}{2}\delta[n] + \delta[n-1] + c\,\delta[n-2]$$

    - ★ Several values for c will be tested: $c = \frac{1}{4}$, $c = \frac{1}{2}$, and $c = \frac{3}{4}$
      - In all cases, optimal delay is $d = 1$.
- Points of the extended constellation generated by ISI

Plot of values for $o[n] = A[n] * p[n] = \frac{1}{2}A[n] + A[n-1] + c\,A[n-2]$

Example - $c = \frac{1}{4}$

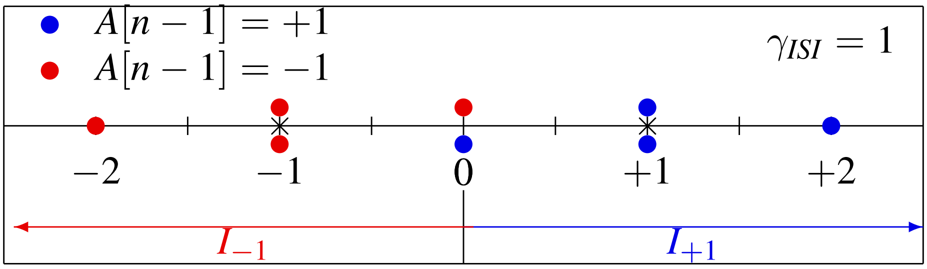
$A[n]$	$A[n-1]$	$A[n-2]$	$o[n]$
$+1$	$+1$	$+1$	$+\frac{7}{4}$
$+1$	$+1$	-1	$+\frac{5}{4}$
$+1$	-1	$+1$	$-\frac{1}{4}$
$+1$	-1	-1	$-\frac{3}{4}$
-1	$+1$	$+1$	$+\frac{3}{4}$
-1	$+1$	-1	$+\frac{1}{4}$
-1	-1	$+1$	$-\frac{5}{4}$
-1	-1	-1	$-\frac{7}{4}$



- Points are still inside of its corresponding decision region

Example - $c = \frac{1}{2}$

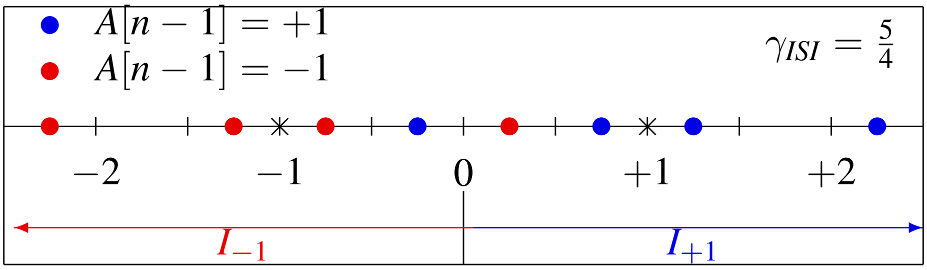
$A[n]$	$A[n-1]$	$A[n-2]$	$o[n]$
$+1$	$+1$	$+1$	$+2$
$+1$	$+1$	-1	$+1$
$+1$	-1	$+1$	0
$+1$	-1	-1	-1
-1	$+1$	$+1$	$+1$
-1	$+1$	-1	0
-1	-1	$+1$	-1
-1	-1	-1	-2



- Points now achieve the limits of its corresponding decision region

Example - $c = \frac{3}{4}$

$A[n]$	$A[n-1]$	$A[n-2]$	$o[n]$
$+1$	$+1$	$+1$	$+\frac{9}{4}$
$+1$	$+1$	-1	$+\frac{3}{4}$
$+1$	-1	$+1$	$+\frac{1}{4}$
$+1$	-1	-1	$-\frac{5}{4}$
-1	$+1$	$+1$	$+\frac{5}{4}$
-1	$+1$	-1	$-\frac{1}{4}$
-1	-1	$+1$	$-\frac{3}{4}$
-1	-1	-1	$-\frac{9}{4}$

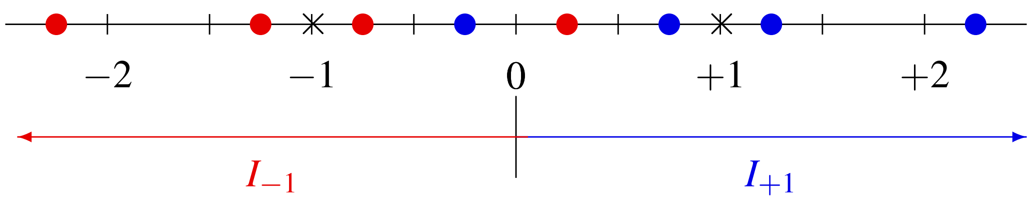


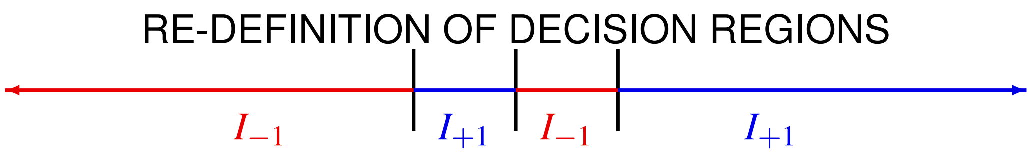
- Some points are now out of its corresponding decision region

Re-defining decision regions

- Decision regions must be re-defined based on the location of points of the extended constellation defined by ISI
  - ● $A[n-1] = +1$
  - ● $A[n-1] = -1$

$\gamma_{ISI} = \frac{5}{4}$





- The new decision regions depend on:
  - Values of points for the extended constellation
  - Variance of the noise sequence

Redefinition of the decision regions

- Design of decision regions depends on:
  - Probabilities of transmission for symbols $p_A(\boldsymbol{a}_i)$
  - Conditional distributions for observation given each symbol $f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}|\boldsymbol{a}_i)$
- Design rules: $\boldsymbol{q}_0 \in I_i$ if for all $j \neq i$
  - General case: *Maximum A Posteriori* (MAP) criterion

$$p_A(\boldsymbol{a}_i)\, f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}_0|\boldsymbol{a}_i) > p_A(\boldsymbol{a}_j)\, f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}_0|\boldsymbol{a}_j)$$

  - Equiprobable symbols ($p_A(\boldsymbol{a}_i) = 1/M$): *Maximum Likelihood* (ML) criterion

$$f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}_0|\boldsymbol{a}_i) > f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}_0|\boldsymbol{a}_j)$$

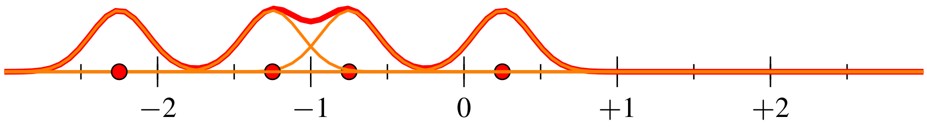
    - ★ Under Gaussian noise and $o[n] = A[n-d]$: *Minimum Euclidean Distance* criterion

$$d(\boldsymbol{q}_0, \boldsymbol{a}_i) < d(\boldsymbol{q}_0, \boldsymbol{a}_j)$$

Conditional distributions $f_{q|A}(q|a_i)$ under Gaussian noise

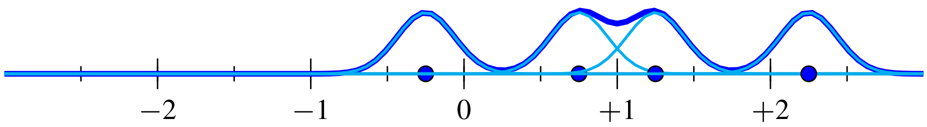
Conditional distribution for $A[n-1]=-1$

$$f_{q|A}(q|-1)=\frac{1}{4}\mathcal{N}\left(-\frac{9}{4},\frac{N_0}{2}\right)+\frac{1}{4}\mathcal{N}\left(-\frac{5}{4},\frac{N_0}{2}\right)+\frac{1}{4}\mathcal{N}\left(-\frac{3}{4},\frac{N_0}{2}\right)+\frac{1}{4}\mathcal{N}\left(+\frac{1}{4},\frac{N_0}{2}\right)$$



Conditional distribution for $A[n-1]=+1$

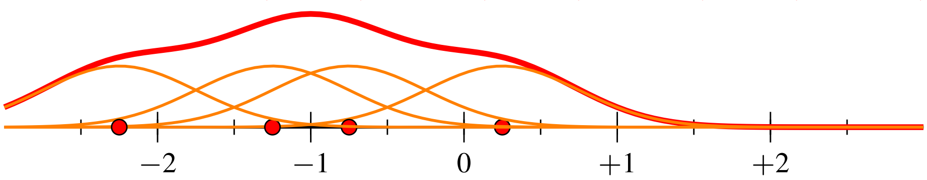
$$f_{q|A}(q|+1)=\frac{1}{4}\mathcal{N}\left(+\frac{9}{4},\frac{N_0}{2}\right)+\frac{1}{4}\mathcal{N}\left(+\frac{5}{4},\frac{N_0}{2}\right)+\frac{1}{4}\mathcal{N}\left(+\frac{3}{4},\frac{N_0}{2}\right)+\frac{1}{4}\mathcal{N}\left(-\frac{1}{4},\frac{N_0}{2}\right)$$



Conditional distributions $f_{q|A}(q|a_i)$ under Gaussian noise (II)

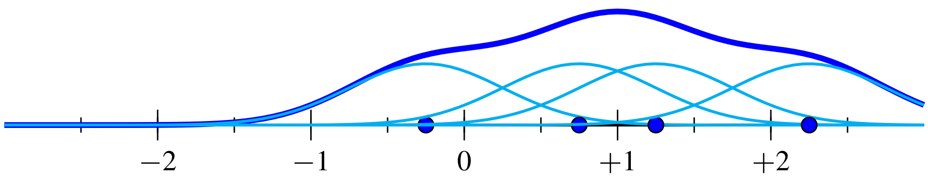
Conditional distribution for $A[n-1]=-1$

$$f_{q|A}(q|-1)=\frac{1}{4}\mathcal{N}\left(-\frac{9}{4},\frac{N_0}{2}\right)+\frac{1}{4}\mathcal{N}\left(-\frac{5}{4},\frac{N_0}{2}\right)+\frac{1}{4}\mathcal{N}\left(-\frac{3}{4},\frac{N_0}{2}\right)+\frac{1}{4}\mathcal{N}\left(+\frac{1}{4},\frac{N_0}{2}\right)$$



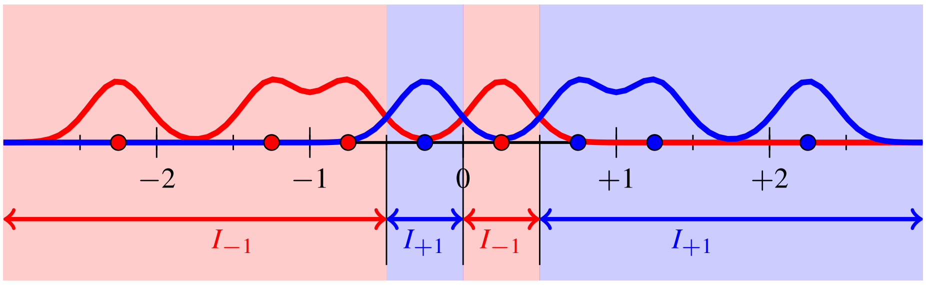
Conditional distribution for $A[n-1]=+1$

$$f_{q|A}(q|+1)=\frac{1}{4}\mathcal{N}\left(+\frac{9}{4},\frac{N_0}{2}\right)+\frac{1}{4}\mathcal{N}\left(+\frac{5}{4},\frac{N_0}{2}\right)+\frac{1}{4}\mathcal{N}\left(+\frac{3}{4},\frac{N_0}{2}\right)+\frac{1}{4}\mathcal{N}\left(-\frac{1}{4},\frac{N_0}{2}\right)$$

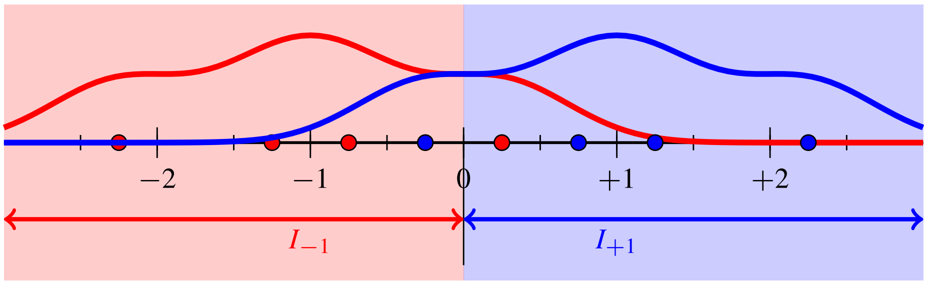


Conditional distributions $f_{q|A}(q|a_i)$ under Gaussian noise (III)

Example for relatively low noise variance



Example for relatively high noise variance



High SNR (low noise) : approximation for decision regions

- Under a high SNR (low noise variance) the design of the new decision regions with a minimum distance criterion with respect to the different points of the extended constellation is a reasonable approximation
- When SNR decreases (noise variance increases), typically the decision regions tend to the initial decision regions designed without ISI

Maximum likelihood sequence detection (MLSD)

- Optimal detection under ISI: MLSD
- Sequence to be detected: L symbols (M^L possible sequences)

$$\boldsymbol{A} = [A[0], A[1], \cdots, A[L-1]]^T$$

- Sufficient statistic for detection: $N_q = L + K_p$ observations

$$\boldsymbol{q} = [q[0], q[1], \cdots, q[N_q-1]], \quad N_q = L + K_p$$

$$\begin{aligned} q[0] &= p[0]\ A[0] + p[1]\ A[-1] + p[2]\ A[-2] + \cdots\ p[K_p]\ A[-K_p] + z[0] \\ q[1] &= p[0]\ A[1] + p[1]\ A[0] + p[2]\ A[-1] \cdots + p[K_p]\ A[1-K_p] + z[1] \\ &\cdots \\ q[L+K_p-1] &= p[0]\ A[L+K_p-1] + p[1]\ A[L+K_p-2] + \cdots \\ &+ p[K_p-1]\ A[L] + p[K_p]\ A[L-1] + z[L+K_p-1] \end{aligned}$$

- Side (additional) information that is necessary:
  - Known channel with memory K_p (K_p+1 coefficients)

$$\mathbf{p} = [p[0], p[1], \cdots, p[K_p]]^T$$

  - Previous K_p symbols and posterior K_p symbols

$$A[-1], A[-2], \cdots, A[-K_p] \text{ and } A[L], A[L+1], \cdots, A[L+K_p-1]$$

    - ★ Cyclic header of K_p symbols between blocks of L information symbols

Maximum Likelihood (ML) sequence

- M^L possible sequences

$$\boldsymbol{A}_i = [A_i[0], A_i[1], \cdots, A_i[L-1]]^T, \ i = 0, 1, \cdots, M^L - 1$$

  - Noiseless output generated by each sequence

$$o_i[n] = A_i[n] * p[n] = \sum_{k=0}^{N} p[k]\ A_i[n-k]$$

    - ★ It is known if $A_i[n]$ and $p[n]$ are known!!!
- ML sequence: the one with the highest likelihood:
  - Sequence

$$\hat{\boldsymbol{A}} = \boldsymbol{A}_i = [A_i[0], A_i[1], \cdots, A_i[L-1]]^T$$

  that satisfies the following condition

$$f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}|\boldsymbol{A}_i) \geq f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}|\boldsymbol{A}_j), \ j = 0, 1, \cdots, M^L - 1, \ \forall j \neq i.$$

Estimation of the maximum likelihood sequence

- Conditional distribution for each observation: $q[n] = o_i[n] + z[n]$

$$f_{q[n]|\boldsymbol{A}}(q[n]|\boldsymbol{A}_i) = \mathcal{N}\left(o_i[n], \sigma_z^2\right) = \frac{1}{\sqrt{2\pi}\sigma_z} \exp\left\{-\frac{1}{2\sigma_z^2}|q[n]-o_i[n]|^2\right\}$$

  - Conditional independency: $f_{q[n]|\boldsymbol{A}}(q[n]|\boldsymbol{a}_i)$ indep. of $f_{q[n]|\boldsymbol{A}}(q[n]|\boldsymbol{a}_k)\ \forall k \neq i$
- Likelihood for data sequence $\boldsymbol{a}_i$

$$f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}|\boldsymbol{A}_i) = \prod_{n=0}^{N_q-1} f_{q[n]|\boldsymbol{A}}(q[n]|\boldsymbol{a}_i)$$

$$f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}|\boldsymbol{a}_i) = \frac{1}{(2\pi\sigma_z^2)^{N_q/2}} \exp\left\{-\frac{1}{2\sigma_z^2}\sum_{n=0}^{Nq-1}|q[n]-o_i[n]|^2\right\}$$

- Maximum likelihood sequence

$$\hat{\boldsymbol{A}} = \arg\min_{\boldsymbol{A}_i} \sum_{n=0}^{N_q-1}|q[n]-o_i[n]|^2, \text{ with } o_i[n] = \sum_{k=0}^{K_p} p[k]\, A_i[n-k]$$

Detection of the ML sequence - Summary

- Sequence to be detected: L symbols (M^L possible solutions)

$$\boldsymbol{A} = [A[0], A[1], \cdots, A[L-1]]^T$$

- Sufficient statistics: $L + K_p$ observations

$$\boldsymbol{q} = [q[0], q[1], \cdots, q[N_q-1]], \quad N_q = L + K_p$$

- Side information: channel and K_p previous and K_p posterior symbols

$$\mathbf{p} = [p[0], p[1], \cdots, p[K_p]]^T$$

$$A[-1], A[-2], \cdots, A[-K_p] \text{ and } A[L], A[L+1], \cdots, A[L+K_p-1]$$

- Maximum likelihood sequence

$$\hat{\boldsymbol{A}} = \arg\min_{\boldsymbol{A}_i} \sum_{n=0}^{N_q-1}|q[n]-o_i[n]|^2$$

$$\boldsymbol{A}_i = [A_i[0], A_i[1], \cdots, A_i[L-1]]^T \text{ and } o_i[n] = \sum_{k=0}^{K_p} p[k]\, A_i[n-k]$$

Example: 2-PAM $K_p = 1$, $L = 4$

- Constellation: 2-PAM (BPSK): $A[n] \in \{\pm 1\}$
- Channel: $p[n] = \delta[n] + 0.5\ \delta[n-1]$, $K_p = 1$
- Sequence to be estimated: $\boldsymbol{A} = [A[0], A[1], A[2], A[3]]$, $L = 4$
- Statistic for detection ($L + \boldsymbol{K}_p$): $\boldsymbol{q} = [q[0], q[1], q[2], q[3], q[4]]$

$$q[-1] = A[-1] + 0.5\ A[-2] + z[-1]$$

$$\begin{aligned} q[0] &= A[0] + 0.5\ A[-1] + z[0] \\ q[1] &= A[1] + 0.5\ A[0] + z[1] \\ q[2] &= A[2] + 0.5\ A[1] + z[2] \\ q[3] &= A[3] + 0.5\ A[2] + z[3] \\ q[4] &= A[4] + 0.5\ A[3] + z[4] \end{aligned}$$

$$q[5] = A[5] + 0.5\ A[4] + z[5]$$

- Assumption: the following values are known $A[-1] = A[4] = +1$
- Problem: to decide the ML sequence when the sequence of observations is

$$q[0] = +0.5\ q[1] = -0.4\ q[2] = +0.1\ q[3] = -1.7\ q[4] = +0.3$$

Brute force detection: comparison of $q[n]$ with $o[n]$

$q[0]$	$q[1]$	$q[2]$	$q[3]$	$q[4]$
$+0.5$	-0.4	$+0.1$	-1.7	$+0.3$

$$o[n] = A[n] + \frac{1}{2}A[n-1], \quad \text{Likelihood Metric} = \sum_{n=0}^{N_q-1} |q[n] - o[n]|^2$$

$A[0]$	$A[1]$	$A[2]$	$A[3]$	$o[0]$	$o[1]$	$o[2]$	$o[3]$	$o[4]$	Likelihood Metric
$+1$	$+1$	$+1$	$+1$	$+1.5$	$+1.5$	$+1.5$	$+1.5$	$+1.5$	18.25
-1	$+1$	$+1$	$+1$	-0.5	$+0.5$	$+1.5$	$+1.5$	$+1.5$	15.45
$+1$	-1	$+1$	$+1$	$+1.5$	-0.5	$+0.5$	$+1.5$	$+1.5$	12.85
-1	-1	$+1$	$+1$	-0.5	-1.5	$+0.5$	$+1.5$	$+1.5$	14.05
$+1$	$+1$	-1	$+1$	$+1.5$	$+1.5$	-0.5	$+0.5$	$+1.5$	11.25
-1	$+1$	-1	$+1$	-0.5	$+0.5$	-0.5	$+0.5$	$+1.5$	8.45
$+1$	-1	-1	$+1$	$+1.5$	-0.5	-1.5	$+0.5$	$+1.5$	9.85
-1	-1	-1	$+1$	-0.5	-1.5	-1.5	$+0.5$	$+1.5$	11.05
$+1$	$+1$	$+1$	-1	$+1.5$	$+1.5$	$+1.5$	-0.5	$+0.5$	8.05
-1	$+1$	$+1$	-1	-0.5	$+0.5$	$+1.5$	-0.5	$+0.5$	5.25
$+1$	-1	$+1$	-1	$+1.5$	-0.5	$+0.5$	-0.5	$+0.5$	2.65
-1	-1	$+1$	-1	-0.5	-1.5	$+0.5$	-0.5	$+0.5$	3.85
$+1$	$+1$	-1	-1	$+1.5$	$+1.5$	-0.5	-1.5	$+0.5$	5.05
-1	$+1$	-1	-1	-0.5	$+0.5$	-0.5	-1.5	$+0.5$	2.25
$+1$	-1	-1	-1	$+1.5$	-0.5	-1.5	-1.5	$+0.5$	3.65
-1	-1	-1	-1	-0.5	-1.5	-1.5	-1.5	$+0.5$	4.85

Brute force detection: comparison of $q[n]$ with $o[n]$

$q[0]$	$q[1]$	$q[2]$	$q[3]$	$q[4]$
-0.4	-1.4	$+0.6$	-0.4	$+0.4$

$$o[n] = A[n] + \frac{1}{2}A[n-1], \quad \text{Likelihood Metric} = \sum_{n=0}^{N_q-1} |q[n] - o[n]|^2$$

$A[0]$	$A[1]$	$A[2]$	$A[3]$	$o[0]$	$o[1]$	$o[2]$	$o[3]$	$o[4]$	Likelihood metric
$+1$	$+1$	$+1$	$+1$	$+1.5$	$+1.5$	$+1.5$	$+1.5$	$+1.5$	17.65
-1	$+1$	$+1$	$+1$	-0.5	$+0.5$	$+1.5$	$+1.5$	$+1.5$	9.25
$+1$	-1	$+1$	$+1$	$+1.5$	-0.5	$+0.5$	$+1.5$	$+1.5$	9.25
-1	-1	$+1$	$+1$	-0.5	-1.5	$+0.5$	$+1.5$	$+1.5$	4.85
$+1$	$+1$	-1	$+1$	$+1.5$	$+1.5$	-0.5	$+0.5$	$+1.5$	15.25
-1	$+1$	-1	$+1$	-0.5	$+0.5$	-0.5	$+0.5$	$+1.5$	6.85
$+1$	-1	-1	$+1$	$+1.5$	-0.5	-1.5	$+0.5$	$+1.5$	10.85
-1	-1	-1	$+1$	-0.5	-1.5	-1.5	$+0.5$	$+1.5$	6.45
$+1$	$+1$	$+1$	-1	$+1.5$	$+1.5$	$+1.5$	-0.5	$+0.5$	12.85
-1	$+1$	$+1$	-1	-0.5	$+0.5$	$+1.5$	-0.5	$+0.5$	4.45
$+1$	-1	$+1$	-1	$+1.5$	-0.5	$+0.5$	-0.5	$+0.5$	4.45
-1	-1	$+1$	-1	-0.5	-1.5	$+0.5$	-0.5	$+0.5$	0.05
$+1$	$+1$	-1	-1	$+1.5$	$+1.5$	-0.5	-1.5	$+0.5$	14.45
-1	$+1$	-1	-1	-0.5	$+0.5$	-0.5	-1.5	$+0.5$	6.05
$+1$	-1	-1	-1	$+1.5$	-0.5	-1.5	-1.5	$+0.5$	10.05
-1	-1	-1	-1	-0.5	-1.5	-1.5	-1.5	$+0.5$	5.65

Example: 2-PAM $K_p = 2$, $L = 4$

- Constellation: 2-PAM (BPSK): $A[n] \in \{\pm 1\}$
- Channel: $p[n] = \delta[n] + 0.5\ \delta[n-1] + 0.25\ \delta[n-2]$, $K_p = 2$
- Sequence to be estimated: $\boldsymbol{A} = [A[0], A[1], A[2], A[3]]$, $L = 4$
- Statistic for detection ($L + \boldsymbol{K}_p$): $\boldsymbol{q} = [q[0], q[1], q[2], q[3], q[4], q[5]]$

$$q[-1] = A[-1] + 0.5\ A[-2] + 0.25\ A[-3] + z[-1]$$

$$\begin{aligned} q[0] &= A[0] + 0.5\ A[-1] + 0.25\ A[-2] + z[0] \\ q[1] &= A[1] + 0.5\ A[0] + 0.25\ A[-1] + z[1] \\ q[2] &= A[2] + 0.5\ A[1] + 0.25\ A[0] + z[2] \\ q[3] &= A[3] + 0.5\ A[2] + 0.25\ A[1] + z[3] \\ q[4] &= A[4] + 0.5\ A[3] + 0.25\ A[2] + z[4] \\ q[5] &= A[5] + 0.5\ A[4] + 0.25\ A[3] + z[4] \end{aligned}$$

$$q[6] = A[6] + 0.5\ A[5] + 0.25\ A[4] + z[5]$$

- Assumption: $A[-1] = A[-2] = +1$ and $A[4] = A[5] = +1$ are known
- Problem: to decide the ML sequence when observations are

$$q[0] = -0.24\ q[1] = -1.15\ q[2] = -1.75\ q[3] = +0.26\ q[4] = +1.27\ q[5] = +1.55$$

Brute force detection: comparison of $q[n]$ with $o[n]$

$q[0]$	$q[1]$	$q[2]$	$q[3]$	$q[4]$	$q[5]$
-0.24	-1.15	-1.75	$+0.26$	$+1.27$	$+1.55$

$$o[n] = A[n] + \frac{1}{2}A[n-1] + \frac{1}{4}A[n-2], \quad \text{Likelihood Metric} = \sum_{n=0}^{N_q-1} |q[n] - o[n]|^2$$

$A[0]$	$A[1]$	$A[2]$	$A[3]$	$o[0]$	$o[1]$	$o[2]$	$o[3]$	$o[4]$	$o[5]$	L. Metric
$+1$	$+1$	$+1$	$+1$	1.75	1.75	1.75	1.75	1.75	1.75	27.1106
-1	$+1$	$+1$	$+1$	-0.25	0.75	1.25	1.75	1.75	1.75	15.1006
$+1$	-1	$+1$	$+1$	1.75	-0.25	0.75	1.25	1.75	1.75	12.2706
-1	-1	$+1$	$+1$	-0.25	-1.25	0.25	1.25	1.75	1.75	5.2606
$+1$	$+1$	-1	$+1$	1.75	1.75	-0.25	0.75	1.25	1.75	14.9006
-1	$+1$	-1	$+1$	-0.25	0.75	-0.75	0.75	1.25	1.75	4.8906
$+1$	-1	-1	$+1$	1.75	-0.25	-1.25	0.25	1.25	1.75	5.0606
-1	-1	-1	$+1$	-0.25	-1.25	-1.75	0.25	1.25	1.75	0.0506
$+1$	$+1$	$+1$	-1	1.75	1.75	1.75	-0.25	0.75	1.25	25.2406
-1	$+1$	$+1$	-1	-0.25	0.75	1.25	-0.25	0.75	1.25	13.2306
$+1$	-1	$+1$	-1	1.75	-0.25	0.75	-0.75	0.75	1.25	12.4006
-1	-1	$+1$	-1	-0.25	-1.25	0.25	-0.75	0.75	1.25	5.3906
$+1$	$+1$	-1	-1	1.75	1.75	-0.25	-1.25	0.25	1.25	18.0306
-1	$+1$	-1	-1	-0.25	0.75	-0.75	-1.25	0.25	1.25	8.0206
$+1$	-1	-1	-1	1.75	-0.25	-1.25	-1.75	0.25	1.25	10.1906
-1	-1	-1	-1	-0.25	-1.25	-1.75	-1.75	0.25	1.25	5.1806

Brute force detection: comparison of $q[n]$ with $o[n]$

$q[0]$	$q[1]$	$q[2]$	$q[3]$	$q[4]$	$q[5]$
$+1.55$	-0.15	$+0.85$	$+1.27$	$+1.65$	$+1.45$

$$o[n] = A[n] + \frac{1}{2}A[n-1] + \frac{1}{4}A[n-2], \quad \text{Likelihood Metric} = \sum_{n=0}^{N_q-1} |q[n] - o[n]|^2$$

$A[0]$	$A[1]$	$A[2]$	$A[3]$	$o[0]$	$o[1]$	$o[2]$	$o[3]$	$o[4]$	$o[5]$	L. Metric
$+1$	$+1$	$+1$	$+1$	1.75	1.75	1.75	1.75	1.75	1.75	4.7904
-1	$+1$	$+1$	$+1$	-0.25	0.75	1.25	1.75	1.75	1.75	4.5404
$+1$	-1	$+1$	$+1$	1.75	-0.25	0.75	1.25	1.75	1.75	0.1604
-1	-1	$+1$	$+1$	-0.25	-1.25	0.25	1.25	1.75	1.75	4.9104
$+1$	$+1$	-1	$+1$	1.75	1.75	-0.25	0.75	1.25	1.75	5.3804
-1	$+1$	-1	$+1$	-0.25	0.75	-0.75	0.75	1.25	1.75	7.1304
$+1$	-1	-1	$+1$	1.75	-0.25	-1.25	0.25	1.25	1.75	5.7504
-1	-1	-1	$+1$	-0.25	-1.25	-1.75	0.25	1.25	1.75	12.5004
$+1$	$+1$	$+1$	-1	1.75	1.75	1.75	-0.25	0.75	1.25	7.6204
-1	$+1$	$+1$	-1	-0.25	0.75	1.25	-0.25	0.75	1.25	7.3704
$+1$	-1	$+1$	-1	1.75	-0.25	0.75	-0.75	0.75	1.25	4.9904
-1	-1	$+1$	-1	-0.25	-1.25	0.25	-0.75	0.75	1.25	9.7404
$+1$	$+1$	-1	-1	1.75	1.75	-0.25	-1.25	0.25	1.25	13.2104
-1	$+1$	-1	-1	-0.25	0.75	-0.75	-1.25	0.25	1.25	14.9604
$+1$	-1	-1	-1	1.75	-0.25	-1.25	-1.75	0.25	1.25	15.5804
-1	-1	-1	-1	-0.25	-1.25	-1.75	-1.75	0.25	1.25	22.3304

Efficient estimation - Definition of system state $\psi[n]$

- To compute the likelihood metric for each possible sequence is uneffient
- Noiseless output is a finite state machine

$$o[n] = A[n]\; p[0] + \sum_{k=1}^{K_p} p[k]\; A[n-k]$$

- Definition of system state at discrete instant n
  Set of K_p previous symbols contributing to the value of $o[n]$

$$\psi[n] = [A[n-1], A[n-2], \cdots, A[n-K_p]]^T$$

  Number of possible estates is M^{K_p}
- Some dependencies

$$o[n] = f(A[n], \psi[n])$$
$$o[n] = g(\psi[n], \psi[n+1])$$
$$\psi[n+1] = f(\psi[n], A[n])$$

State diagram / Trellis diagram

Drawing of the evolution of system state under ISI

$$\psi[n] = [A[n-1], A[n-2], \cdots, A[n-K_p+1], A[n-K_p]]^T$$

$$\psi[n+1] = [A[n], A[n-1], A[n-2], \cdots, A[n-K_p+1]]^T$$

- The diagram depends on two parameters
  - The constellation that is transmited
  - The equivalent discrete channel $p[n]$
- There are M^{K_p} possible states
- M arrows go out of (and arrive at) each state
  - Each arrow is labelled with the following two-sided label

    Label : $A[n] \mid o[n]$ Diagram : $\psi[n] \xrightarrow{A[n] \mid o[n]} \psi[n+1]$

    - ★ $A[n]$: the value of current symbol that forces the state transition
    - ★ $o[n]$: noiseless output that is generated in that case
  - Arrows going out of an state: one for each possible value of $A[n]$
  - Arrows arriving at each state: all of them with the same value of $A[n]$

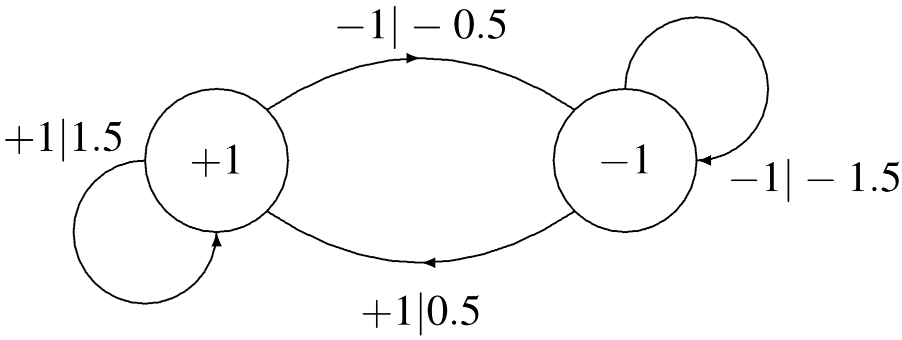
State diagram - Example A

- Diagram for $A[n] \in \{\pm 1\}$ and $p[n] = \delta[n] + \frac{1}{2}\delta[n-1]$
  - States at n and at $n+1$

$$\psi[n] = A[n-1], \qquad \psi[n+1] = A[n]$$

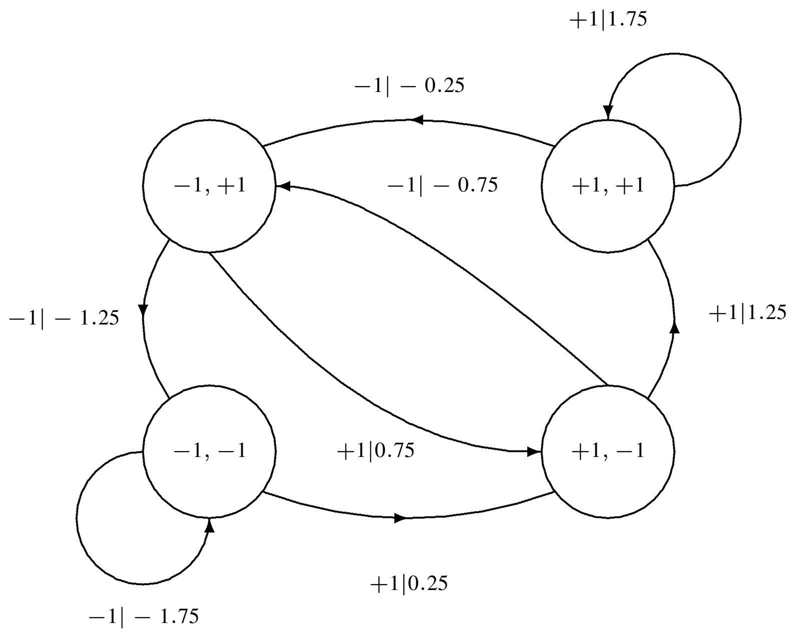
  - Noiseless output

$$o[n] = A[n] + \frac{1}{2}A[n-1]$$



State diagram - Example B

- $A[n] \in \{\pm 1\}$, $p[n] = \delta[n] + \frac{1}{2}\delta[n-1] + \frac{1}{4}\delta[n-2]$
  - $\psi[n] = [A[n-1], A[n-2]]^T$, $\psi[n+1] = [A[n], A[n-1]]^T$
  - $o[n] = A[n] + \frac{1}{2}A[n-1] + \frac{1}{4}A[n-2]$



Trellis diagram - Example A

- Represents the time evolution of states

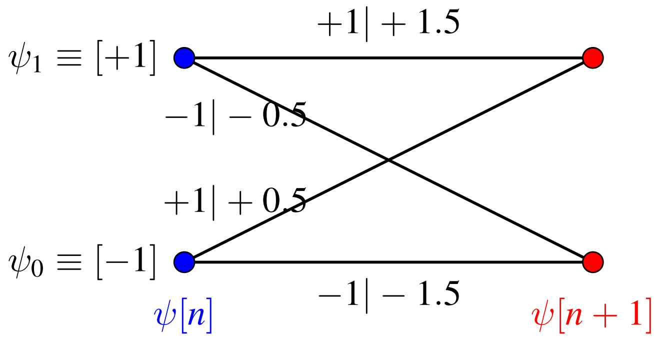
$$\psi[n] \xrightarrow{A[n]\,|\,o[n]} \psi[n+1]$$

- Example: $A[n] \in \{\pm 1\}$, $p[n] = \delta[n] + \frac{1}{2}\delta[n-1]$
  - Definition of state: $\psi[n] = A[n-1]$
  - Transition from states: $\psi[n] = A[n-1] \rightarrow \psi[n+1] = A[n]$
  - Labels: $A[n]|o[n]$. In this case $o[n] = A[n] * p[n] = A[n] + \frac{1}{2}A[n-1]$

$$\psi[n] = A[n-1] \xrightarrow{A[n]\,|\,o[n]} \psi[n+1] = A[n]$$

$A[n]$	$A[n-1]$	$o[n]$
$+1$	$+1$	$+1.5$
-1	$+1$	-0.5
$+1$	-1	$+0.5$
-1	-1	-1.5

$o[n] = A[n] + \frac{1}{2}A[n-1]$



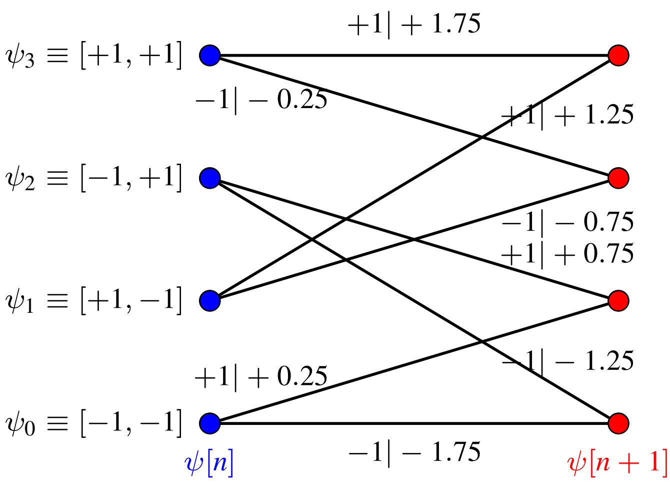
Trellis diagram - Example B

2-PAM and $p[n] = \delta[n] + \frac{1}{2}\delta[n-1] + \frac{1}{4}\delta[n-2]$ $(K_p = 2)$

$$\psi[n] = [A[n-1], A[n-2]] \xrightarrow{A[n]\,|\,o[n]} \psi[n+1] = [A[n], A[n-1]]$$

$o[n] = A[n] + \frac{1}{2}A[n-1] + \frac{1}{4}A[n-2]$

$A[n]$	$A[n-1]$	$A[n-2]$	$o[n]$
$+1$	$+1$	$+1$	$+1.75$
-1	$+1$	$+1$	-0.25
$+1$	-1	$+1$	$+0.75$
-1	-1	$+1$	-1.25
$+1$	$+1$	-1	$+1.25$
-1	$+1$	-1	-0.75
$+1$	-1	-1	$+0.25$
-1	-1	-1	-1.75

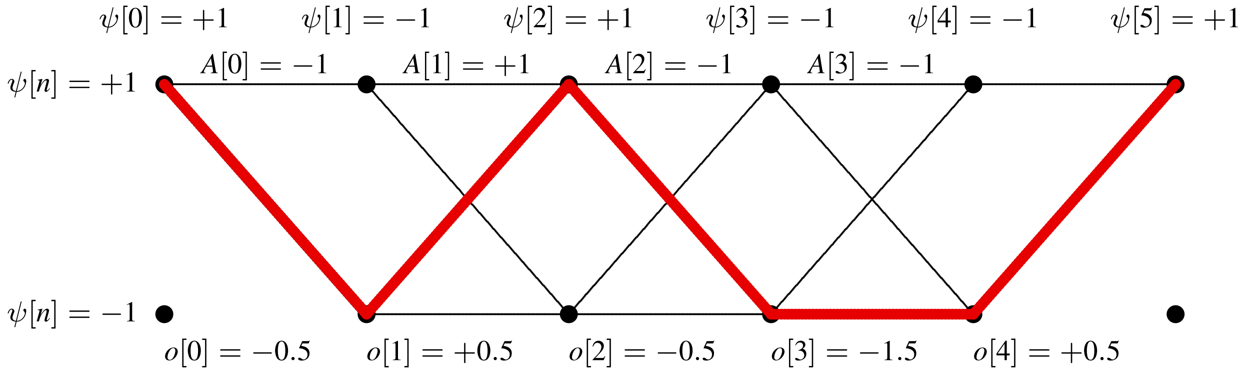


Trellis diagram - Representation of a sequence

Example, 2-PAM :

n	0	1	2	3
$A[n]$	-1	$+1$	-1	-1

- A given sequence can be drawn as a path through the trellis
  - Example for $L = 4$ (Channel of Example A)
  - Periodical header: $A[-1] = +1$, $A[4] = +1$
  - Initial state: $\psi[0] = +1$ / Final state: $\psi[5] = +1$

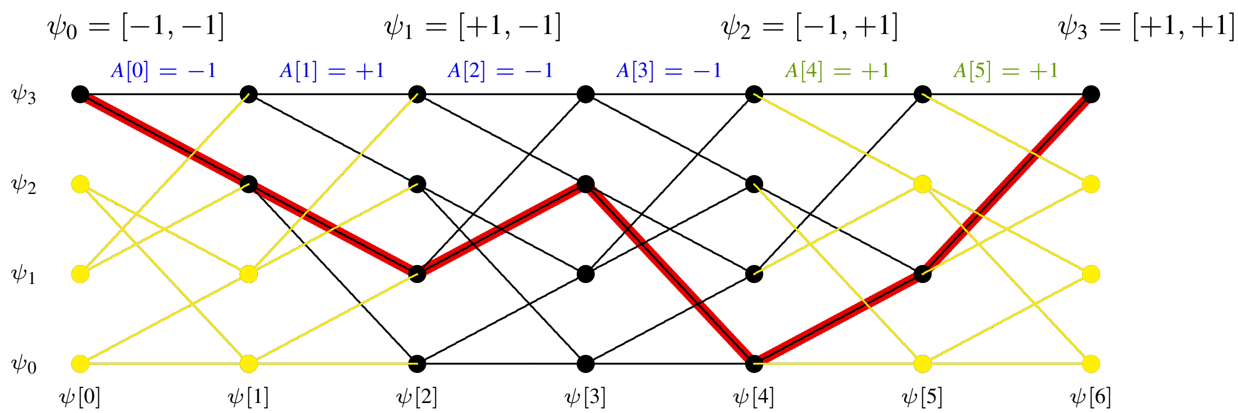


Trellis diagram - Representation of a sequence (II)

Example, 2-PAM :

n	0	1	2	3
$A[n]$	-1	$+1$	-1	-1

- A given sequence can be drawn as a path through the trellis
  - Example for $L = 4$ (Channel of Example B)
  - Periodical header: $A[-2] = A[-1] = +1$, $A[4] = A[5] = +1$
  - Initial state: $\psi[0] = +1$ / Final state: $\psi[6] = +1$



Brute force detection: comparison of $q[n]$ with $o[n]$

$q[0]$	$q[1]$	$q[2]$	$q[3]$	$q[4]$
$+0.5$	-0.4	$+0.1$	-1.7	$+0.3$

$$o[n] = A[n] + \frac{1}{2}A[n-1], \quad \text{Likelihood Metric} = \sum_{n=0}^{N_q-1} |q[n] - o[n]|^2$$

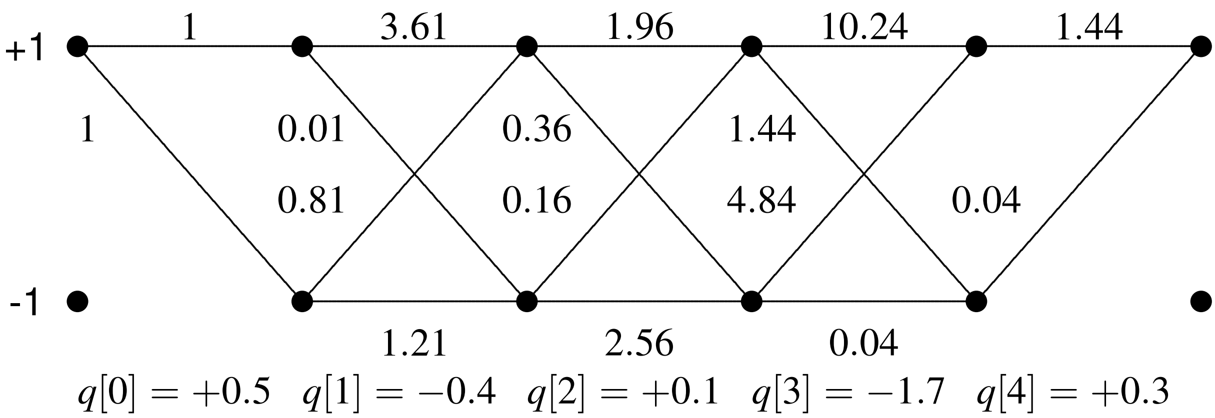
$A[0]$	$A[1]$	$A[2]$	$A[3]$	$o[0]$	$o[1]$	$o[2]$	$o[3]$	$o[4]$	Likelihood Metric
$+1$	$+1$	$+1$	$+1$	$+1.5$	$+1.5$	$+1.5$	$+1.5$	$+1.5$	18.25
-1	$+1$	$+1$	$+1$	-0.5	$+0.5$	$+1.5$	$+1.5$	$+1.5$	15.45
$+1$	-1	$+1$	$+1$	$+1.5$	-0.5	$+0.5$	$+1.5$	$+1.5$	12.85
-1	-1	$+1$	$+1$	-0.5	-1.5	$+0.5$	$+1.5$	$+1.5$	14.05
$+1$	$+1$	-1	$+1$	$+1.5$	$+1.5$	-0.5	$+0.5$	$+1.5$	11.25
-1	$+1$	-1	$+1$	-0.5	$+0.5$	-0.5	$+0.5$	$+1.5$	8.45
$+1$	-1	-1	$+1$	$+1.5$	-0.5	-1.5	$+0.5$	$+1.5$	9.85
-1	-1	-1	$+1$	-0.5	-1.5	-1.5	$+0.5$	$+1.5$	11.05
$+1$	$+1$	$+1$	-1	$+1.5$	$+1.5$	$+1.5$	-0.5	$+0.5$	8.05
-1	$+1$	$+1$	-1	-0.5	$+0.5$	$+1.5$	-0.5	$+0.5$	5.25
$+1$	-1	$+1$	-1	$+1.5$	-0.5	$+0.5$	-0.5	$+0.5$	2.65
-1	-1	$+1$	-1	-0.5	-1.5	$+0.5$	-0.5	$+0.5$	3.85
$+1$	$+1$	-1	-1	$+1.5$	$+1.5$	-0.5	-1.5	$+0.5$	5.05
-1	$+1$	-1	-1	-0.5	$+0.5$	-0.5	-1.5	$+0.5$	2.25
$+1$	-1	-1	-1	$+1.5$	-0.5	-1.5	-1.5	$+0.5$	3.65
-1	-1	-1	-1	-0.5	-1.5	-1.5	-1.5	$+0.5$	4.85

MLSD using the trellis diagram

- Maximum likelihood sequence

$$\hat{\boldsymbol{A}} = \arg \min_{\boldsymbol{a}_i} \sum_{n=0}^{N_q-1} \left| q[n] - \underbrace{\sum_{k=0}^{N} p[k]\, A_i[n-k]}_{o_i[n]} \right|^2$$

  - New labels in the trellis - branch metric: $|q[n] - o_i[n]|^2$
  - Likelihood metric for a sequence: addition of branch metrics of its path through the trellis

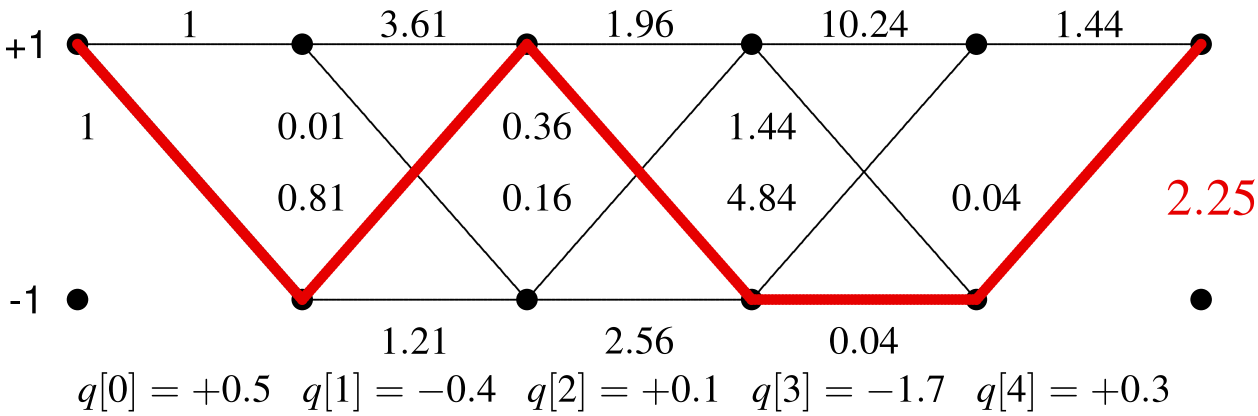


MLSD using the trellis diagram

- Maximum likelihood sequence

$$\hat{A} = \arg \min_{a_i} \sum_{n=0}^{N_q-1} \left| q[n] - \underbrace{\sum_{k=0}^{N} p[k]\, A_i[n-k]}_{o_i[n]} \right|^2$$

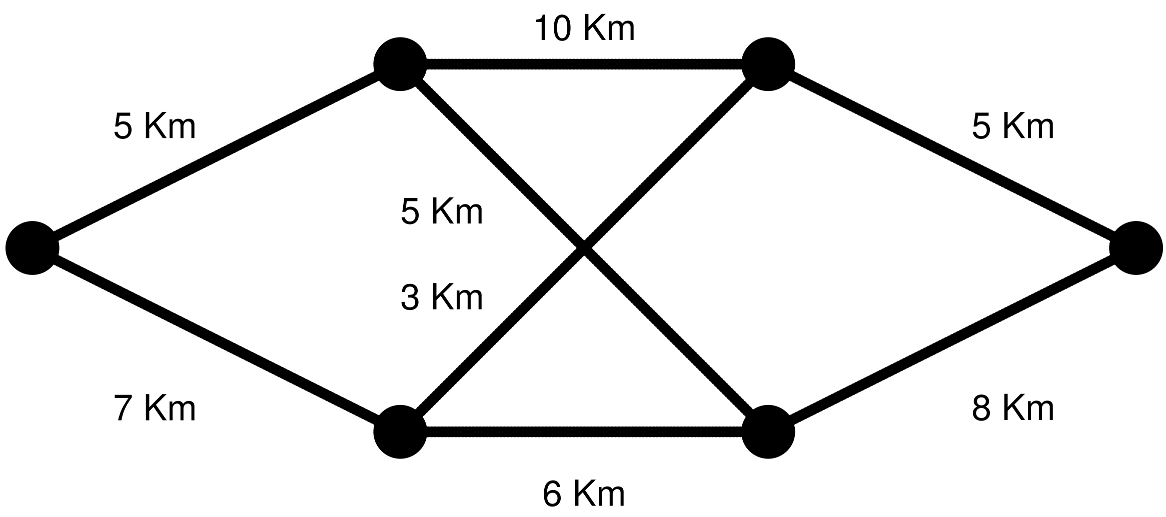
  - New labels in the trellis - branch metric: $|q[n] - o_i[n]|^2$
  - Likelihood metric for a sequence: addition of branch metrics of its path through the trellis



Obtention of the ML sequence - Viterbi algorithm

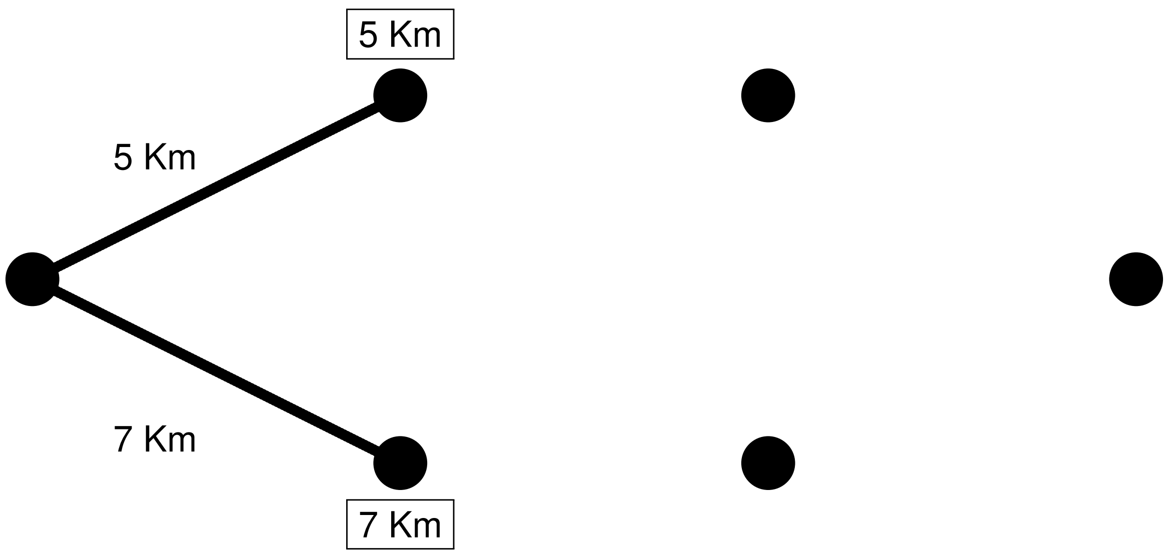
- Evaluation of the likelihood metric for the M^L possible sequences
  - Analytically, or by means of the path metric of M^L paths through $L + K_p$ transitions of the trellis diagram
  - Computationally expensive
- Efficient obtention of the ML sequence - Viterbi algorithm
  - Efficient obtention of the shortest path through a trellis
- Basic foundations of Viterbi algorithm
  - A trellis includes a set of nodes (states in our problem) and branches linking nodes
  - Branch metric: defines the metric associated to each branch
  - Path metric: the addition of branch metrics for all branches in a path
  - Survival path for a node: the path arriving at that node having the lowest accumulated path metric
  - Accumulated metric of a state: the metric of its survival path

Viterbi algorithm - A simple example



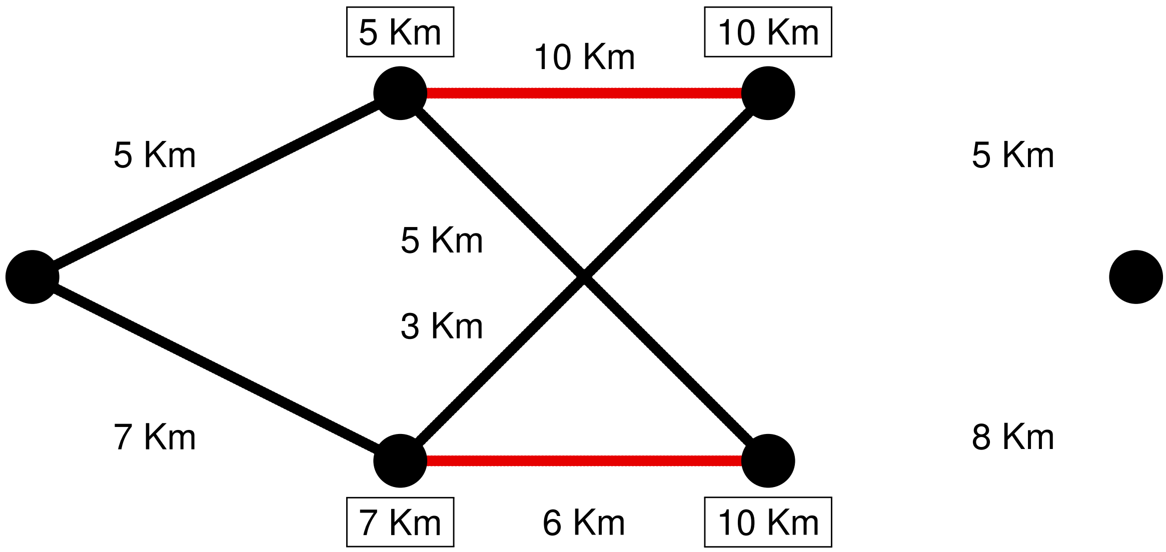
- Goal: finding the shortest path through a trellis
- Some example metrics

Viterbi algorithm - A simple example



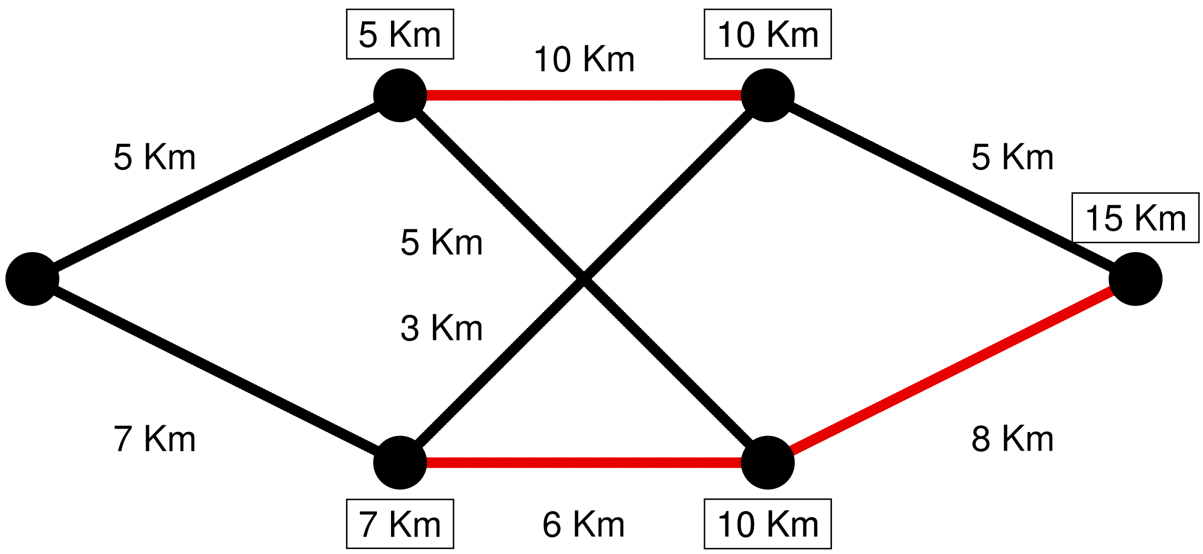
- First step
  - Computing the metrics at first stage after opening the trellis
  - Accumulated metric for the nodes are framed in a squared box

Viterbi algorithm - A simple example



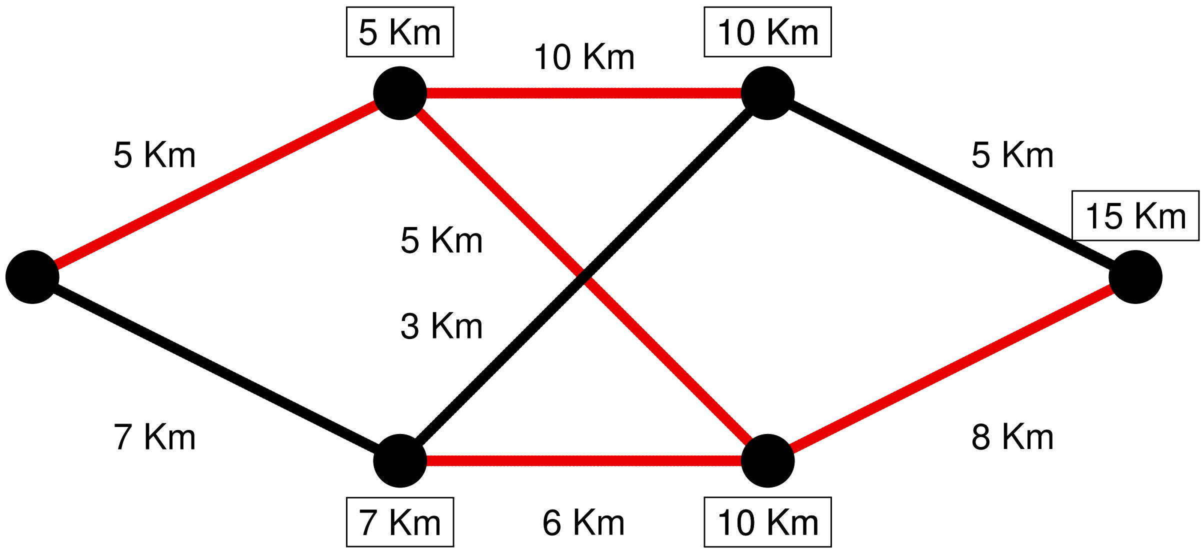
- Second step
  - Computing survival paths (black) for each node in next stage
    - ⋆ Upper node: [7 Km] + 3 Km is shorter than [5 Km] + 10 Km
    - ⋆ Lower node: [5 Km] + 5 Km is shorter than [7 Km] + 6 Km

Viterbi algorithm - A simple example



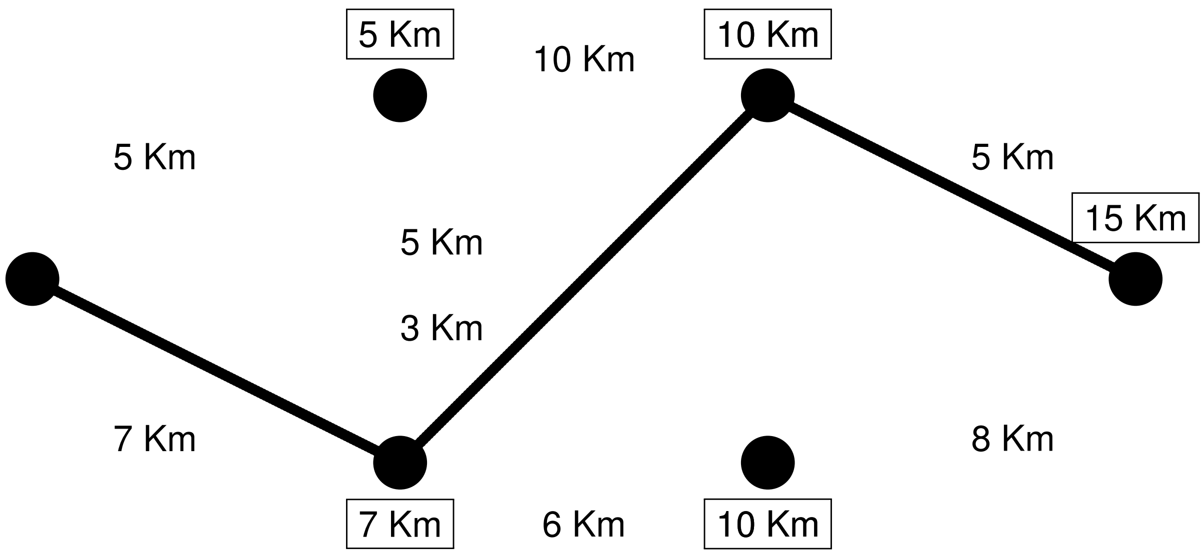
- Third step
  - Computing survival path (black) for node in final stage
    - ⋆ [10 Km] + 5 Km is shorter than [10 Km] + 8 Km

Viterbi algorithm - A simple example



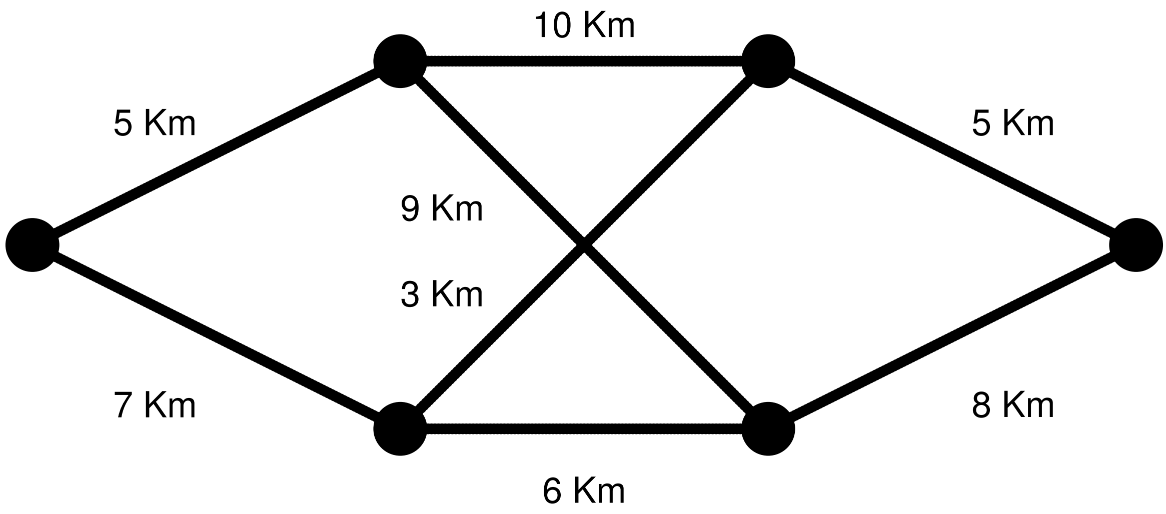
- Fourth step
  - Going back through the survival path
    - ★ Identification of the only survival path (black)
    - ★ Remove branches linking forward non-survival paths (red)

Viterbi algorithm - A simple example



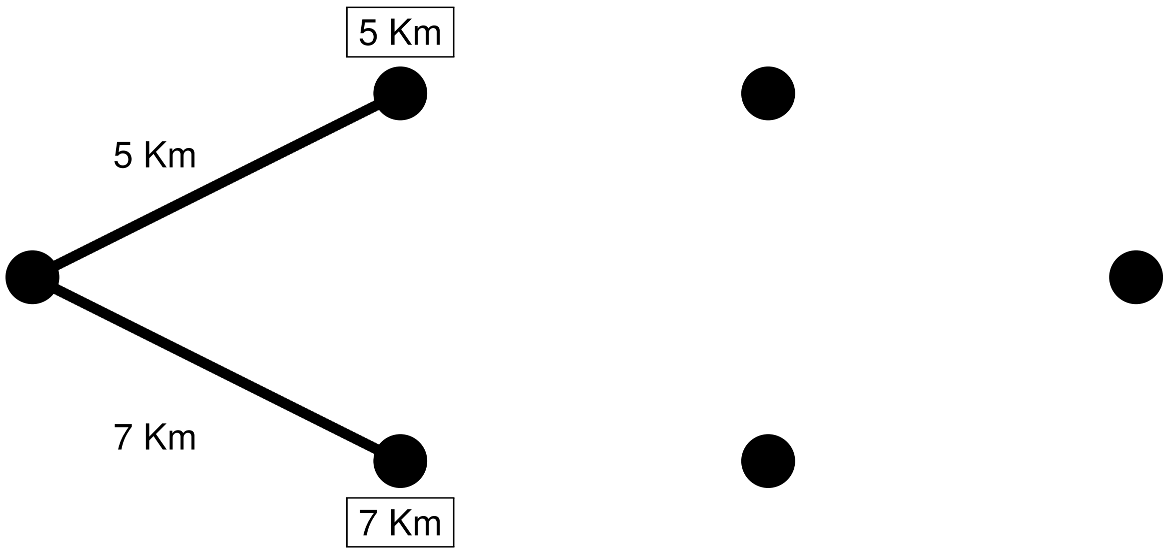
- Final step
  - The shortest path is identified

Viterbi algorithm - A simple example - Different metrics



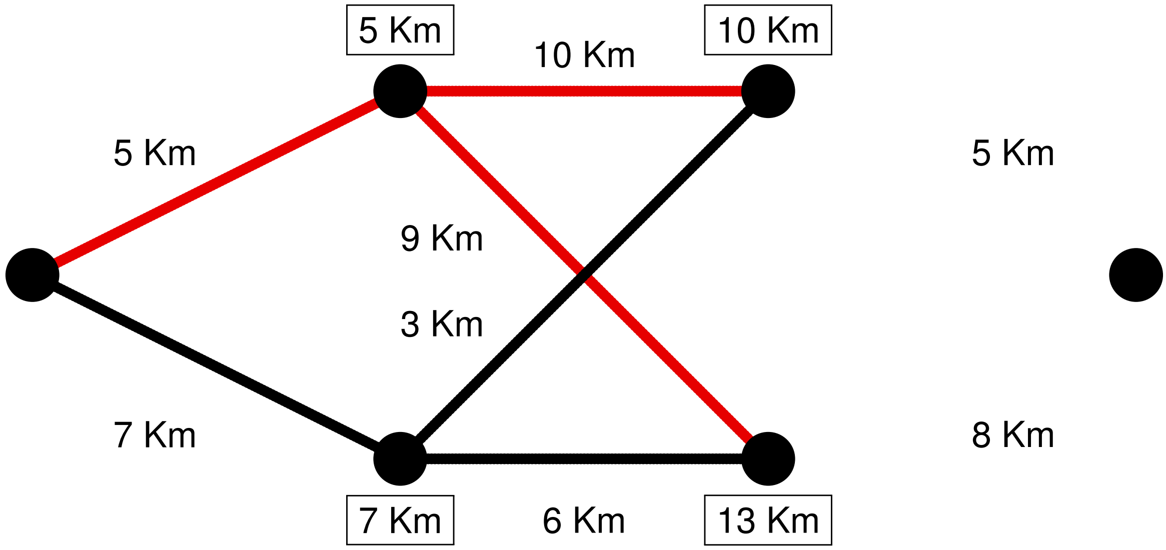
- In previous example, identification of each branch of the survival path requires to process up to the final node
  - Partial paths can be identified previously under some conditions

Viterbi algorithm - A simple example - Different metrics



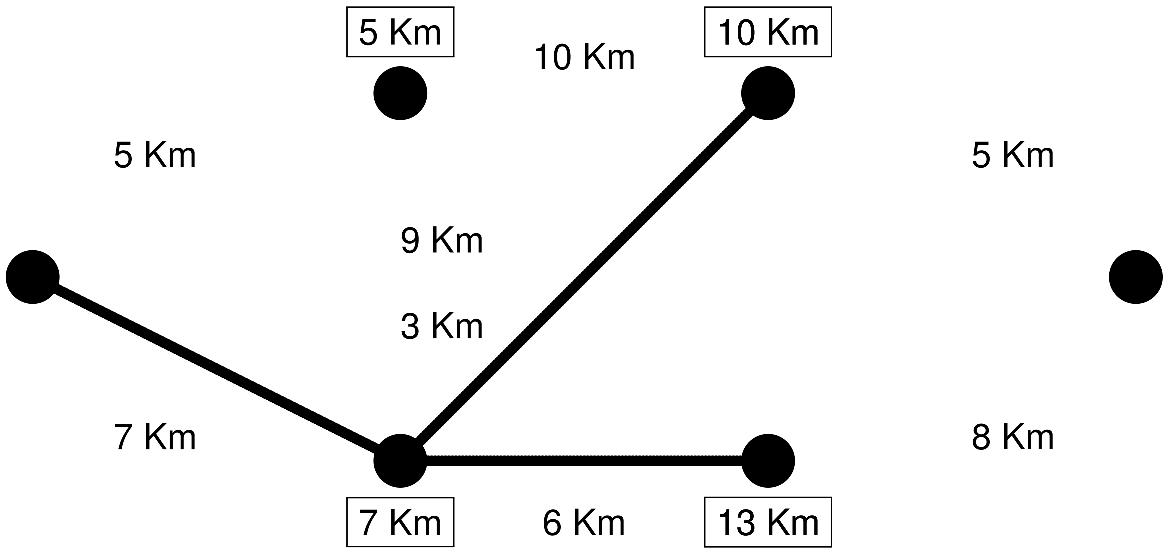
- First step
  - Computing the metrics at first stage after opening the trellis

Viterbi algorithm - A simple example - Different metrics



- Second step
  - Computing survival paths (black) for each node in next stage
    - ★ Upper node: 7 Km + 3 Km is shorter than 5 Km + 10 Km
    - ★ Lower node: 7 Km + 6 Km is shorter than 5 Km + 9 Km

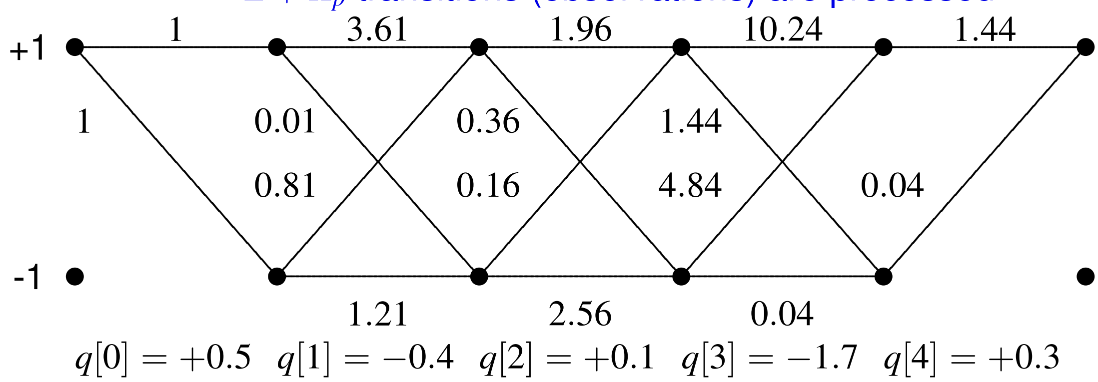
Viterbi algorithm - A simple example - Different metrics



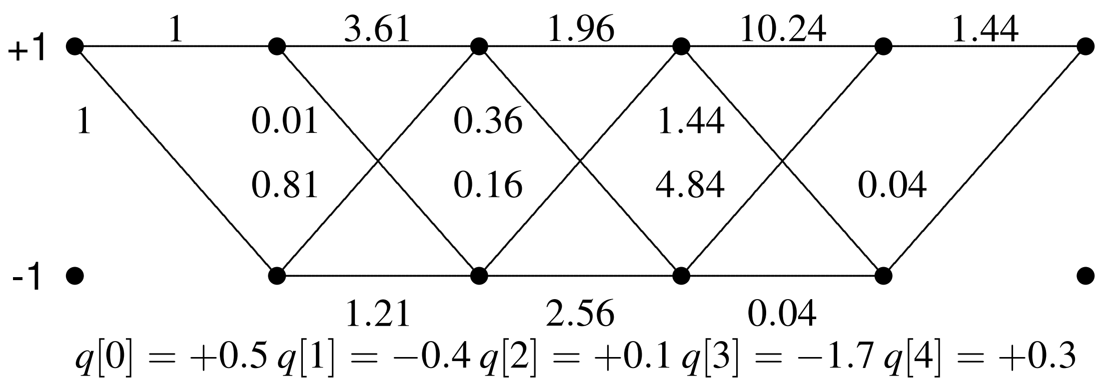
- At this point the survival path are fused at the lowest node of first stage (the one with metric 7 Km)
  - We now know which is the first branch of the shortest path !!!
  - This is known without processing the last stage

Viterbi applied to ISI receiver - Example

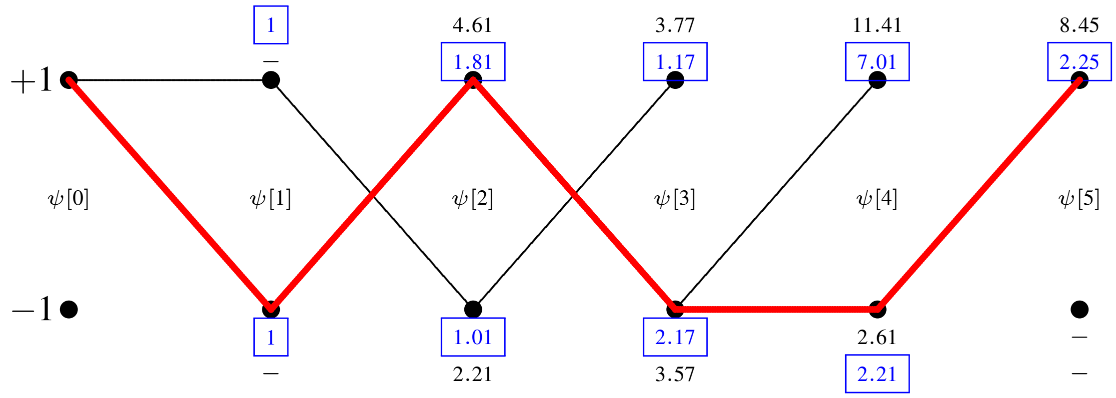
- Previous example with the following parameters
  - Equivalent discrete channel with memory $K_p = 1$
  - Sequence to be decoded: length $L = 4$ symbols
    $$A[0],\ A[1],\ A[2],\ A[3]$$
  - Cyclic header of K_p symbols determining initial ($\psi[0]$) and final ($\psi[K_p + L]$) states
    $$\text{Header } A[-1] = A[L] = +1 \rightarrow \psi[0] = +1, \psi[5] = +1$$
  - Branch metrics for each observation ($|q[n] - o[n]|^2$)
    - ⋆ $L + K_p$ transitions (observations) are processed



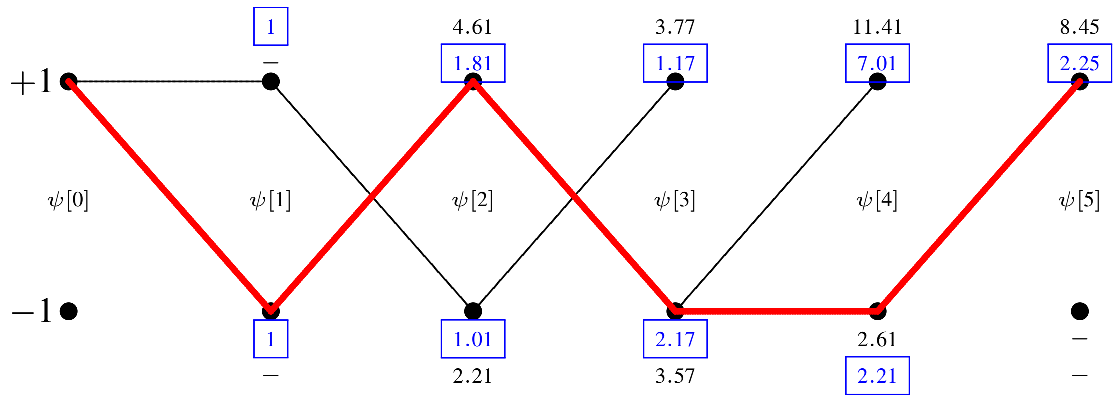
Viterbi applied to ISI receiver - Example (II)



- Application: survival paths and accumulated metrics for each state



Viterbi applied to ISI receiver - Example (III)



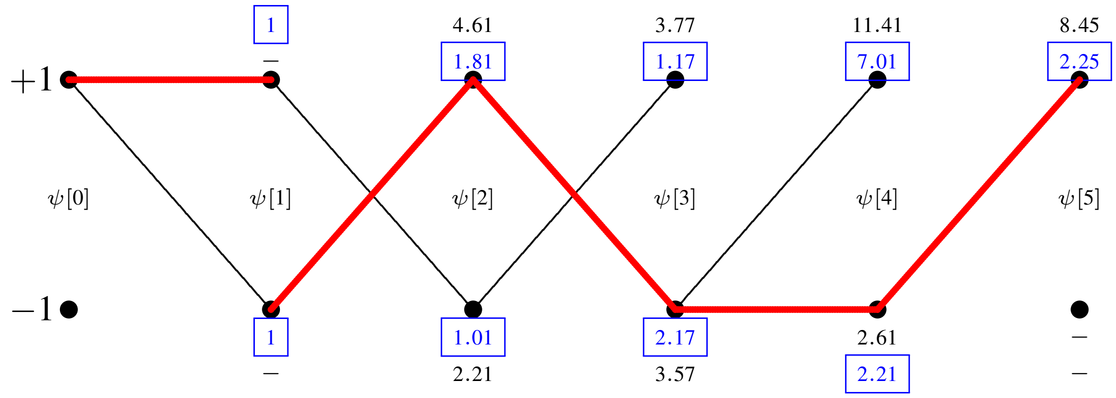
- Metric of survival path for each state is highlighted
- Path associated to maximum likelihood sequences is in red
  - This defines the associated transmitted sequence

$$\hat{A}[0] = -1,\ \hat{A}[1] = +1,\ \hat{A}[2] = -1,\ \hat{A}[3] = -1$$

Truncated Viterbi algorithm

- To ensure proper decoding of L data symbols
  - Initial and final states have to be known
    - Cyclic header of K_p known symbols between data sequences of L symbols (which defines initial and final state)
    - $L + K_p$ transitions of trellis are processed to decode L data symbols
    - Delay and memory constraints
- Decision for symbol $A[n]$ before processing the $L + K_p$ observations requires the merging of all survival paths at $\psi[n+1]$
  - Introduction of arbitrary delay
  - Need for information storing
- Truncated algorithm with truncation length (depth) d
  - After processing observation at discrete instant n (transition from $\psi[n]$ and $\psi[n+1]$ in the trellis), a decision is made for estimation of symbol $A[n-d]$
    - Choice of symbol associated to survival path with lowest length

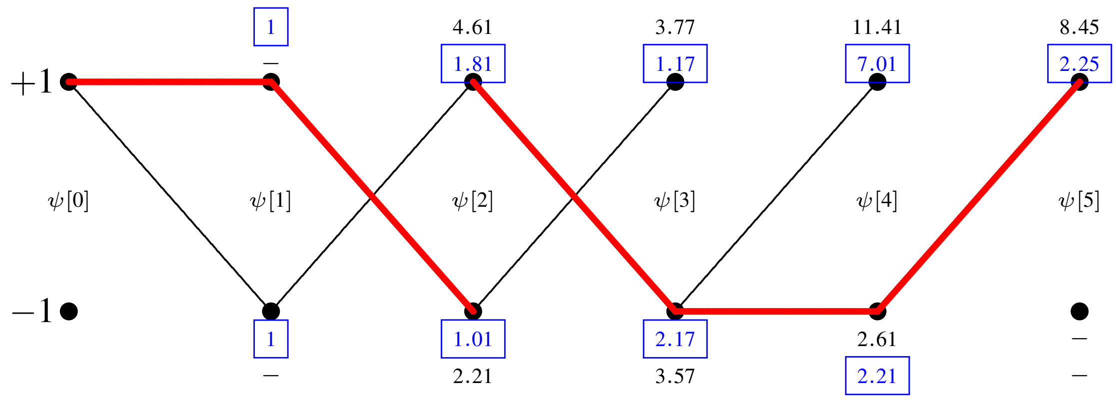
Truncated Viterbi - Example $d = 2$



- Metrics of survival paths are maintained
- Decided sequence may not be the ML sequence
  - Discontinuity in the path trough the trellis

$$\hat{A}[0] = +1,\ \hat{A}[1] = +1,\ \hat{A}[2] = -1,\ \hat{A}[3] = -1$$

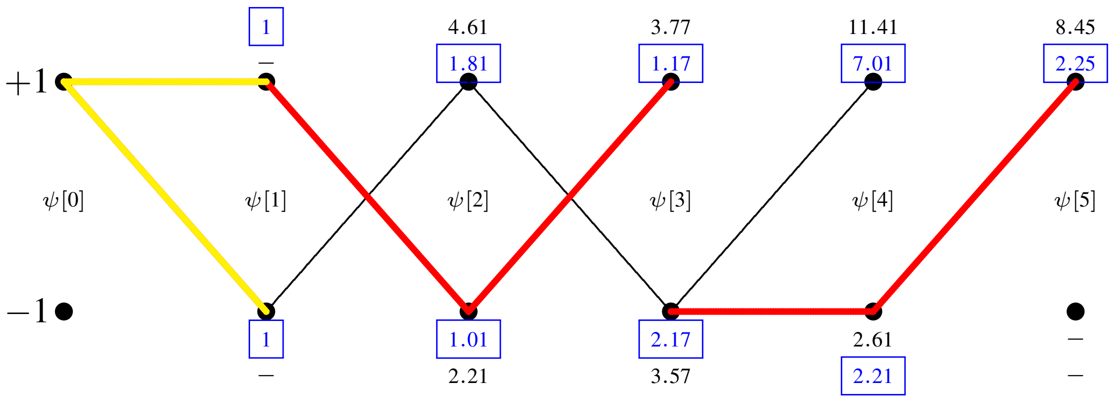
Truncated Viterbi - Example $d = 1$



- Metrics of survival paths are maintained
- Decided sequence may not be the ML sequence
  - Discontinuity in the path trough the trellis

$$\hat{A}[0] = +1,\ \hat{A}[1] = -1,\ \hat{A}[2] = -1,\ \hat{A}[3] = -1$$

Truncated Viterbi - Example $d = 0$



- Metrics of survival paths are maintained
- Decided sequence may not be the ML sequence
  - Discontinuity in the path trough the trellis

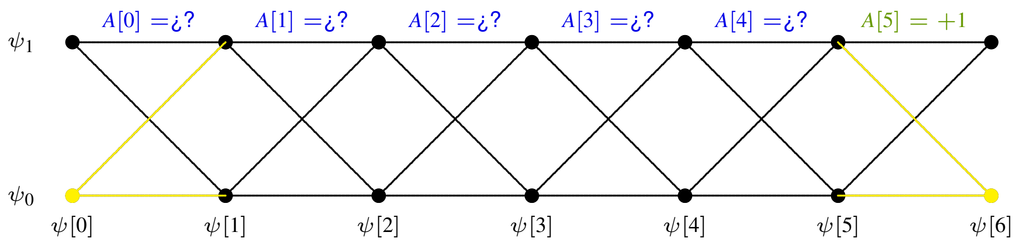
$$\hat{A}[0] = \pm 1,\ \hat{A}[1] = -1,\ \hat{A}[2] = +1,\ \hat{A}[3] = -1$$

REMARK: For $\hat{A}[0]$ both values have same likelihood (random decision)

Detection of a sequence of L symbols

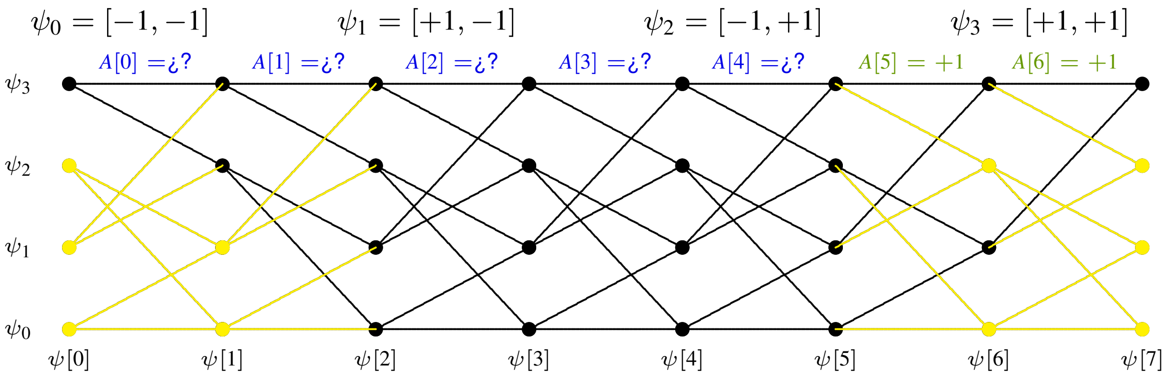
- Equivament discrete channel with memory K_p
  - $K_p + L$ observations are processed
    - ⋆ $K_p + L$ transitions through the trellis
  - Cyclic header of K_p symbols is transmitted
    - ⋆ Determines the initial estate
    - ⋆ Determines the final state
- Example for $K_p = 1$, $L = 5$ and 2-PAM $A[n] \in \{\pm 1\}$
  Sequence $A[0], A[1], A[2], A[3], A[4]$
  - Header:$[+1]$
    $A[-1] = A[5] = +1$
    - ⋆ Initial and final states: $\psi[0] = \psi[K_p + L] = +1$

$\psi_0 = -1$ $\psi_1 = +1$



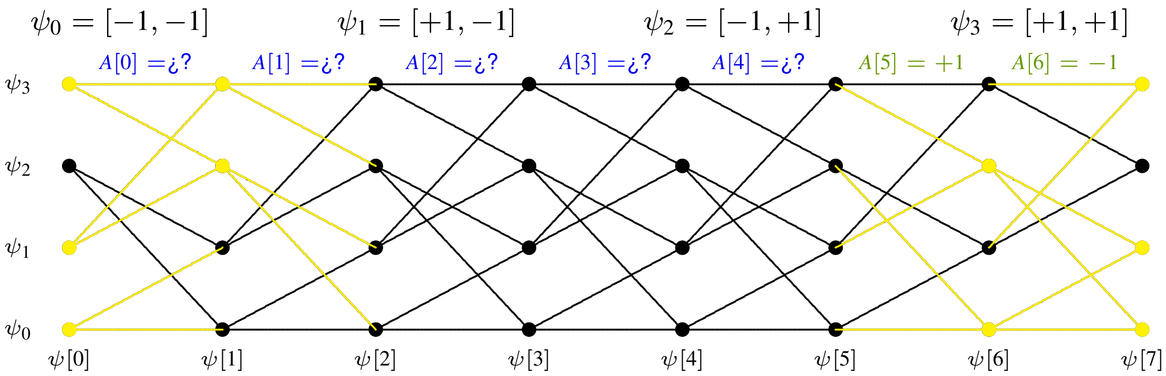
Detection of a sequence of L symbols

- Example for $K_p = 2$, $L = 5$ and 2-PAM $A[n] \in \{\pm 1\}$
  Sequence $A[0], A[1], A[2], A[3], A[4]$
  - Header: $[+1, +1]$
    $A[-2] = A[-1] = A[5] = A[6] = +1$
    - Initial and final states: $\psi[0] = \psi[K_p + L] = [+1, +1] \equiv \psi_3$



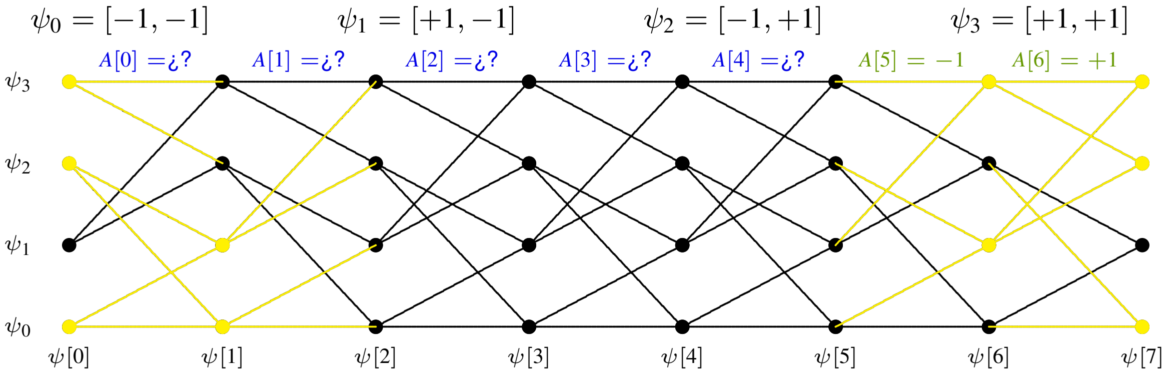
Detection of a sequence of L symbols

- Example for $K_p = 2$, $L = 5$ and 2-PAM $A[n] \in \{\pm 1\}$
  Sequence $A[0], A[1], A[2], A[3], A[4]$
  - Header: $[+1, -1]$
    $A[-2] = +1, = A[-1] = -1, \; A[5] = +1, \; A[6] = -1$
    - Initial and final states: $\psi[0] = \psi[K_p + L] = [-1, +1] \equiv \psi_2$



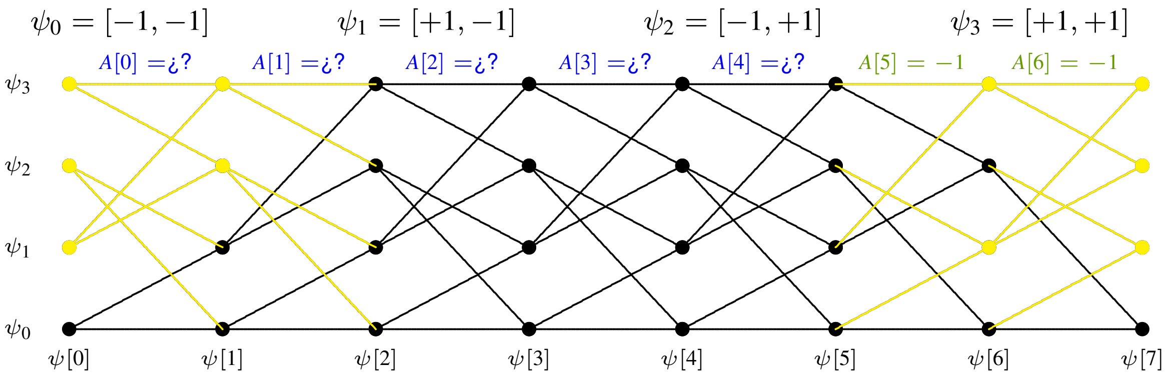
Detection of a sequence of L symbols

- Example for $K_p = 2$, $L = 5$ and 2-PAM $A[n] \in \{\pm 1\}$
  Sequence $A[0], A[1], A[2], A[3], A[4]$
  - Header: $[-1, +1]$
    $A[-2] = -1, \ = A[-1] = +1, \ A[5] = -1, \ A[6] = +1$
    - Initial and final states: $\psi[0] = \psi[K_p + L] = [+1, -1] \equiv \psi_1$



Detection of a sequence of L symbols

- Example for $K_p = 2$, $L = 5$ and 2-PAM $A[n] \in \{\pm 1\}$
  Sequence $A[0], A[1], A[2], A[3], A[4]$
  - Header: $[-1, -1]$
    $A[-2] = A[-1] = A[5] = A[6] = -1$
    - Initial and final states: $\psi[0] = \psi[K_p + L] = [-1, -1] \equiv \psi_0$

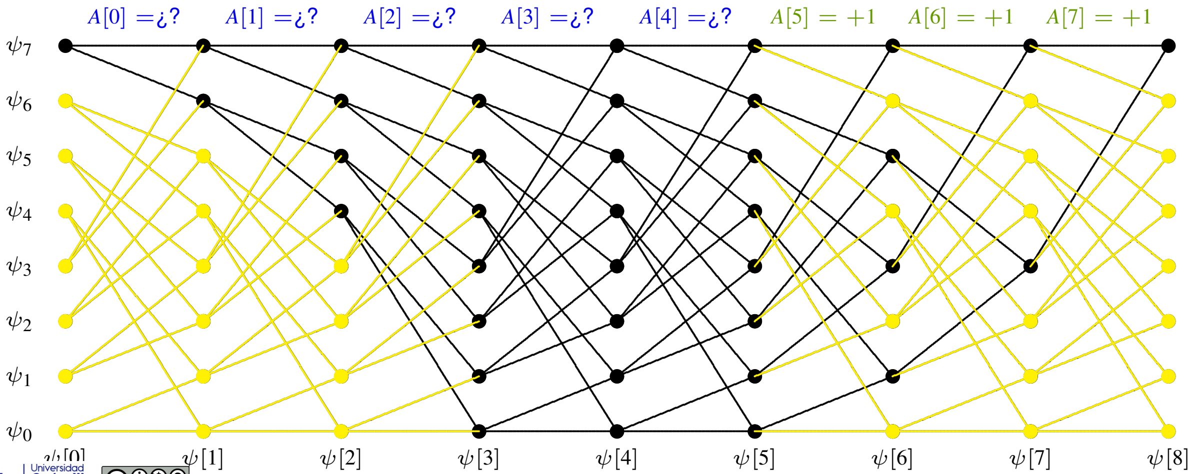


Detection of a sequence of L symbols

- Example for $K_p = 3$, $L = 5$ and 2-PAM $A[n] \in \{\pm 1\}$
  Sequence $A[0], A[1], A[2], A[3], A[4]$
  - Header: $[+1, +1, +1]$
    $A[-3] = A[-2] = A[-1] = A[5] = A[6] = A[7] + 1$
    - Initial and final states: $\psi[0] = \psi[K_p + L] = [+1, +1, +1] \equiv \psi_7$

$\psi_4 = [-1, -1, +1]$ $\psi_5 = [+1, -1, +1]$ $\psi_6 = [-1, +1, +1]$ $\psi_7 = [+1, +1, +1]$

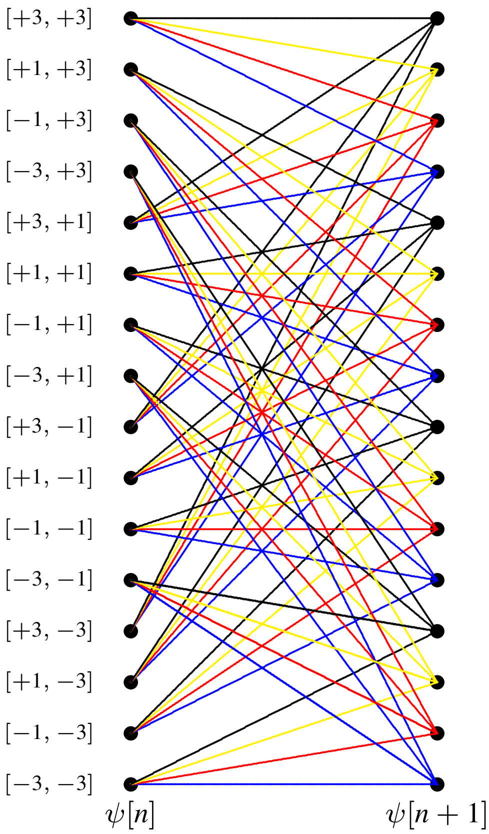
$\psi_0 = [-1, -1, -1]$ $\psi_1 = [+1, -1, -1]$ $\psi_2 = [-1, +1, -1]$ $\psi_3 = [+1, +1, -1]$



Trellis for $M = 4$ in a channel with memory $K_p = 2$

$A[n]$	$A[n-1]$	$A[n-2]$	$o[n]$
+3	+3	+3	a_0
+1	+3	+3	a_1
−1	+3	+3	a_2
−3	+3	+3	a_3
+3	+1	+3	b_0
+1	+1	+3	b_1
−1	+1	+3	b_2
−3	+1	+3	b_3
+3	−1	+3	c_0
+1	−1	+3	c_1
−1	−1	+3	c_2
−3	−1	+3	c_3
+3	−3	+3	d_0
+1	−3	+3	d_1
−1	−3	+3	d_2
−3	−3	+3	d_3
+3	+3	+1	e_0
+1	+3	+1	e_1
−1	+3	+1	e_2
−3	+3	+1	e_3
+3	+1	+1	f_0
+1	+1	+1	f_1
−1	+1	+1	f_2
−3	+1	+1	f_3
+3	−1	+1	g_0
+1	−1	+1	g_1
−1	−1	+1	g_2
−3	−1	+1	g_3
+3	−3	+1	h_0
+1	−3	+1	h_1
−1	−3	+1	h_2
−3	−3	+1	h_3

$A[n]$	$A[n-1]$	$A[n-2]$	$o[n]$
+3	+3	−1	i_0
+1	+3	−1	i_1
−1	+3	−1	i_2
−3	+3	−1	i_3
+3	+1	−1	j_0
+1	+1	−1	j_1
−1	+1	−1	j_2
−3	+1	−1	j_3
+3	−1	−1	k_0
+1	−1	−1	k_1
−1	−1	−1	k_2
−3	−1	−1	k_3
+3	−3	−1	l_0
+1	−3	−1	l_1
−1	−3	−1	l_2
−3	−3	−1	l_3
+3	+3	−3	m_0
+1	+3	−3	m_1
−1	+3	−3	m_2
−3	+3	−3	m_3
+3	+1	−3	n_0
+1	+1	−3	n_1
−1	+1	−3	n_2
−3	+1	−3	n_3
+3	−1	−3	o_0
+1	−1	−3	o_1
−1	−1	−3	o_2
−3	−1	−3	o_3
+3	−3	−3	p_0
+1	−3	−3	p_1
−1	−3	−3	p_2
−3	−3	−3	p_3



Probability of error - Erroneous event

- A path through the trellis (associated to a given symbol sequence) can be described by a sequence of states through the trellis

$$\boldsymbol{\psi} = [\psi[0], \psi[1], \psi[2], \cdots]$$

- Erroneous event: different sequence of states for transmitted sequence and detected sequence

$$\boldsymbol{e} = (\boldsymbol{\psi}, \hat{\boldsymbol{\psi}})$$

- Each erroneous event has two associated parameters
  - Length of the event ($\ell(\boldsymbol{e})$)
  - Number of erroneous symbols in the event ($w(\boldsymbol{e})$)
- Definition of erroneous event of length $\ell(\boldsymbol{e})$
  Number of different states between two paths: example, $\ell(\boldsymbol{e}) = \ell$
  - $\psi[m] = \hat{\psi}[m]$
  - $\psi[m + \ell + 1] = \hat{\psi}[m + \ell + 1]$
  - $\psi[n] \neq \hat{\psi}[n]$ para $m < n \leq m + \ell$
- The number of errors of an event, $w(\boldsymbol{e})$, fulfills $1 \leq w(\boldsymbol{e}) \leq \ell$

Probability of detecting an erroneous sequence

- Approximation

$$P\{\text{erroneous sequence}\} \approx k\ Q\left(\frac{D_{min}/2}{\sqrt{N_0/2}}\right)$$

  - D_{min}: minimum euclidean distance between noiseless outputs of two different symbol sequences $\{o_i[n], o_j[n]\},\ j \neq i$
  - k: maximum number of sequences whose noiseless outputs are at distance D_{min} of the noiseless output of the noiseless output of a symbol sequence
- When L grows up, k grows up, and therefore the probability of detecting an erroneous sequence goes to infinity when L grows up

Probability of symbol errors

- In general it is more usefull to estimate

$$P_e = P\{\hat{A}[n] \neq A[n]\}$$

- Definition for ML sequence detection

$$P_e = \frac{1}{L} \sum_{\boldsymbol{e} \in \mathcal{E}} w(\boldsymbol{e})\ P\{\boldsymbol{e}\}$$

  - $P\{\boldsymbol{e}\}$: the probability of erroneous event $\boldsymbol{e}$
  - $\mathcal{E}$: set of all erroneous events that can happen in the trellis

- Probability of an erroneous event $\boldsymbol{e} = (\psi, \hat{\psi})$

$$P\{\boldsymbol{e}\} = P\{\hat{\psi}|\psi\}\ P\{\psi\}$$

  - Difficult to evaluate $\rightarrow$ Bounds and approximations for P_e

Bounds and approximation for P_e

- Bounds for the probability error for symbols

$$k_2\ Q\left(\frac{D_{min}}{2\sqrt{N_0/2}}\right) \leq P_e \leq k_1\ Q\left(\frac{D_{min}}{2\sqrt{N_0/2}}\right)$$

  - k_2 is the ratio of trellis paths having associated an erroneous event at distance D_{min}. It always fulfills $k_2 \leq 1$
  - k_1 averages the number of errors produced in the erroneous events with minimum distance $k_1 = \sum_{\boldsymbol{e} \in \mathcal{E}_{min}} w(\boldsymbol{e})\ P\{\psi\}$

- Aproximation for P_e

$$P_e \approx k_0\ Q\left(\frac{D_{min}}{2\sqrt{N_0/2}}\right)$$

  - k_0: constant such that $k_2 \leq k_0 \leq k_1$.
  - Both k_1 and k_2 are independent of the noise variance

Minimum euclidean distance with respect to the noiseless output of a given sequence

- Reference sequence $\boldsymbol{A} = \boldsymbol{A}_i$

$$D_{min}(\boldsymbol{A}_i) = \min_{\substack{\boldsymbol{A}_j \\ j \neq i}} \sqrt{\sum_{n=0}^{N_q-1} \left| o_i[n] - \underbrace{\sum_{k=0}^{K_p} p[k]\, A_j[n-k]}_{o_j[n]} \right|^2}$$

- It can be found by using Viterbi algorithm
  - Branch metric: $|o_i[n] - o_j[n]|^2$
  - Reference: $o_i[n]$
  - Algorithm looks for the erroneous event including the reference sequence having lowest distance

Minimum distance D_{min}

- It is the minimum value of $D_{min}(\boldsymbol{A}_i)$, for $i = 0, 1, \cdots, M^L - 1$

$$D_{min} = \min_{\substack{\boldsymbol{A}_i, \boldsymbol{A}_j \\ j \neq i}} \sqrt{\sum_{n=0}^{N_q-1} \left| \sum_{k=0}^{K_p} p[k]\ (A_i[n-k] - A_j[n-k]) \right|^2}$$

  i.e., the minimum distance between the noiseless output of any two different sequences

- If trellis diagram is symmetric
  - Calculation using a single reference sequence
- In general, D_{min} depends on $\left| \sum_{k=0}^{K_p} p[k]\ (A_i[n-k] - A_j[n-k]) \right|^2$
  - Noiseless output if transmitted sequence is $\xi[n] = A_i[n] - A_j[n]$
- A trellis diagram can be defined using this error constellation

$$\xi[n] = A_i[n] - A_j[n]$$

  - Reference: sequence of zeros (associated to sequence detection without errors)

Example A ($L = 4$, $K_p = 1$)

$q[0]$	$q[1]$	$q[2]$	$q[3]$	$q[4]$
$+0.5$	-0.4	$+0.1$	-1.7	$+0.3$

$$o[n] = A[n] + \frac{1}{2}A[n-1], \quad \text{Likelihood Metric} = \sum_{n=0}^{N_q-1} |q[n] - o[n]|^2$$

$A[0]$	$A[1]$	$A[2]$	$A[3]$	$o[0]$	$o[1]$	$o[2]$	$o[3]$	$o[4]$	Likelihood Metric
$+1$	$+1$	$+1$	$+1$	$+1.5$	$+1.5$	$+1.5$	$+1.5$	$+1.5$	18.25
-1	$+1$	$+1$	$+1$	-0.5	$+0.5$	$+1.5$	$+1.5$	$+1.5$	15.45
$+1$	-1	$+1$	$+1$	$+1.5$	-0.5	$+0.5$	$+1.5$	$+1.5$	12.85
-1	-1	$+1$	$+1$	-0.5	-1.5	$+0.5$	$+1.5$	$+1.5$	14.05
$+1$	$+1$	-1	$+1$	$+1.5$	$+1.5$	-0.5	$+0.5$	$+1.5$	11.25
-1	$+1$	-1	$+1$	-0.5	$+0.5$	-0.5	$+0.5$	$+1.5$	8.45
$+1$	-1	-1	$+1$	$+1.5$	-0.5	-1.5	$+0.5$	$+1.5$	9.85
-1	-1	-1	$+1$	-0.5	-1.5	-1.5	$+0.5$	$+1.5$	11.05
$+1$	$+1$	$+1$	-1	$+1.5$	$+1.5$	$+1.5$	-0.5	$+0.5$	8.05
-1	$+1$	$+1$	-1	-0.5	$+0.5$	$+1.5$	-0.5	$+0.5$	5.25
$+1$	-1	$+1$	-1	$+1.5$	-0.5	$+0.5$	-0.5	$+0.5$	2.65
-1	-1	$+1$	-1	-0.5	-1.5	$+0.5$	-0.5	$+0.5$	3.85
$+1$	$+1$	-1	-1	$+1.5$	$+1.5$	-0.5	-1.5	$+0.5$	5.05
-1	$+1$	-1	-1	-0.5	$+0.5$	-0.5	-1.5	$+0.5$	2.25
$+1$	-1	-1	-1	$+1.5$	-0.5	-1.5	-1.5	$+0.5$	3.65
-1	-1	-1	-1	-0.5	-1.5	-1.5	-1.5	$+0.5$	4.85

Example A - Calculating D_{min} by brute force

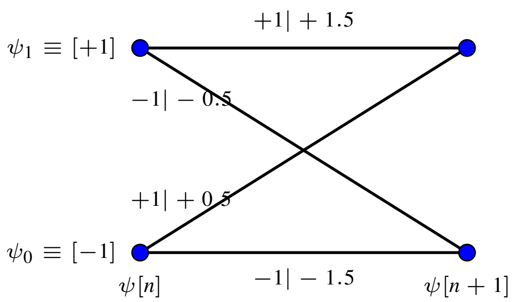
i/j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	5	5	14	5	10	14	23	5	10	10	19	14	19	23	32
1	5	0	6	5	10	5	15	14	10	5	11	10	19	14	24	23
2	5	6	0	5	6	7	5	10	10	11	5	10	15	16	14	19
3	14	5	5	0	15	6	10	5	19	10	10	5	24	15	19	14
4	5	10	6	15	0	5	5	14	6	11	7	16	5	10	10	19
5	10	5	7	6	5	0	6	5	11	6	8	7	10	5	11	10
6	14	15	5	10	5	6	0	5	15	16	6	11	10	11	5	10
7	23	14	10	5	14	5	5	0	24	15	11	6	19	10	10	5
8	5	10	10	19	6	11	15	24	0	5	5	14	5	10	14	23
9	10	5	11	10	11	6	16	15	5	0	6	5	10	5	15	14
10	10	11	5	10	7	8	6	11	5	6	0	5	6	7	5	10
11	19	10	10	5	16	7	11	6	14	5	5	0	15	6	10	5
12	14	19	15	24	5	10	10	19	5	10	6	15	0	5	5	14
13	19	14	16	15	10	5	11	10	10	5	7	6	5	0	6	5
14	23	24	14	19	10	11	5	10	14	15	5	10	5	6	0	5
15	32	23	19	14	19	10	10	5	23	14	10	5	14	5	5	0

$$\text{Value for } \sum_{n=0}^{N_q-1} |o_i[n] - o_j[n]|^2$$

$$D_{min}(\boldsymbol{A}_0) = \sqrt{5}$$

$$D_{min} = \sqrt{5}$$

Calculating D_{min} using the trellis - Example A



Header defines initial/final state: $[+1]$

Reference sequence: $o_{ref}[n]$

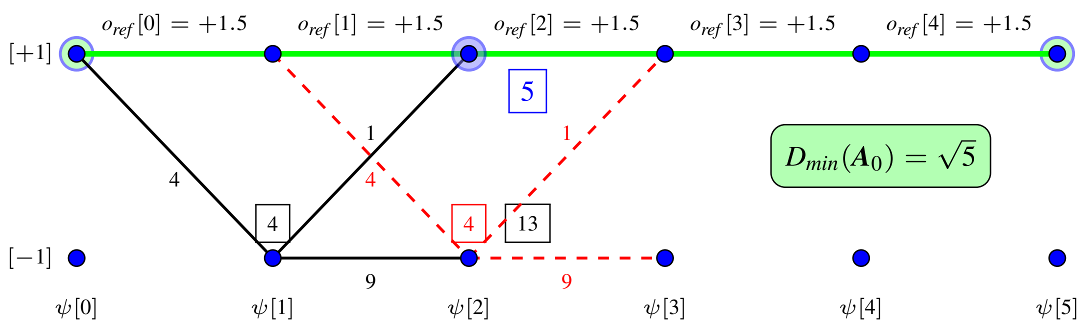
D_{min}: branch metric

$$|o_{ref}[n] - o_i[n]|^2$$

$|+1.5-(-0.5)|^2 = 4$

$|+1.5-(+0.5)|^2 = 1$

$|+1.5-(-1.5)|^2 = 9$



Trellis for error constellation - Example A

- Error constellation for a 2-PAM

$$\xi[n] \in \{+2, 0, -2\}$$

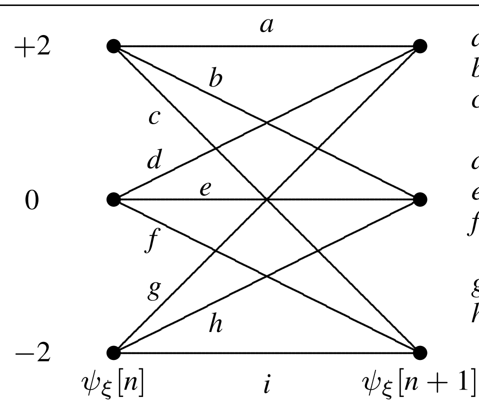
- Noiseless output for $p[n] = \delta[n] + \frac{1}{2}\,\delta[n-1]$

$$o_\xi[n] = p[n] * \xi[n] = \xi[n] + \frac{1}{2}\,\xi[n-1]$$

- Definition of states over the error constellation

$$\psi_\xi[n] = \xi[n-1],\ \psi_\xi[n+1] = \xi[n]$$

$\xi[n]$	$\xi[n-1]$	$o_\xi[n]$
$+2$	$+2$	$+3$
$+0$	$+2$	$+1$
-2	$+2$	-1
$+2$	$+0$	$+2$
$+0$	$+0$	$+0$
-2	$+0$	-2
$+2$	-2	$+1$
$+0$	-2	-1
-2	-2	-3



$a \equiv +2 \mid +3$
$b \equiv +0 \mid +1$
$c \equiv -2 \mid -1$

$d \equiv +2 \mid +2$
$e \equiv +0 \mid +0$
$f \equiv -2 \mid -2$

$g \equiv +2 \mid +1$
$h \equiv +0 \mid -1$
$i \equiv -2 \mid -3$

Example B ($L = 4$, $K_p = 2$)

$q[0]$	$q[1]$	$q[2]$	$q[3]$	$q[4]$	$q[5]$
-0.24	-1.15	-1.75	$+0.26$	$+1.27$	$+1.55$

$$o[n] = A[n] + \frac{1}{2}A[n-1] + \frac{1}{4}A[n-2], \quad \text{Likelihood Metric} = \sum_{n=0}^{N_q-1} |q[n] - o[n]|^2$$

$A[0]$	$A[1]$	$A[2]$	$A[3]$	$o[0]$	$o[1]$	$o[2]$	$o[3]$	$o[4]$	$o[5]$	L. Metric
$+1$	$+1$	$+1$	$+1$	1.75	1.75	1.75	1.75	1.75	1.75	27.1106
-1	$+1$	$+1$	$+1$	-0.25	0.75	1.25	1.75	1.75	1.75	15.1006
$+1$	-1	$+1$	$+1$	1.75	-0.25	0.75	1.25	1.75	1.75	12.2706
-1	-1	$+1$	$+1$	-0.25	-1.25	0.25	1.25	1.75	1.75	5.2606
$+1$	$+1$	-1	$+1$	1.75	1.75	-0.25	0.75	1.25	1.75	14.9006
-1	$+1$	-1	$+1$	-0.25	0.75	-0.75	0.75	1.25	1.75	4.8906
$+1$	-1	-1	$+1$	1.75	-0.25	-1.25	0.25	1.25	1.75	5.0606
-1	-1	-1	$+1$	-0.25	-1.25	-1.75	0.25	1.25	1.75	0.0506
$+1$	$+1$	$+1$	-1	1.75	1.75	1.75	-0.25	0.75	1.25	25.2406
-1	$+1$	$+1$	-1	-0.25	0.75	1.25	-0.25	0.75	1.25	13.2306
$+1$	-1	$+1$	-1	1.75	-0.25	0.75	-0.75	0.75	1.25	12.4006
-1	-1	$+1$	-1	-0.25	-1.25	0.25	-0.75	0.75	1.25	5.3906
$+1$	$+1$	-1	-1	1.75	1.75	-0.25	-1.25	0.25	1.25	18.0306
-1	$+1$	-1	-1	-0.25	0.75	-0.75	-1.25	0.25	1.25	8.0206
$+1$	-1	-1	-1	1.75	-0.25	-1.25	-1.75	0.25	1.25	10.1906
-1	-1	-1	-1	-0.25	-1.25	-1.75	-1.75	0.25	1.25	5.1806

Example B - Computation of D_{min} by brute force

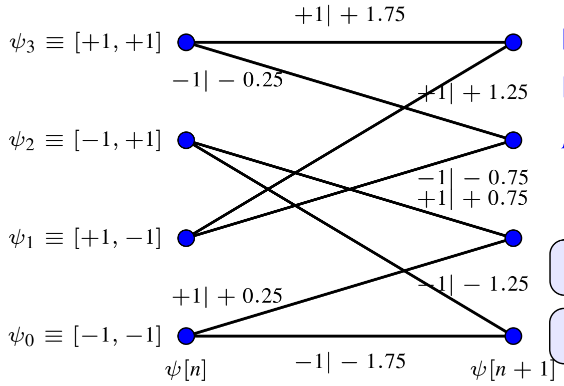
i/j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	5.25	5.25	15.5	5.25	12.5	15.5	27.75	5.25	10.5	12.5	22.75	15.5	22.75	27.75	40
1	5.25	0	5.5	5.25	8.5	5.25	13.75	15.5	10.5	5.25	12.75	12.5	18.75	15.5	26	27.75
2	5.25	5.5	0	5.25	5.5	7.75	5.25	12.5	8.5	8.75	5.25	10.5	13.75	16	15.5	22.75
3	15.5	5.25	5.25	0	13.75	5.5	8.5	5.25	18.75	8.5	10.5	5.25	22	13.75	18.75	15.5
4	5.25	8.5	5.5	13.75	0	5.25	5.25	15.5	5.5	8.75	7.75	16	5.25	10.5	12.5	22.75
5	12.5	5.25	7.75	5.5	5.25	0	5.5	5.25	12.75	5.5	10	7.75	10.5	5.25	12.75	12.5
6	15.5	13.75	5.25	8.5	5.25	5.5	0	5.25	13.75	12	5.5	8.75	8.5	8.75	5.25	10.5
7	27.75	15.5	12.5	5.25	15.5	5.25	5.25	0	26	13.75	12.75	5.5	18.75	8.5	10.5	5.25
8	5.25	10.5	8.5	18.75	5.5	12.75	13.75	26	0	5.25	5.25	15.5	5.25	12.5	15.5	27.75
9	10.5	5.25	8.75	8.5	8.75	5.5	12	13.75	5.25	0	5.5	5.25	8.5	5.25	13.75	15.5
10	12.5	12.75	5.25	10.5	7.75	10	5.5	12.75	5.25	5.5	0	5.25	5.5	7.75	5.25	12.5
11	22.75	12.5	10.5	5.25	16	7.75	8.75	5.5	15.5	5.25	5.25	0	13.75	5.5	8.5	5.25
12	15.5	18.75	13.75	22	5.25	10.5	8.5	18.75	5.25	8.5	5.5	13.75	0	5.25	5.25	15.5
13	22.75	15.5	16	13.75	10.5	5.25	8.75	8.5	12.5	5.25	7.75	5.5	5.25	0	5.5	5.25
14	27.75	26	15.5	18.75	12.5	12.75	5.25	10.5	15.5	13.75	5.25	8.5	5.25	5.5	0	5.25
15	40	27.75	22.75	15.5	22.75	12.5	10.5	5.25	27.75	15.5	12.5	5.25	15.5	5.25	5.25	0

$$\text{Value for } \sum_{n=0}^{N_q-1} |o_i[n] - o_j[n]|^2$$

$$D_{min}(\mathbf{A}_0) = \sqrt{5.25}$$

$$D_{min} = \sqrt{5.25}$$

Calculating D_{min} using the trellis - Example B



Header defines initial/final state: $[+1,+1]$

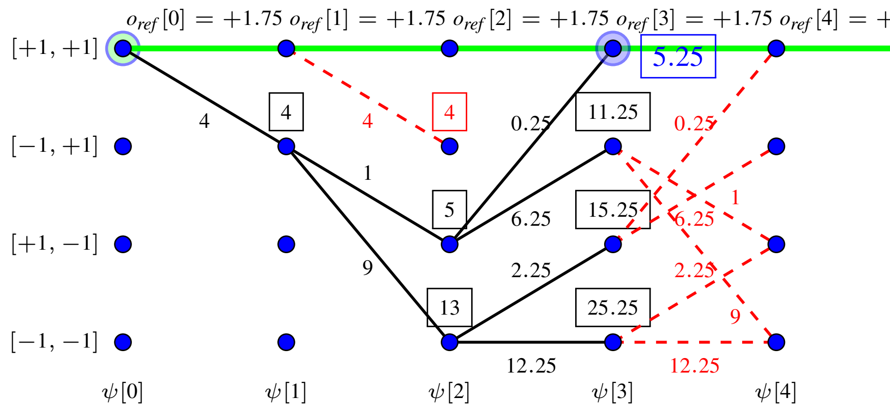
Reference sequence: $o_{ref}[n]$

D_{min}: branch metric

$$\left|o_{ref}[n] - o_i[n]\right|^2$$

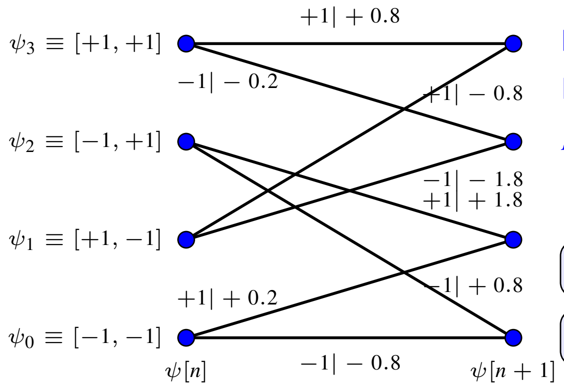
$|+1.75-(+1.25)|^2 = 0.25$ $\quad$ $|+1.75-(+0.25)|^2 = 2.25$

$|+1.75-(-0.75)|^2 = 6.25$ $\quad$ $|+1.75-(-1.75)|^2 = 12.25$



$D_{min}(A_0) = \sqrt{5.25}$

Calculating D_{min} using the trellis - Exercise 2



Header defines initial/final state: $[+1,+1]$

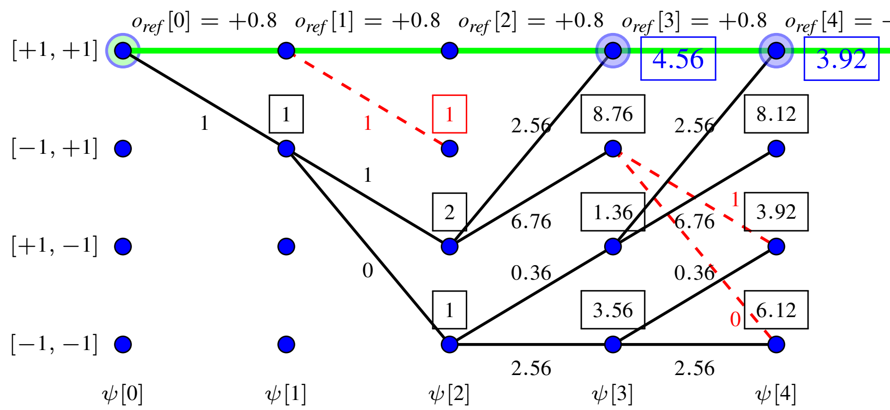
Reference sequence: $o_{ref}[n]$

D_{min}: branch metric

$$\left|o_{ref}[n] - o_i[n]\right|^2$$

$|+0.8-(-0.8)|^2 = 2.56$ $\quad$ $|+0.8-(+0.2)|^2 = 0.36$

$|+0.8-(-1.8)|^2 = 6.76$ $\quad$ $|+0.8-(-0.8)|^2 = 2.56$



$D_{min}(A_0) = \sqrt{3.92}$

Matched filter bound

- Provides a bound for probability of symbol error under ISI

$$P_e \geq k\ Q\left(\frac{d_{min}}{2}\ \frac{||\mathbf{p}||}{\sqrt{N_0/2}}\right)$$

  - k: maximun number of symbols at minimun distance d_{min} of any symbol in the constellation
  - Channel norm is defined as follows: $||\mathbf{p}|| = \sqrt{\sum_{k=0}^{K_p} |p[k]|^2}$
- This allows to bound D_{min}

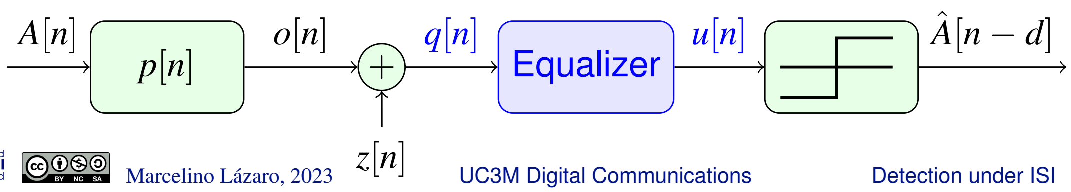
$$D_{min} \leq d_{min}\ ||\mathbf{p}||$$

- The increase in signal to noise ratio required to achieve the same P_e than in a system without ISI is

$$\Delta\text{SNR} = 20 \log_{10} \frac{d_{min}||\mathbf{p}||}{D_{min}}$$

Channel equalizers

- Simplest solution for a receiver under ISI
  - Memoryless symbol-by-symbol detector (optimal delay + re-design of decision regions)
  - Low performance for high level of ISI
- Optimal solution
  - Maximum likelihood sequence detection (MLSD)
  - Complexity of Viterbi algorithm is exponential with M and K_p
    - ★ There are M^{K_p} states
    - ★ M branches go out of each state
    - ★ M branches arrive to each state
- Sub-optimal alternative
  - Channel equalizer + memoryless symbol-by-symbol detector
  - Performance is lower than with MLSD, but higher than with a memoryless symbol-by-symbol detector



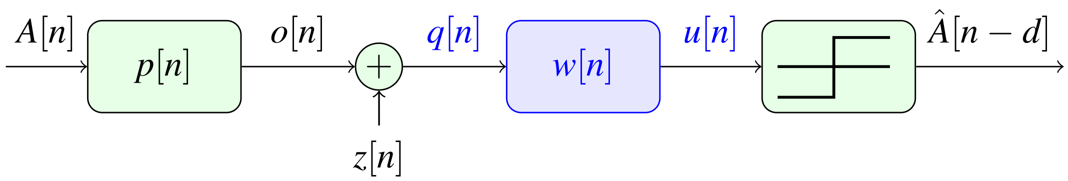
Equalizer structure

- Linear equalizer
  - LTE: *Linear Transversal Equalizer*
  - Equalizer is a linear filter
- Nonlinear equalizer with decision feedback
  - DFE: *Decision Feedback Equalizer*
- Other nonlinear structures
  - Bayesian equalizer
  - Neural networks (MLP. RBF, etc.)
  - Support vector machines
  - ...

Blind / Non-blind equalization

- Non-blind equalization
  - Channel is known ($p[n]$ is known)
  - A reference sequence for transmitted symbols is available
- Blind equalization
  - Channel is unknown
  - A reference sequence for transmitted symbols is not available
  - Available information is reduced to statistical information about $A[n]$

Non-blind linear equalization



- The channel, $p[n]$, is assumed to be known (non-blind)
- Structure of the equalizer (linear)
  - Linear system: causal, with $K_w + 1$ coefficients (memory K_w)

$$u[n] = q[n] * w[n] = \sum_{k=0}^{K_w} w[k]\ q[n-k] = \mathbf{w}^T\ \boldsymbol{q}_n$$

- Criteria for equalizer design (to obtain the equalizer parameters)
  - Zero forcing (ZF)
  - Minimum mean squared error (MMSE)

Linear equalizer

- Equalizer is a causal discrete time linear filter of $K_w + 1$ coefficients, $w[n]$

$$u[n] = q[n] * w[n] = (A[n] * p[n] + z[n]) * w[n]$$

$$u[n] = A[n] * p[n] * w[n] + z[n] * w[n]$$

- Definition: joint channel-equalizer response

$$c[n] = w[n] * p[n], \quad 0 \leq n \leq K_p + K_w$$

Causal response with $K_p + K_w + 1$ coefficients (memory $K_p + K_w$)

$$u[n] = A[n] * c[n] + z[n] * w[n] = \sum_{k=0}^{K_p+K_w} c[k]\ A[n-k] + \sum_{k=0}^{K_w} w[k]\ z[n-k]$$

- Output of the equalizer - delay d in the decision

$$u[n] = \underbrace{c[d]\ A[n-d]}_{\text{desired term}} + \underbrace{\sum_{\substack{k=0\\k\neq d}}^{K_p+K_w} c[k]\ A[n-k]}_{\text{residual ISI}} + \underbrace{\sum_{k=0}^{K_w} w[k]\ z[n-k]}_{\text{filtered noise } z'[n]}$$

Design criteria for linear equalizers

- Zero forcing (ZF) criterion
  - Aims at eliminating intersymbol interference (ISI)
  - Mathematically, this means to look for an ideal joint channel-equalizer response

$$c[n] = p[n] * w[n] = \delta[n-d], \text{ for some arbitrary delay } d$$

Zeros are forced in the joint channel-equalizer response

- Minimum mean squared error (MMSE) criterion
  - Aims at minimizing the joint effect of resigual ISI and filtered noise
  - Error at the output of the equalizer for a delay d

$$e_d[n] = u[n] - A[n-d]$$

Difference between equalizer output and the transmitted symbol (considering d)

  - Mathematically, MMSE minimizes expected energy of observation error $e_d[n]$

$$\text{mín}\, E[|e_d[n]|^2]$$

Design using ZF criterion with unlimited equalizer length

- Ideal response (in time domain)

$$c[n] = p[n] * w[n] = \delta[n-d]$$

- Equalizer ideal response can easily be obtained in frequency domain

$$C(e^{j\omega}) = P(e^{j\omega})\ W(e^{j\omega}) = e^{-j\omega d} \rightarrow W(e^{j\omega}) = \frac{e^{-j\omega d}}{P(e^{j\omega})}$$

- Selection of delay d
  - Decomposition of $P(z)$ in minimum and maximum phase systems

$$P(z) = P_0 \underbrace{\prod_{k=1}^{K_1} \left(1 - \alpha_k\, z^{-1}\right)}_{P_{min}(z)} \underbrace{\prod_{\ell=1}^{K_2} \left(1 - \beta_\ell\, z^{-1}\right)}_{P_{max}(z)}$$

$$|\alpha_k| < 1, \text{ para } 1 \leq k \leq K_1, \ |\beta_\ell| > 1, \text{ para } 1 \leq \ell \leq K_2$$

  - $P_{min}(z)$, a minimum phase system, has an stable causal inverse
  - Stable inverse of $P_{max}(z)$, a maximum phase system, in non-causal
  - Choice of d to have a causal stable inverse of the channel

Design using ZF criterion with unlimited equalizer length (II)

- Procedure to obtain the ZF unlimited linear equalizer:
  - Obtain the inverse (frequency) response of channel $p[n]$

$$W^{(0)}\left(e^{j\omega}\right)=\frac{1}{P\left(e^{j\omega}\right)}$$

  - Obtain its corresponding time response

$$w^{(0)}[n]=\mathcal{FT}^{-1}\left\{W^{(0)}\left(e^{j\omega}\right)\right\}$$

    - ★ If this is a non-causal response, evaluate the length of non-causal side, i.e., look for the maximum value for k such that $w^{(0)}[-k]\neq 0$
    - ★ Then, choose delay as $d=k$.
  - Obtain the causal stable ZF equalizer as follows

$$w[n]=w^{(0)}[n-d]$$

Main drawback of ZF equalizer

- ZF equalizer basically inverts the channel frequency response
  - The equalizer affects the transmitted data signal but also has an effect on the noise
- Power espectral density of the filtered noise $z'[n]$

$$S_{z'}^{ZF}(e^{j\omega})=S_z(e^{j\omega})\ \left|W(e^{j\omega})\right|^2=\frac{\sigma_z^2}{|P(e^{j\omega})|^2}$$

- Power of noise sequence $z[n]$ is

$$\sigma_{z'}^2|_{ZF}=\frac{1}{2\pi}\int_{-\pi}^{\pi}S_z^{ZF}(e^{j\omega})\,d\omega=\sigma_z^2\times\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{|P(e^{j\omega})|^2}\,d\omega$$

  - Noise amplification can happen if channel has strong attenuation for some frequencies

    NOTE: Infinite noise power is analitycaly obtained for channels with spectral nulls

Design using ZF criterion with $K_w + 1$ coefficients

- Equation system given by the joint channel-equalizer response

$$c[n] = \sum_{k=0}^{K_p} p[k]\; w[n-k]$$

  - There are $K_p + K_w + 1$ equations, one for each value of n

- Equation system

$$\begin{aligned}
c[0] &= w[0]\; p[0] \\
c[1] &= w[0]\; p[1] + w[1]\; p[0] \\
c[2] &= w[0]\; p[2] + w[1]\; p[1] + w[2]\; p[0] \\
c[3] &= w[0]\; p[3] + w[1]\; p[2] + w[2]\; p[1] + w[3]\; p[0] \\
&\;\;\vdots \quad \vdots \qquad \vdots \\
c[K_p + K_w] &= w[K_w]\; p[K_p]
\end{aligned}$$

Design using ZF criterion with $K_w + 1$ coefficients (II)

- Equation system given by the joint channel-equalizer response

$$c[n] = \sum_{k=0}^{K_p} p[k]\; w[n-k]$$

  - There are $K_p + K_w + 1$ equations, one for each value of n

- Equation system in matrix notation

$$\underbrace{\begin{bmatrix} c[0] \\ c[1] \\ \vdots \\ c[K_p+K_w] \end{bmatrix}}_{\mathbf{c}\equiv(K_p+K_w+1)\times 1} = \underbrace{\begin{bmatrix} p[0] & 0 & 0 & \cdots & 0 \\ p[1] & p[0] & 0 & \cdots & 0 \\ p[2] & p[1] & p[0] & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p[K_p] & p[K_p-1] & p[K_p-2] & \cdots & 0 \\ 0 & p[K_p] & p[K_p-1] & \cdots & p[0] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & p[K_p] \end{bmatrix}}_{\mathbf{P}\equiv(K_p+K_w+1)\times(K_w+1)} \underbrace{\begin{bmatrix} w[0] \\ w[1] \\ \vdots \\ w[K_w] \end{bmatrix}}_{\mathbf{w}\equiv(K_w+1)\times 1}$$

Matrix $\mathbf{P}$ is called CHANNEL CONVOLUTION MATRIX or simply CHANNEL MATRIX

Design using ZF criterion with $K_w + 1$ coefficients (III)

- Desired response for joint response with delay d

$$c[n] = \delta[n-d] \rightarrow \mathbf{c}_d = [\underbrace{00\cdots 0}_{d}10\cdots 0]^T$$

Equation system for this ideal response $\mathbf{c}_d = \mathbf{P}\ \mathbf{w}$

- This is an overdetermined equation system
  - $K_p + K_w + 1$ equations (one per each n in $c[n]$)
  - $K_w + 1$ unknowns (one per equalizer coefficient $w[n]$)
- Least squares (LS) solution

$$\mathbf{w}_d^{ZF} = \arg\min_{\mathbf{w}} ||\mathbf{c}_d - \mathbf{P}\ \mathbf{w}||^2 = \mathbf{P}^{\#}\ \mathbf{c}_d$$

  - Solution is provided by Moore-Penrose pseudo-inverse

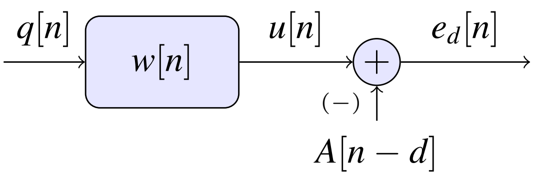
$$\mathbf{P}^{\#} = (\mathbf{P}^H\ \mathbf{P})^{-1}\ \mathbf{P}^H$$

The obtained solution does not fulfills all equations, i.e.

Joint response $\mathbf{c}_d^{ZF} = \mathbf{P}\ \mathbf{w}_d^{ZF} \neq \mathbf{c}_d$

Some residual ISI is still present because of limitation in number of coefficients

Design using MMSE criterion with unlimited equalizer length



- Error sequence at the equalizer output for delay d

$$e_d[n] = u[n] - A[n-d]$$

- MMSE optimal linear filtering: minimization of $E\left[|e_d[n]|^2\right]$
- MMSE solution - Orthogonality principle:
  - Error sequence $e_d[n]$ is orthogonal to system output $u[n]$
  - Error sequence $e_d[n]$ is orthogonal to system input $q[n]$

$$E[\underbrace{(u[n] - A[n-d])}_{e_d[n]}\ q^*[\ell]] = 0,\ \forall \ell$$

This equation can be written as follows

$$E[A[n-d]\ q^*[\ell]] = E[u[n]\ q^*[\ell]],\ \forall \ell$$

Orthogonality principle - first term

- Initial assumptions
  - Data sequence $A[n]$ is white: $R_A[k] = E_s\ \delta[k]$
  - Noise sequence $z[n]$ is white: $R_z[k] = \sigma_z^2\ \delta[k]$
  - Data and noise sequences, $A[n]$ and $z[n]$, are independent

This means that $R_{A,z}[k] = E\left[A[n+k]\ z^*[n]\right] = 0,\ \forall k$

- Development of the first term of orthogonality principle

$$\begin{aligned}
E[A[n-d]\ q^*[\ell]] =& E\left[A[n-d]\ \left(\sum_{k=0}^{K_p} p[k]\ A[\ell-k] + z[\ell]\right)^*\right] \\
=& \sum_{k=0}^{K_p} p^*[k]\ \underbrace{E\left[A[n-d]\ A^*[\ell-k]\right]}_{R_A[n-d-\ell+k]} \\
&+ \underbrace{E\left[A[n-d]\ z^*[\ell]\right]}_{R_{A,z}[n-d-\ell]} \\
=& E_s\ p^*[\ell+d-n]
\end{aligned}$$

Note that because $R_A[k] = E_s\ \delta[n]$, then $R_A[n-d-\ell+k] \neq 0$ just for index $k = \ell+d-n$

Orthogonality principle - Second term

$$\begin{aligned}
E\left[u[n]\ q^*[\ell]\right] &= E\left[\left(\sum_{k=0}^{K_w} w[k]\ q[n-k]\right)\ q^*[\ell]\right] \\
&= \sum_{k=0}^{K_w} w[k]\ \underbrace{E[q[n-k\ q^*[\ell]]}_{R_q[n-k-\ell]} = \left(w[k] * R_q[k]\right)\big|_{k=n-\ell}
\end{aligned}$$

- Autocorrelation function for observations $q[n]$

$$\begin{aligned}
R_q[n] &= E\left[q[\ell+n]\ q^*[\ell]\right] \\
&= E\left[\left(\sum_{k=0}^{K_p} p[k]\ A[\ell+n-k] + z[\ell+n]\right)\ \left(\sum_{j=0}^{K_p} p[j]\ A[\ell-j] + z[\ell]\right)^*\right] \\
&= \sum_{k=0}^{K_p}\sum_{j=0}^{K_p} p[k]\ p^*[j]\ \underbrace{E\left[A[\ell+n-k]\ A^*[\ell-j]\right]}_{R_A[n-k+j]} + \underbrace{E\left[z[\ell+n]\ z^*[\ell]\right]}_{R_z[n]} \\
&= E_s\ \sum_{k=0}^{K_p} p[k]\ p^*[k-n] + \sigma_z^2\ \delta[n] = E_s\ \left(p[n] * p^*[-n]\right) + \sigma_z^2\ \delta[n]
\end{aligned}$$

Note that because $R_A[k] = E_s\ \delta[n]$, then $R_A[n-k+j] \neq 0$ just for index $j = k-n$

Orthogonality principle - MMSE equalizer

- Combining both terms

$$E_s\, p^*[\underbrace{\ell+d-n}_{-(n-\ell-d)}] = w[n] * \left[E_s \;\; (p[k]*p^*[-k])|_{k=n-\ell} + \sigma_z^2\;\delta[n-\ell]\right]$$

- Making change of variable $k=n-\ell$, and dividing by E_s

$$p^*[-(k-d)] = w[k] * \left[(p[k]*p^*[-k]) + \frac{\sigma_z^2}{E_s}\;\delta[k]\right]$$

  - This is equivalent in the frequency domain to

$$P^*(e^{j\omega})\; e^{-j\omega d} = W(e^{j\omega}) \times \left[P(e^{j\omega})\; P^*(e^{j\omega}) + \frac{\sigma_z^2}{E_s}\right]$$

- The expression of equalizer in the frequency domain becomes

$$W(e^{j\omega}) = \frac{P^*(e^{j\omega})\; e^{-j\omega d}}{P(e^{j\omega})\; P^*(e^{j\omega}) + \frac{\sigma_z^2}{E_s}}$$

Delay d: causal implementation of $w[n]$

Linear MMSE equalizer without limitation in K_w

- Solution expressed in the frequency domain

$$W(e^{j\omega}) = \frac{P^*(e^{j\omega})\; e^{-j\omega d}}{P(e^{j\omega})\; P^*(e^{j\omega}) + \frac{\sigma_z^2}{E_s}} = \frac{P^*(e^{j\omega})\; e^{-j\omega d}}{|P(e^{j\omega})|^2 + \frac{\sigma_z^2}{E_s}}$$

Delay d: causal implemtation of $w[n]$

- For $\lambda = \frac{\sigma_z^2}{E_s} = 0$ (Null power noise for $z[n]$, $\sigma_z^2 = 0$)

$$W(e^{j\omega}) = \frac{P^*(e^{j\omega})\; e^{-j\omega d}}{P(e^{j\omega})\; P^*(e^{j\omega})} = \frac{e^{-j\omega d}}{P(e^{j\omega})}$$

MMSE equalizer becomes ZF equalizer!!!

Design of linear MMSE equalizer without limitation in K_w

- Procedure to obtain the MMSE unlimited linear equalizer:
  - Frequency response without delay

$$W^{(0)}\left(e^{j\omega}\right) = \frac{P^*\left(e^{j\omega}\right)}{P\left(e^{j\omega}\right)\, P^*\left(e^{j\omega}\right) + \frac{\sigma_z^2}{E_s}}$$

  - Time domain response is obtained

$$w^{(0)}[n] = \mathcal{TF}^{-1}\left\{W^{(0)}\left(e^{j\omega}\right)\right\}$$

    - ★ If this is a non-causal response, evaluate the length of non-causal side, i.e., look for the maximum value for k such that $w^{(0)}[-k] \neq 0$
    - ★ Then, choose delay as $d = k$.
  - Obtain the causal stable ZF equalizer as follows

$$w[n] = w^{(0)}[n-d]$$

Design using MMSE criterion with $K_w + 1$ coefficients

- Orthogonality principle can be written as follows

$$R_{A,q}[n-d] = \sum_{k=0}^{K_w} w[k]\, R_q[n-k]$$

  - System with $K_w + 1$ equations for the $K_w + 1$ unknowns

$$\boldsymbol{r}_{A,q}^{d} = \boldsymbol{R}_q\, \mathbf{w} \rightarrow \mathbf{w}_d^{MMSE} = (\boldsymbol{R}_q)^{-1}\, \boldsymbol{r}_{A,q}^{d}$$

REMARK: definition of involved vectors and matrix are in next slide

- Solution can also be found through channel matrix $\mathbf{P}$

$$\mathbf{w}_d^{MMSE} = \left(\mathbf{P}^H\, \mathbf{P} + \lambda\, \mathbf{I}\right)^{-1}\, \mathbf{P}^H\, \mathbf{c}_d \qquad \lambda = \frac{\sigma_z^2}{E_s}$$

Regularized pseudo-inverse: $P_\lambda^{\#} = \left(\mathbf{P}^H\, \mathbf{P} + \lambda\, \mathbf{I}\right)^{-1}\, \mathbf{P}^H$

$$\lambda = 0 \rightarrow \mathbf{w}_d^{MMSE} = \mathbf{w}_d^{ZF} \qquad \left(P_{\lambda=0}^{\#} = P^{\#}\right)$$

Matrix equation system

$$\underbrace{\begin{bmatrix} R_{A,q}[-d] \\ R_{A,q}[-(d-1)] \\ R_{A,q}[-(d-2)] \\ \vdots \\ R_{A,q}[K_w-d] \end{bmatrix}}_{\mathbf{r}_{A,q}^{d}} =$$

$$\underbrace{\begin{bmatrix} R_q[0] & R_q^*[1] & R_q^*[2] & \cdots & R_q^*[K_W] \\ R_q[1] & R_q[0] & R_q^*[1] & \cdots & R_q^*[K_W-1] \\ R_q[2] & R_q[1] & R_q[0] & \cdots & R_q^*[K_W-2] \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ R_q[K_W] & R_q[K_w-1] & R_q[K_w-2] & \cdots & R_q[0] \end{bmatrix}}_{\mathbf{R}_q} \underbrace{\begin{bmatrix} w[0] \\ w[1] \\ w[2] \\ \vdots \\ w[K_w] \end{bmatrix}}_{\mathbf{w}}$$

Estimation of performance for equalizers

- Ideal situation, without ISI

$$q[n] = A[n-d] + z[n] \Rightarrow P_e \approx k\ Q\left(\frac{d_{min}}{2\sigma_z}\right)$$

  - d_{min}: minimum distance between symbols in the constellation
  - k: maximum number of symbols at distance d_{min} of a symbol
  - Gaussian noise $z[n]$ has variance σ_z^2 for each dimension

- Equalizers with unlimited length

$$u[n] = A[n-d] + e_d[n] \Rightarrow P_e \approx k\ Q\left(\frac{d_{min}}{2\sigma_{e_d}}\right)$$

  - It is assumed that $e_d[n]$ zero mean Faussian distribution with variance $\sigma_{e_d}^2$

- Equalizers with a limited number of coefficients $(Kw+1)$

$$u[n] = A[n-d]\ c[d] + ISI + z'[n] \Rightarrow P_e \approx k\ Q\left(\frac{d_{min}|c[d]|}{2\sigma_{ISI+z'}}\right)$$

  - It is assumed that the addition of ISI and filtered noise has a zero mean Gaussian distribution with variance $\sigma_{ISI+z'}^2$

Asymptotic performance of linear equalizers

- Analysis based on the characterization of $e_d[n] = u[n] - A[n-d]$
- Equation output

$$u[n] = A[n] * p[n] * w[n] + z[n] * w[n]$$

- Error sequence at equalizer output

$$e_d[n] = A[n] * (w[n] * p[n] - \delta[n-d]) + z[n] * w[n]$$

- Power spectral density of of error sequence

$$S_{e_d}\left(e^{j\omega}\right) = S_A\left(e^{j\omega}\right)\ \left|W\left(e^{j\omega}\right)\ P\left(e^{j\omega}\right) - e^{-j\omega d}\right|^2 + S_z\left(e^{j\omega}\right)\ \left|W\left(e^{j\omega}\right)\right|^2$$

- For white data sequences and white noise

$$S_{e_d}\left(e^{j\omega}\right) = E_s\ \left|W\left(e^{j\omega}\right)\ P\left(e^{j\omega}\right) - e^{-j\omega d}\right|^2 + \sigma_z^2\ \left|W\left(e^{j\omega}\right)\right|^2$$

- Power of the error sequence

$$\sigma_{e_d}^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{e_d}\left(e^{j\omega}\right)\ d\omega$$

Asymptotic performance of linear equalizers (II)

- Power spectral density of the error sequence

$$S_{e_d}\left(e^{j\omega}\right) = E_s\ \left|W\left(e^{j\omega}\right)\ P\left(e^{j\omega}\right) - e^{-j\omega d}\right|^2 + \sigma_z^2\ \left|W\left(e^{j\omega}\right)\right|^2$$

- ZF and MMSE equalizers

$$ZF \Rightarrow W(e^{j\omega}) = \frac{e^{-j\omega d}}{P(e^{j\omega})},\ MMSE \Rightarrow W(e^{j\omega}) = \frac{P^*\left(e^{j\omega}\right)\ e^{-j\omega d}}{|P(e^{j\omega})|^2 + \frac{\sigma_z^2}{E_s}}$$

- Replacing $W\left(e^{j\omega}\right)$ for ZF equalizer

$$S_{e_d}\left(e^{j\omega}\right) = E_s\ \left|\frac{e^{-j\omega d}}{P(e^{j\omega})}\ P\left(e^{j\omega}\right) - e^{-j\omega d}\right|^2 + \sigma_z^2\ \left|\frac{e^{-j\omega d}}{P(e^{j\omega})}\right|^2 = \sigma_z^2 \frac{1}{|P(e^{j\omega})|^2}$$

- Replacing $W\left(e^{j\omega}\right)$ for MMSE equalizer

$$\begin{aligned}
S_{e_d}\left(e^{j\omega}\right) =& E_s\left|\frac{P^*\left(e^{j\omega}\right)\ e^{-j\omega d}}{|P(e^{j\omega})|^2 + \frac{\sigma_z^2}{E_s}}\ P\left(e^{j\omega}\right) - e^{-j\omega d}\right|^2 + \sigma_z^2\left|\frac{P^*\left(e^{j\omega}\right)\ e^{-j\omega d}}{|P(e^{j\omega})|^2 + \frac{\sigma_z^2}{E_s}}\right|^2 \\
=& E_s\left|\frac{\left|P(e^{j\omega})\right|^2}{|P(e^{j\omega})|^2 + \frac{\sigma_z^2}{E_s}} - 1\right|^2 + \sigma_z^2\left|\frac{P^*\left(e^{j\omega}\right)}{|P(e^{j\omega})|^2 + \frac{\sigma_z^2}{E_s}}\right|^2 \\
=& E_s\left|\frac{\frac{\sigma_z^2}{E_s}}{|P(e^{j\omega})|^2 + \frac{\sigma_z^2}{E_s}}\right|^2 + \sigma_z^2\left|\frac{P^*\left(e^{j\omega}\right)}{|P(e^{j\omega})|^2 + \frac{\sigma_z^2}{E_s}}\right|^2 = \sigma_z^2\frac{\frac{\sigma_z^2}{E_s} + \left|P(e^{j\omega})\right|^2}{\left||P(e^{j\omega})|^2 + \frac{\sigma_z^2}{E_s}\right|^2} = \sigma_z^2\frac{1}{|P(e^{j\omega})|^2 + \frac{\sigma_z^2}{E_s}}
\end{aligned}$$

Asymptotic performance of linear equalizers (III)

- Replacing $W\left(e^{j\omega}\right)$ for ZF criterion

$$\sigma_{e_d}^2(ZF) = \sigma_z^2 \times \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{\left|P\left(e^{j\omega}\right)\right|^2}\, d\omega$$

- Replacing $W\left(e^{j\omega}\right)$ for MMSE criterion

$$\sigma_{e_d}^2(MMSE) = \sigma_z^2 \times \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{\left|P\left(e^{j\omega}\right)\right|^2 + \frac{\sigma_z^2}{E_s}}\, d\omega$$

- Probability of error - Approximation for P_e

$$P_e \approx k\, Q\left(\frac{d_{min}}{2\sigma_{e_d}}\right)$$

  - d_{min}: minimum distance between symbols in the constellation
  - k: maximum number of symbols at d_{min} of a symbol in the constellation

Performance of linear equalizers with $K_w + 1$ coefficients

- Equalizer output is now

$$u[n] = \underbrace{c[d]}_{\text{gain}}\ A[n-d] + \underbrace{\sum_{\substack{k=0\\k\neq d}}^{K_p+K_w} c[k]\ A[n-k]}_{\text{residual ISI}} + \underbrace{\sum_{k=0}^{K_w} w[k]\ z[n-k]}_{\text{filtered noise } z'[n]}$$

- Assumptions
  - ISI and filtered noise are independent
  - Distribution for ISI is Gaussian
- Approximation for the probability of error

$$P_e \approx k\, Q\left(\frac{d_{min}\ |c[d]|}{2\sqrt{\sigma_{z'}^2 + \sigma_{ISI}^2}}\right)$$

k: maximum number of symbols at minimum distance of another symbol in the constellation

Mean y variance for filtered noise $z'[n]$

- Mean of $z'[n]$

$$E\left[z'[n]\right] = \sum_{k=0}^{K_w} w[k]\ E[z[n-k]] = 0$$

- Variance of $z'[n]$

$$\begin{aligned}\sigma_{z'}^2 &= E\left[\left(\sum_{k=0}^{K_w} w[k]\ z[n-k]\right)\ \left(\sum_{j=0}^{K_w} w^*[j]\ z^*[n-j]\right)\right]\\ &= \sum_{k=0}^{K_w}\sum_{j=0}^{K_w} w[k]\ w^*[j]\ \underbrace{E\left[z[n-k]\ z^*[n-j]\right]}_{R_z[j-k]=\sigma_z^2\ \delta[j-k]}\\ &= \sigma_z^2\ \sum_{k=0}^{K_w} |w[k]|^2\end{aligned}$$

Mean and variance of ISI term

- Mean of ISI term

$$E\left[ISI\right] = \sum_{\substack{k=0\\k\neq d}}^{K_p+K_w} c[k]\ E[A[n-k]] = 0$$

- Variance of ISI term

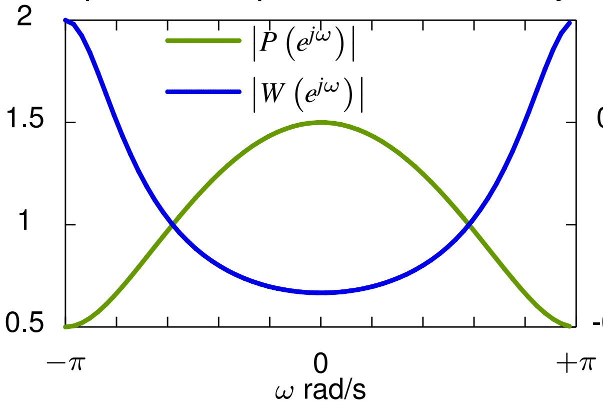
$$\begin{aligned}\sigma_{ISI}^2 &= E\left[\left(\sum_{\substack{k=0\\k\neq d}}^{K_p+K_w} c[k]\ A[n-k]\right)\ \left(\sum_{\substack{j=0\\j\neq d}}^{K_p+K_w} c^*[j]\ A^*[n-j]\right)\right]\\ &= \sum_{\substack{k=0\\k\neq d}}^{K_p+K_w}\ \sum_{\substack{j=0\\j\neq d}}^{K_p+K_w} c[k]\ c^*[j]\ \underbrace{E\left[A[n-k]\ A^*[n-j]\right]}_{R_A[j-k]=E_s\ \delta[j-k]}\\ &= E_s\ \sum_{\substack{k=0\\k\neq d}}^{K_p+K_w} |c[k]|^2\end{aligned}$$

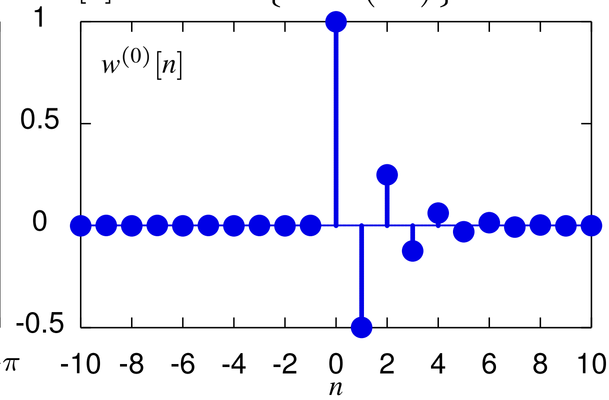
Equalizer ZF without limits - Example - Channel A

- Channel $p[n] = \delta[n] + \frac{1}{2}\,\delta[n-1]$
- Channel frequency response

$$P\left(e^{j\omega}\right) = \sum_n p[n]\,e^{-j\omega n} = 1 + \frac{1}{2}\,e^{-j\omega},\;\; W^{(0)}\left(e^{j\omega}\right) = \frac{1}{P\left(e^{j\omega}\right)}$$

- Equalizer response without delay: $w^{(0)}[n] = \mathcal{TF}^{-1}\left\{W^{(0)}\left(e^{j\omega}\right)\right\}$





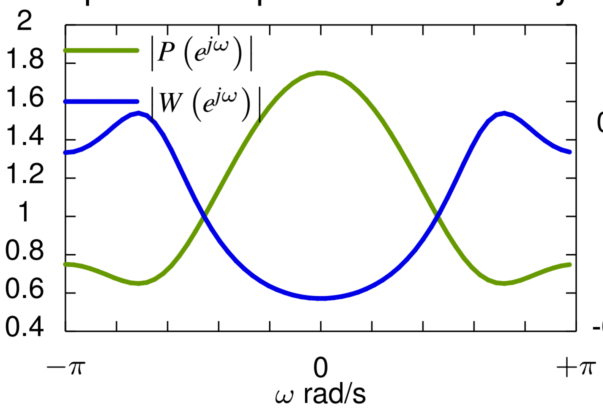
  - Causal response - Delay $d = 0$
  - Relevant coefficients ≈ 7 ($K_w \approx 6$)

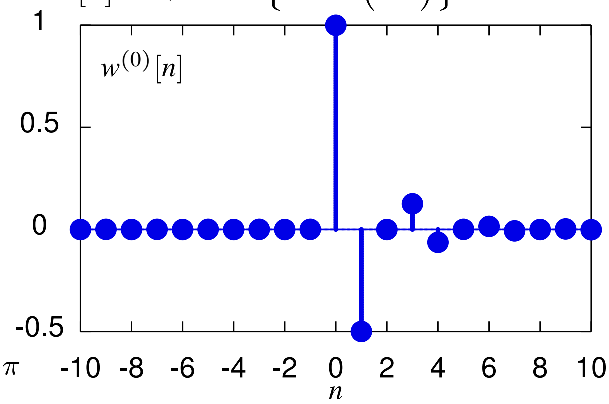
Equalizer ZF without limits - Example - Channel B

- Channel $p[n] = \delta[n] + \frac{1}{2}\,\delta[n-1] + \frac{1}{4}\,\delta[n-2]$
- Channel frequency response

$$P\left(e^{j\omega}\right) = 1 + \frac{1}{2}\,e^{-j\omega} + \frac{1}{4}\,e^{-j\omega 2},\;\; W^{(0)}\left(e^{j\omega}\right) = \frac{1}{P\left(e^{j\omega}\right)}$$

- Equalizer response without delay: $w^{(0)}[n] = \mathcal{TF}^{-1}\left\{W^{(0)}\left(e^{j\omega}\right)\right\}$





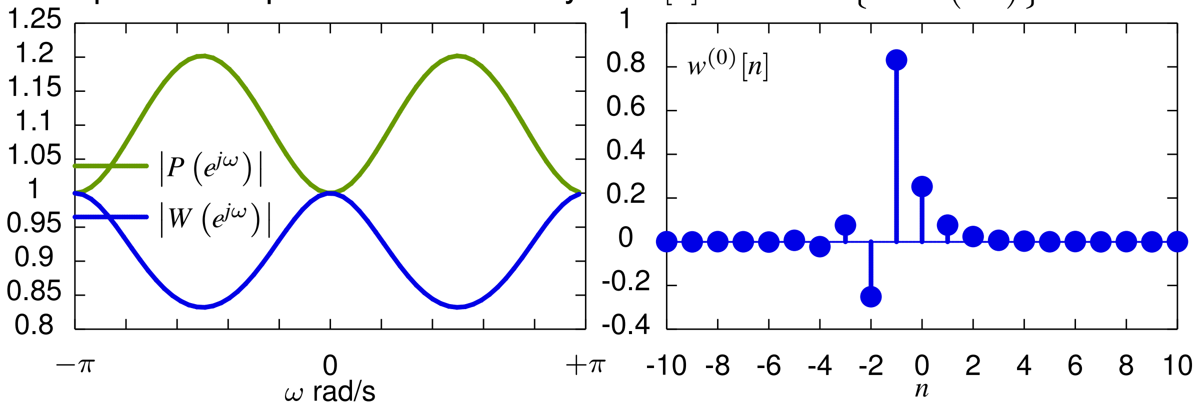
  - Causal response - Delay $d = 0$
  - Relevant coefficients ≈ 7 ($K_w \approx 6$)

Equalizer ZF without limits - Example - Channel C

- Channel $p[n] = \frac{1}{3}\,\delta[n] + \delta[n-1] - \frac{1}{3}\,\delta[n-2]$
- Channel frequency response

$$P\left(e^{j\omega}\right) = \frac{1}{3} + e^{-j\omega} - \frac{1}{3}\,e^{-j\omega 2},\ W^{(0)}\left(e^{j\omega}\right) = \frac{1}{P\left(e^{j\omega}\right)}$$

- Equalizer response without delay: $w^{(0)}[n] = \mathcal{TF}^{-1}\left\{W^{(0)}\left(e^{j\omega}\right)\right\}$



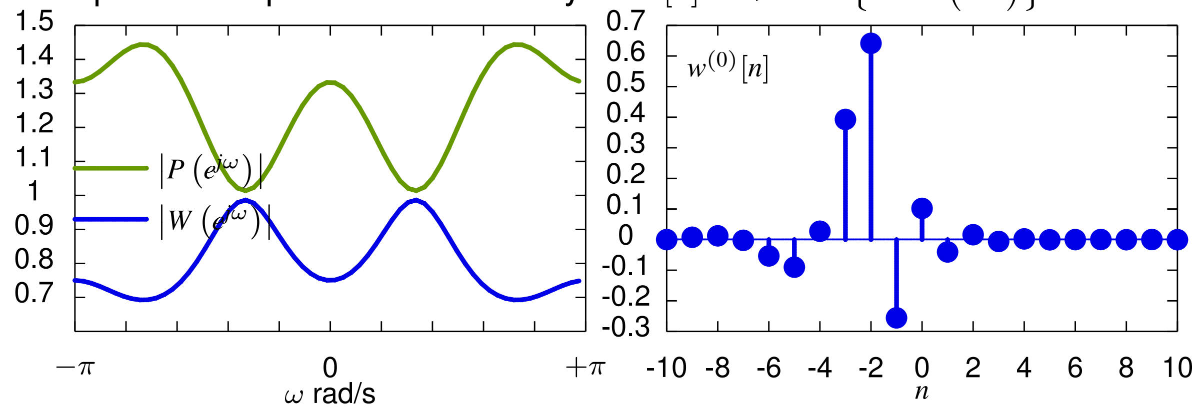
  - Non-causal response - delay $d \approx 5$
  - Relevant coefficients $\approx$ 9 ($K_w \approx 8$)

Equalizer ZF without limits - Example - Channel D

- Channel $p[n] = \frac{1}{3}\,\delta[n] - \frac{1}{2}\,\delta[n-1] + \delta[n-2] + \frac{1}{2}\,\delta[n-3]$
- Channel frequency response

$$P\left(e^{j\omega}\right) = \frac{1}{3} - \frac{1}{2}\,e^{-j\omega} + e^{-j\omega 2} + \frac{1}{2}\,e^{-j\omega 3},\ W^{(0)}\left(e^{j\omega}\right) = \frac{1}{P\left(e^{j\omega}\right)}$$

- Equalizer response without delay: $w^{(0)}[n] = \mathcal{TF}^{-1}\left\{W^{(0)}\left(e^{j\omega}\right)\right\}$



  - Non-causal response - delay $d \approx 9$
  - Relevant coefficients $\approx$ 13 ($K_w \approx 12$)

Equalizer ZF with 4 coefficients ($K_w = 3$) - Channel D - $d = 0$

- Channel $p[n] = \frac{1}{3}\,\delta[n] - \frac{1}{2}\,\delta[n-1] + \delta[n-2] + \frac{1}{2}\,\delta[n-3]$
- Involved matrices and vectors - delay $d = 0$

$$\mathbf{c}_d = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{7\times 1}, \ \mathbf{P} = \underbrace{\begin{bmatrix} +\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{2} & +\frac{1}{3} & 0 & 0 \\ +1 & -\frac{1}{2} & +\frac{1}{3} & 0 \\ +\frac{1}{2} & +1 & -\frac{1}{2} & +\frac{1}{3} \\ 0 & +\frac{1}{2} & +1 & -\frac{1}{2} \\ 0 & 0 & +\frac{1}{2} & +1 \\ 0 & 0 & 0 & +\frac{1}{2} \end{bmatrix}}_{7\times 4}$$

- ZF solution

$$\mathbf{w}_d^{ZF} = \mathbf{P}^{\#}\,\mathbf{c}_d = \begin{bmatrix} +0.2123 \\ +0.0221 \\ -0.0112 \\ -0.0243 \end{bmatrix}, \ \mathbf{c}_d^{ZF} = \mathbf{P}\,\mathbf{w} = \begin{bmatrix} +0.0708 \\ -0.0988 \\ +0.1975 \\ +0.1257 \\ +0.0119 \\ -0.0299 \\ -0.0121 \end{bmatrix}$$

$$\sigma_{ISI}^2 = E_s \times 0.0657$$

$$\sigma_{z'}^2 = \sigma_z^2 \times 0.0463$$

$$\frac{|c[d]|}{2\sqrt{\sigma_{ISI}^2 + \sigma_{z'}^2}} = 0.1334$$

Equalizer ZF with 4 coefficients ($K_w = 3$) - Channel D - $d = 1$

- Channel $p[n] = \frac{1}{3}\,\delta[n] - \frac{1}{2}\,\delta[n-1] + \delta[n-2] + \frac{1}{2}\,\delta[n-3]$
- Involved matrices and vectors - delay $d = 1$

$$\mathbf{c}_d = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{P} = \begin{bmatrix} +\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{2} & +\frac{1}{3} & 0 & 0 \\ +1 & -\frac{1}{2} & +\frac{1}{3} & 0 \\ +\frac{1}{2} & +1 & -\frac{1}{2} & +\frac{1}{3} \\ 0 & +\frac{1}{2} & +1 & -\frac{1}{2} \\ 0 & 0 & +\frac{1}{2} & +1 \\ 0 & 0 & 0 & +\frac{1}{2} \end{bmatrix}$$

- ZF solution

$$\mathbf{w}_d^{ZF} = \mathbf{P}^{\#}\,\mathbf{c}_d = \begin{bmatrix} -0.2963 \\ +0.1787 \\ +0.0364 \\ +0.0252 \end{bmatrix}, \ \mathbf{c}_d^{ZF} = \mathbf{P}\,\mathbf{w} = \begin{bmatrix} -0.0988 \\ +0.2077 \\ -0.3736 \\ +0.0207 \\ +0.1132 \\ +0.0434 \\ +0.0126 \end{bmatrix}$$

$$\sigma_{ISI}^2 = E_s \times 0.1646$$

$$\sigma_{z'}^2 = \sigma_z^2 \times 0.1217$$

$$\frac{|c[d]|}{2\sqrt{\sigma_{ISI}^2 + \sigma_{z'}^2}} = 0.2471$$

Equalizer ZF with 4 coefficients ($K_w = 3$) - Channel D - $d = 2$

- Channel $p[n] = \frac{1}{3}\,\delta[n] - \frac{1}{2}\,\delta[n-1] + \delta[n-2] + \frac{1}{2}\,\delta[n-3]$
- Involved matrices and vectors - delay $d = 2$

$$\mathbf{c}_d = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} +\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{2} & +\frac{1}{3} & 0 & 0 \\ +1 & -\frac{1}{2} & +\frac{1}{3} & 0 \\ +\frac{1}{2} & +1 & -\frac{1}{2} & +\frac{1}{3} \\ 0 & +\frac{1}{2} & +1 & -\frac{1}{2} \\ 0 & 0 & +\frac{1}{2} & +1 \\ 0 & 0 & 0 & +\frac{1}{2} \end{bmatrix}$$

- ZF solution

$$\mathbf{w}_d^{ZF} = \mathbf{P}^{\#}\,\mathbf{c}_d = \begin{bmatrix} +0.5925 \\ -0.2319 \\ +0.1488 \\ -0.0339 \end{bmatrix}, \quad \mathbf{c}_d^{ZF} = \mathbf{P}\,\mathbf{w} = \begin{bmatrix} +0.1975 \\ -0.3736 \\ +0.7581 \\ -0.0213 \\ +0.0498 \\ +0.0405 \\ -0.0170 \end{bmatrix}$$

$$\sigma_{ISI}^2 = E_s \times 0.1834$$

$$\sigma_{z'}^2 = \sigma_z^2 \times 0.4281$$

$$\frac{|c[d]|}{2\sqrt{\sigma_{ISI}^2 + \sigma_{z'}^2}} = 0.7969$$

Equalizer ZF with 4 coefficients ($K_w = 3$) - Channel D - $d = 3$

- Channel $p[n] = \frac{1}{3}\,\delta[n] - \frac{1}{2}\,\delta[n-1] + \delta[n-2] + \frac{1}{2}\,\delta[n-3]$
- Involved matrices and vectors - delay $d = 3$

$$\mathbf{c}_d = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} +\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{2} & +\frac{1}{3} & 0 & 0 \\ +1 & -\frac{1}{2} & +\frac{1}{3} & 0 \\ +\frac{1}{2} & +1 & -\frac{1}{2} & +\frac{1}{3} \\ 0 & +\frac{1}{2} & +1 & -\frac{1}{2} \\ 0 & 0 & +\frac{1}{2} & +1 \\ 0 & 0 & 0 & +\frac{1}{2} \end{bmatrix}$$

- ZF solution

$$\mathbf{w}_d^{ZF} = \mathbf{P}^{\#}\,\mathbf{c}_d = \begin{bmatrix} +0.3771 \\ +0.6278 \\ -0.2536 \\ +0.1092 \end{bmatrix}, \quad \mathbf{c}_d^{ZF} = \mathbf{P}\,\mathbf{w} = \begin{bmatrix} +0.1257 \\ +0.0207 \\ -0.0213 \\ +0.9796 \\ +0.0057 \\ -0.0176 \\ +0.0546 \end{bmatrix}$$

$$\sigma_{ISI}^2 = E_s \times 0.0200$$

$$\sigma_{z'}^2 = \sigma_z^2 \times 0.6126$$

$$\frac{|c[d]|}{2\sqrt{\sigma_{ISI}^2 + \sigma_{z'}^2}} = 1.7181$$

Equalizer ZF with 4 coefficients ($K_w = 3$) - Channel D - $d = 4$

- Channel $p[n] = \frac{1}{3}\,\delta[n] - \frac{1}{2}\,\delta[n-1] + \delta[n-2] + \frac{1}{2}\,\delta[n-3]$
- Involved matrices and vectors - delay $d = 4$

$$\mathbf{c}_d = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{P} = \begin{bmatrix} +\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{2} & +\frac{1}{3} & 0 & 0 \\ +1 & -\frac{1}{2} & +\frac{1}{3} & 0 \\ +\frac{1}{2} & +1 & -\frac{1}{2} & +\frac{1}{3} \\ 0 & +\frac{1}{2} & +1 & -\frac{1}{2} \\ 0 & 0 & +\frac{1}{2} & +1 \\ 0 & 0 & 0 & +\frac{1}{2} \end{bmatrix}$$

- ZF solution

$$\mathbf{w}_d^{ZF} = \mathbf{P}^{\#}\ \mathbf{c}_d = \begin{bmatrix} +0.0358 \\ +0.3933 \\ +0.6317 \\ -0.2690 \end{bmatrix}, \ \mathbf{c}_d^{ZF} = \mathbf{P}\ \mathbf{w} = \begin{bmatrix} +0.0119 \\ +0.1132 \\ +0.0498 \\ +0.0057 \\ +0.9629 \\ +0.0468 \\ -0.1345 \end{bmatrix} \quad \begin{array}{l} \sigma_{ISI}^2 = E_s \times 0.0357 \\ \sigma_{z'}^2 = \sigma_z^2 \times 0.6274 \\ \frac{|c[d]|}{2\sqrt{\sigma_{ISI}^2+\sigma_{z'}^2}} = 1.5341 \end{array}$$

Equalizer ZF with 4 coefficients ($K_w = 3$) - Channel D - $d = 5$

- Channel $p[n] = \frac{1}{3}\,\delta[n] - \frac{1}{2}\,\delta[n-1] + \delta[n-2] + \frac{1}{2}\,\delta[n-3]$
- Involved matrices and vectors - delay $d = 5$

$$\mathbf{c}_d = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{P} = \begin{bmatrix} +\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{2} & +\frac{1}{3} & 0 & 0 \\ +1 & -\frac{1}{2} & +\frac{1}{3} & 0 \\ +\frac{1}{2} & +1 & -\frac{1}{2} & +\frac{1}{3} \\ 0 & +\frac{1}{2} & +1 & -\frac{1}{2} \\ 0 & 0 & +\frac{1}{2} & +1 \\ 0 & 0 & 0 & +\frac{1}{2} \end{bmatrix}$$

- ZF solution

$$\mathbf{w}_d^{ZF} = \mathbf{P}^{\#}\ \mathbf{c}_d = \begin{bmatrix} -0.0896 \\ -0.0042 \\ +0.3838 \\ +0.6699 \end{bmatrix}, \ \mathbf{c}_d^{ZF} = \mathbf{P}\ \mathbf{w} = \begin{bmatrix} -0.0299 \\ +0.0434 \\ +0.0405 \\ -0.0176 \\ +0.0468 \\ +0.8618 \\ +0.3349 \end{bmatrix} \quad \begin{array}{l} \sigma_{ISI}^2 = E_s \times 0.1191 \\ \sigma_{z'}^2 = \sigma_z^2 \times 0.6041 \\ \frac{|c[d]|}{2\sqrt{\sigma_{ISI}^2+\sigma_{z'}^2}} = 1.0170 \end{array}$$

Equalizer ZF with 4 coefficients ($K_w = 3$) - Channel D - $d = 6$

- Channel $p[n] = \frac{1}{3}\,\delta[n] - \frac{1}{2}\,\delta[n-1] + \delta[n-2] + \frac{1}{2}\,\delta[n-3]$
- Involved matrices and vectors - delay $d = 6$

$$\mathbf{c}_d = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} +\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{2} & +\frac{1}{3} & 0 & 0 \\ +1 & -\frac{1}{2} & +\frac{1}{3} & 0 \\ +\frac{1}{2} & +1 & -\frac{1}{2} & +\frac{1}{3} \\ 0 & +\frac{1}{2} & +1 & -\frac{1}{2} \\ 0 & 0 & +\frac{1}{2} & +1 \\ 0 & 0 & 0 & +\frac{1}{2} \end{bmatrix}$$

- ZF solution

$$\mathbf{w}_d^{ZF} = \mathbf{P}^{\#}\,\mathbf{c}_d = \begin{bmatrix} -0.0364 \\ -0.0168 \\ +0.0331 \\ +0.3184 \end{bmatrix}, \quad \mathbf{c}_d^{ZF} = \mathbf{P}\,\mathbf{w} = \begin{bmatrix} -0.0121 \\ +0.0126 \\ -0.0170 \\ +0.0546 \\ -0.1345 \\ +0.3349 \\ +0.1592 \end{bmatrix}$$

$$\sigma_{ISI}^2 = E_s \times 0.1339$$

$$\sigma_{z'}^2 = \sigma_z^2 \times 0.1041$$

$$\frac{|c[d]|}{2\sqrt{\sigma_{ISI}^2 + \sigma_{z'}^2}} = 0.2096$$

Channel D - Pseudo-inverse matrix

- Channel $p[n] = \frac{1}{3}\,\delta[n] - \frac{1}{2}\,\delta[n-1] + \delta[n-2] + \frac{1}{2}\,\delta[n-3]$
- Matriz de canal

$$\mathbf{P} = \underbrace{\begin{bmatrix} +\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{2} & +\frac{1}{3} & 0 & 0 \\ +1 & -\frac{1}{2} & +\frac{1}{3} & 0 \\ +\frac{1}{2} & +1 & -\frac{1}{2} & +\frac{1}{3} \\ 0 & +\frac{1}{2} & +1 & -\frac{1}{2} \\ 0 & 0 & +\frac{1}{2} & +1 \\ 0 & 0 & 0 & +\frac{1}{2} \end{bmatrix}}_{7\times 4 \equiv (K_p+K_w+1)\times(K_w+1)}$$

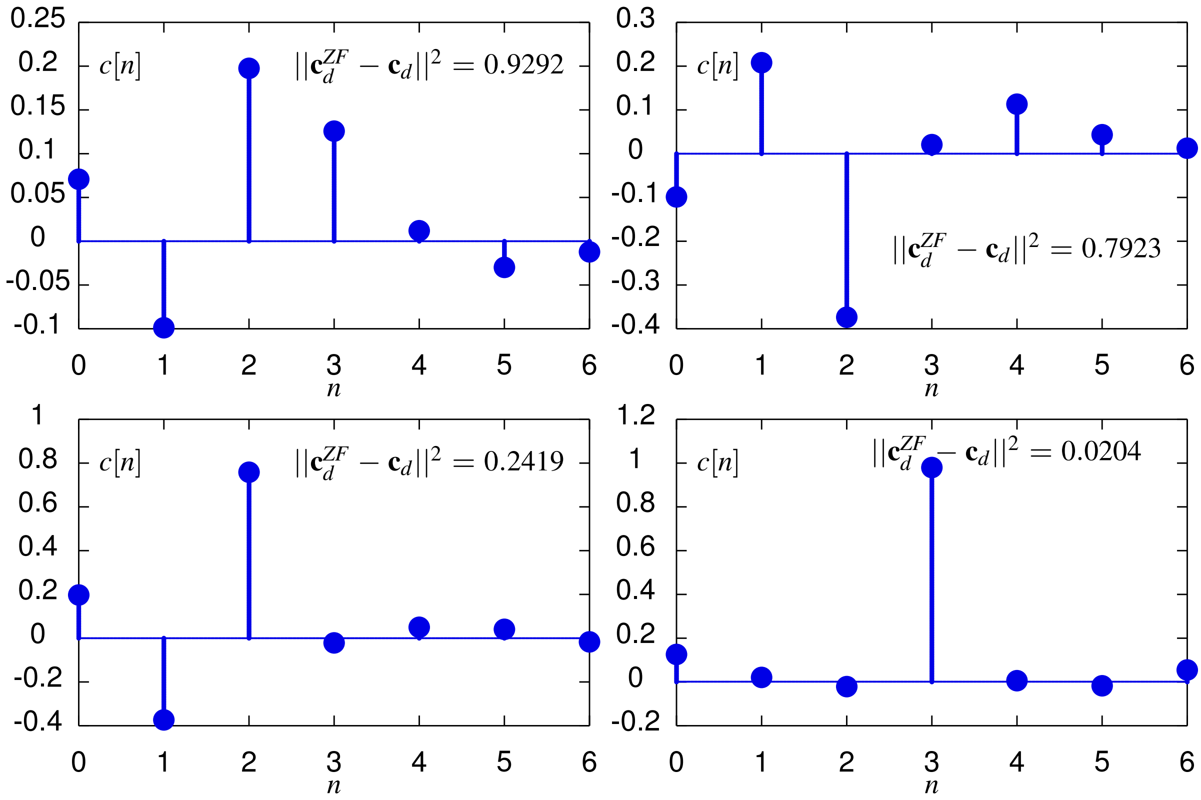
- Moore-Penrose pseudo-inverse matrix

$$\mathbf{P}^{\#} = (\mathbf{P}^H\,\mathbf{P})^{-1}\,\mathbf{P}^H$$

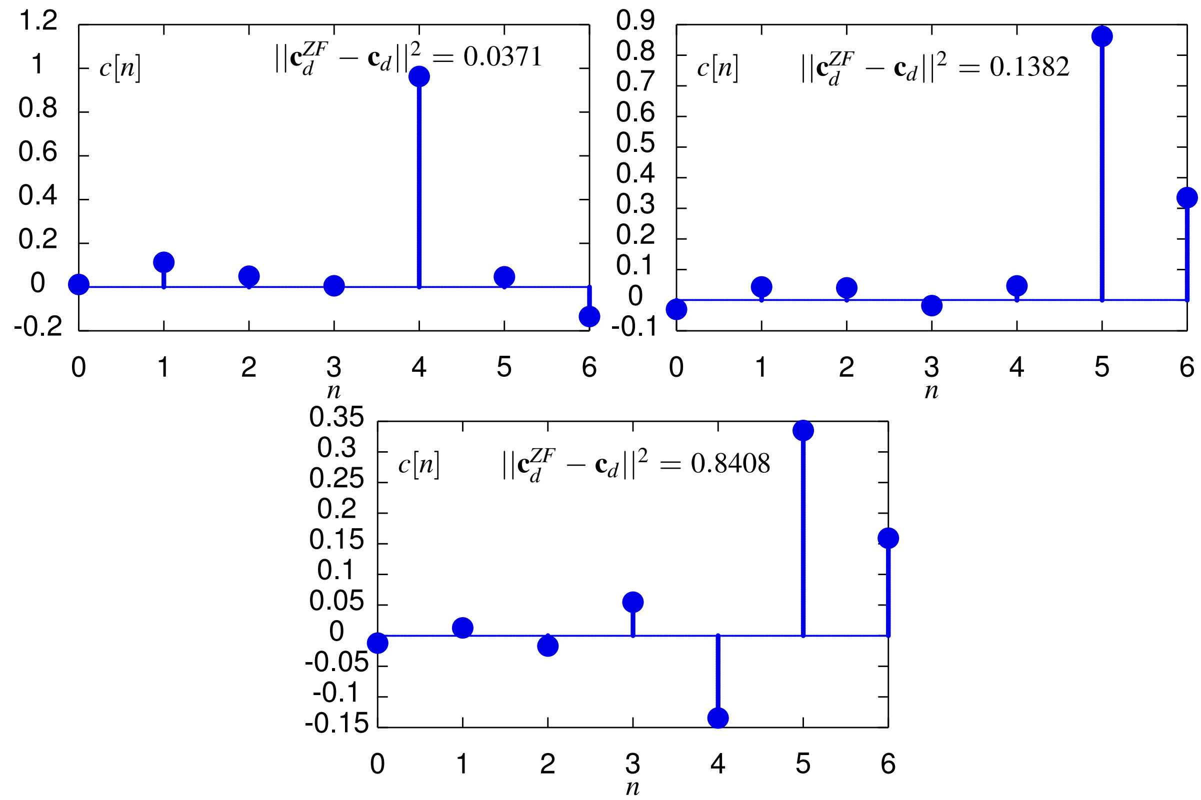
$$= \underbrace{\begin{bmatrix} +0.2123 & -0.2963 & +0.5925 & +0.3771 & +0.0358 & -0.0896 & -0.0364 \\ +0.0221 & +0.1787 & -0.2319 & +0.6278 & +0.3933 & -0.0042 & -0.0168 \\ -0.0112 & +0.0364 & +0.1488 & -0.2536 & +0.6317 & +0.3838 & +0.0331 \\ -0.0243 & +0.0252 & -0.0339 & +0.1092 & -0.2690 & +0.6699 & +0.3184 \end{bmatrix}}_{4\times 7 \equiv (K_w+1)\times(K_p+K_w+1)}$$

  - ZF solutions are given by columms of this matrix (according to delay d

Joint responses for delays $d \in \{0, 1, 2, 3\}$



Joint responses for delays $d \in \{4, 5, 6\}$

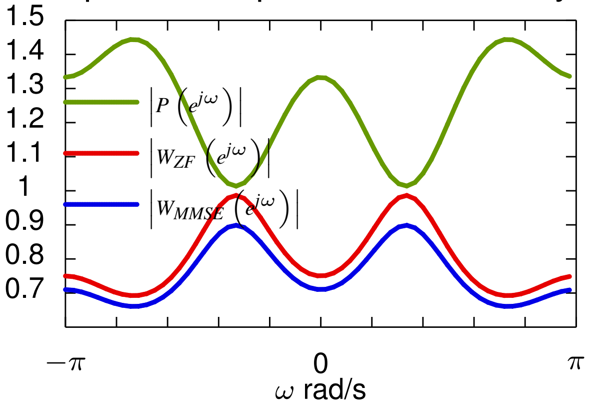


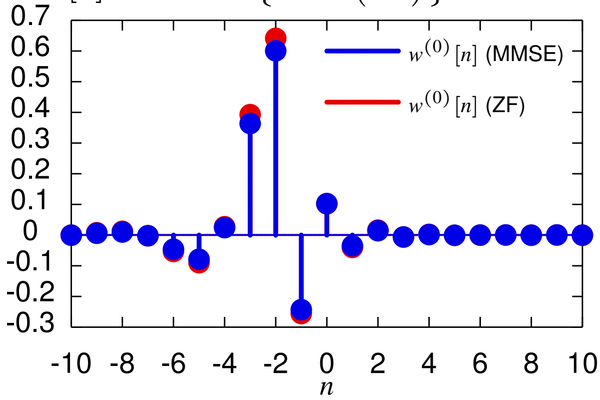
Equalizer MMSE without limits - Example - Channel D

- Channel $p[n] = \frac{1}{3}\,\delta[n] - \frac{1}{2}\,\delta[n-1] + \delta[n-2] + \frac{1}{2}\,\delta[n-3]$
- Channel frequency response

$$P\left(e^{j\omega}\right) = \frac{1}{3} - \frac{1}{2}\,e^{-j\omega} + e^{-j\omega 2} + \frac{1}{2}\,e^{-j\omega 3},\ W^{(0)}\left(e^{j\omega}\right) = \frac{P^*\left(e^{j\omega}\right)}{\left|P\left(e^{j\omega}\right)\right|^2 + \frac{\sigma_z^2}{E_s}}$$

- Equalizer response without delay: $w^{(0)}[n] = \mathcal{TF}^{-1}\left\{W^{(0)}\left(e^{j\omega}\right)\right\}$





  - Non-causal response - delay $d \approx 9$
  - Relevant coefficients $\approx$ 13 ($K_w \approx 12$)

MMSE with 4 coefficients ($K_w = 3$) - Channel D - $d = 0$

- Channel $p[n] = \frac{1}{3}\,\delta[n] - \frac{1}{2}\,\delta[n-1] + \delta[n-2] + \frac{1}{2}\,\delta[n-3]$
- Noise variance $\sigma_z^2 = 0.1$, 2-PAM con $E_s = 1$
- Involved matrices and vectors - delay $d = 0$

$$\mathbf{c}_d = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{7\times 1},\ \mathbf{P} = \underbrace{\begin{bmatrix} +\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{2} & +\frac{1}{3} & 0 & 0 \\ +1 & -\frac{1}{2} & +\frac{1}{3} & 0 \\ +\frac{1}{2} & +1 & -\frac{1}{2} & +\frac{1}{3} \\ 0 & +\frac{1}{2} & +1 & -\frac{1}{2} \\ 0 & 0 & +\frac{1}{2} & +1 \\ 0 & 0 & 0 & +\frac{1}{2} \end{bmatrix}}_{7\times 4},\ \lambda = \frac{\sigma_z^2}{E_s} = 0.1,\ \mathbf{I} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{4\times 4}$$

- MMSE solution

$$\mathbf{w}_d^{MMSE} = (\mathbf{P}^H\,\mathbf{P} + \lambda\,\mathbf{I})^{-1}\,\mathbf{P}^H\,\mathbf{c}_d = \begin{bmatrix} +0.1993 \\ +0.0195 \\ -0.0099 \\ -0.0213 \end{bmatrix},\ \mathbf{c}_d^{MMSE} = \begin{bmatrix} +0.0664 \\ -0.0931 \\ +0.1862 \\ +0.1169 \\ +0.0105 \\ -0.0263 \\ -0.0107 \end{bmatrix}$$

$$\sigma_{ISI}^2 = E_s \times 0.0579$$

$$\sigma_{z'}^2 = \sigma_z^2 \times 0.0406$$

$$\frac{|c[d]|}{2\sqrt{\sigma_{ISI}^2 + \sigma_{z'}^2}} = 0.1334$$

(ZF: 0.1334)

MMSE with 4 coefficients ($K_w = 3$) - Channel D - $d = 1$

- Channel $p[n] = \frac{1}{3}\,\delta[n] - \frac{1}{2}\,\delta[n-1] + \delta[n-2] + \frac{1}{2}\,\delta[n-3]$
- Noise variance $\sigma_z^2 = 0.1$, 2-PAM con $E_s = 1$
- Involved matrices and vectors - delay $d = 1$

$$\mathbf{c}_d = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} +\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{2} & +\frac{1}{3} & 0 & 0 \\ +1 & -\frac{1}{2} & +\frac{1}{3} & 0 \\ +\frac{1}{2} & +1 & -\frac{1}{2} & +\frac{1}{3} \\ 0 & +\frac{1}{2} & +1 & -\frac{1}{2} \\ 0 & 0 & +\frac{1}{2} & +1 \\ 0 & 0 & 0 & +\frac{1}{2} \end{bmatrix},\ \lambda = \frac{\sigma_z^2}{E_s} = 0.1,\ \ \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- MMSE solution

$$\mathbf{w}_d^{MMSE} = (\mathbf{P}^H\ \mathbf{P} + \lambda\ \mathbf{I})^{-1}\ \mathbf{P}^H\ \mathbf{c}_d = \begin{bmatrix} -0.2794 \\ +0.1697 \\ +0.0323 \\ +0.0221 \end{bmatrix},\ \mathbf{c}_d^{MMSE} = \begin{bmatrix} -0.0931 \\ +0.1963 \\ -0.3535 \\ +0.0212 \\ +0.1061 \\ +0.0382 \\ +0.0110 \end{bmatrix}$$

$$\sigma_{ISI}^2 = E_s \times 0.1469$$

$$\sigma_{z'}^2 = \sigma_z^2 \times 0.1084$$

$$\frac{|c[d]|}{2\sqrt{\sigma_{ISI}^2 + \sigma_{z'}^2}} = 0.2471$$

(ZF: 0.2471)

MMSE with 4 coefficients ($K_w = 3$) - Channel D - $d = 2$

- Channel $p[n] = \frac{1}{3}\,\delta[n] - \frac{1}{2}\,\delta[n-1] + \delta[n-2] + \frac{1}{2}\,\delta[n-3]$
- Noise variance $\sigma_z^2 = 0.1$, 2-PAM con $E_s = 1$
- Involved matrices and vectors - delay $d = 2$

$$\mathbf{c}_d = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} +\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{2} & +\frac{1}{3} & 0 & 0 \\ +1 & -\frac{1}{2} & +\frac{1}{3} & 0 \\ +\frac{1}{2} & +1 & -\frac{1}{2} & +\frac{1}{3} \\ 0 & +\frac{1}{2} & +1 & -\frac{1}{2} \\ 0 & 0 & +\frac{1}{2} & +1 \\ 0 & 0 & 0 & +\frac{1}{2} \end{bmatrix},\ \lambda = \frac{\sigma_z^2}{E_s} = 0.1,\ \ \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- MMSE solution

$$\mathbf{w}_d^{MMSE} = (\mathbf{P}^H\ \mathbf{P} + \lambda\ \mathbf{I})^{-1}\ \mathbf{P}^H\ \mathbf{c}_d = \begin{bmatrix} +0.5587 \\ -0.2224 \\ +0.1430 \\ -0.0297 \end{bmatrix},\ \mathbf{c}_d^{MMSE} = \begin{bmatrix} +0.1862 \\ -0.3535 \\ +0.7176 \\ -0.0245 \\ +0.0467 \\ +0.0419 \\ -0.0148 \end{bmatrix}$$

$$\sigma_{ISI}^2 = E_s \times 0.1644$$

$$\sigma_{z'}^2 = \sigma_z^2 \times 0.3829$$

$$\frac{|c[d]|}{2\sqrt{\sigma_{ISI}^2 + \sigma_{z'}^2}} = 0.7970$$

(ZF: 0.7969)

MMSE with 4 coefficients ($K_w = 3$) - Channel D - $d = 3$

- Channel $p[n] = \frac{1}{3}\,\delta[n] - \frac{1}{2}\,\delta[n-1] + \delta[n-2] + \frac{1}{2}\,\delta[n-3]$
- Noise variance $\sigma_z^2 = 0.1$, 2-PAM con $E_s = 1$
- Involved matrices and vectors - delay $d = 3$

$$\mathbf{c}_d = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; \mathbf{P} = \begin{bmatrix} +\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{2} & +\frac{1}{3} & 0 & 0 \\ +1 & -\frac{1}{2} & +\frac{1}{3} & 0 \\ +\frac{1}{2} & +1 & -\frac{1}{2} & +\frac{1}{3} \\ 0 & +\frac{1}{2} & +1 & -\frac{1}{2} \\ 0 & 0 & +\frac{1}{2} & +1 \\ 0 & 0 & 0 & +\frac{1}{2} \end{bmatrix}, \; \lambda = \frac{\sigma_z^2}{E_s} = 0.1, \;\; \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- MMSE solution

$$\mathbf{w}_d^{MMSE} = (\mathbf{P}^H\,\mathbf{P} + \lambda\,\mathbf{I})^{-1}\,\mathbf{P}^H\,\mathbf{c}_d = \begin{bmatrix} +0.3508 \\ +0.5898 \\ -0.2413 \\ +0.1084 \end{bmatrix}, \; \mathbf{c}_d^{MMSE} = \begin{bmatrix} +0.1169 \\ +0.0212 \\ -0.0245 \\ +0.9220 \\ -0.0006 \\ -0.0122 \\ +0.0542 \end{bmatrix}$$

$$\sigma_{ISI}^2 = E_s \times 0.0178$$

$$\sigma_{z'}^2 = \sigma_z^2 \times 0.5409$$

$$\frac{|c[d]|}{2\sqrt{\sigma_{ISI}^2 + \sigma_{z'}^2}} = 1.7192$$

(ZF: 1.7181)

MMSE with 4 coefficients ($K_w = 3$) - Channel D - $d = 4$

- Channel $p[n] = \frac{1}{3}\,\delta[n] - \frac{1}{2}\,\delta[n-1] + \delta[n-2] + \frac{1}{2}\,\delta[n-3]$
- Noise variance $\sigma_z^2 = 0.1$, 2-PAM con $E_s = 1$
- Involved matrices and vectors - delay $d = 4$

$$\mathbf{c}_d = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \; \mathbf{P} = \begin{bmatrix} +\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{2} & +\frac{1}{3} & 0 & 0 \\ +1 & -\frac{1}{2} & +\frac{1}{3} & 0 \\ +\frac{1}{2} & +1 & -\frac{1}{2} & +\frac{1}{3} \\ 0 & +\frac{1}{2} & +1 & -\frac{1}{2} \\ 0 & 0 & +\frac{1}{2} & +1 \\ 0 & 0 & 0 & +\frac{1}{2} \end{bmatrix}, \; \lambda = \frac{\sigma_z^2}{E_s} = 0.1, \;\; \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- MMSE solution

$$\mathbf{w}_d^{MMSE} = (\mathbf{P}^H\,\mathbf{P} + \lambda\,\mathbf{I})^{-1}\,\mathbf{P}^H\,\mathbf{c}_d = \begin{bmatrix} +0.0316 \\ +0.3655 \\ +0.5936 \\ -0.2553 \end{bmatrix}, \; \mathbf{c}_d^{MMSE} = \begin{bmatrix} +0.0105 \\ +0.1061 \\ +0.0467 \\ -0.0006 \\ +0.9040 \\ +0.0415 \\ -0.1276 \end{bmatrix}$$

$$\sigma_{ISI}^2 = E_s \times 0.0316$$

$$\sigma_{z'}^2 = \sigma_z^2 \times 0.5522$$

$$\frac{|c[d]|}{2\sqrt{\sigma_{ISI}^2 + \sigma_{z'}^2}} = 1.5345$$

(ZF: 1.5341)

MMSE with 4 coefficients ($K_w = 3$) - Channel D - $d = 5$

- Channel $p[n] = \frac{1}{3}\,\delta[n] - \frac{1}{2}\,\delta[n-1] + \delta[n-2] + \frac{1}{2}\,\delta[n-3]$
- Noise variance $\sigma_z^2 = 0.1$, 2-PAM con $E_s = 1$
- Involved matrices and vectors - delay $d = 5$

$$\mathbf{c}_d = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} +\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{2} & +\frac{1}{3} & 0 & 0 \\ +1 & -\frac{1}{2} & +\frac{1}{3} & 0 \\ +\frac{1}{2} & +1 & -\frac{1}{2} & +\frac{1}{3} \\ 0 & +\frac{1}{2} & +1 & -\frac{1}{2} \\ 0 & 0 & +\frac{1}{2} & +1 \\ 0 & 0 & 0 & +\frac{1}{2} \end{bmatrix},\ \lambda = \frac{\sigma_z^2}{E_s} = 0.1,\quad \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- MMSE solution

$$\mathbf{w}_d^{MMSE} = (\mathbf{P}^H\,\mathbf{P} + \lambda\,\mathbf{I})^{-1}\,\mathbf{P}^H\,\mathbf{c}_d = \begin{bmatrix} -0.0788 \\ -0.0035 \\ +0.3568 \\ +0.6270 \end{bmatrix},\ \mathbf{c}_d^{MMSE} = \begin{bmatrix} -0.0263 \\ +0.0382 \\ +0.0419 \\ -0.0122 \\ +0.0415 \\ +0.8054 \\ +0.3135 \end{bmatrix}$$

$$\sigma_{ISI}^2 = E_s \times 0.1041$$

$$\sigma_{z'}^2 = \sigma_z^2 \times 0.5267$$

$$\frac{|c[d]|}{2\sqrt{\sigma_{ISI}^2 + \sigma_{z'}^2}} = 1.0172$$

(ZF: 1.0170)

MMSE with 4 coefficients ($K_w = 3$) - Channel D - $d = 6$

- Channel $p[n] = \frac{1}{3}\,\delta[n] - \frac{1}{2}\,\delta[n-1] + \delta[n-2] + \frac{1}{2}\,\delta[n-3]$
- Noise variance $\sigma_z^2 = 0.1$, 2-PAM con $E_s = 1$
- Involved matrices and vectors - delay $d = 6$

$$\mathbf{c}_d = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} +\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{2} & +\frac{1}{3} & 0 & 0 \\ +1 & -\frac{1}{2} & +\frac{1}{3} & 0 \\ +\frac{1}{2} & +1 & -\frac{1}{2} & +\frac{1}{3} \\ 0 & +\frac{1}{2} & +1 & -\frac{1}{2} \\ 0 & 0 & +\frac{1}{2} & +1 \\ 0 & 0 & 0 & +\frac{1}{2} \end{bmatrix},\ \lambda = \frac{\sigma_z^2}{E_s} = 0.1,\quad \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- MMSE solution

$$\mathbf{w}_d^{MMSE} = (\mathbf{P}^H\,\mathbf{P} + \lambda\,\mathbf{I})^{-1}\,\mathbf{P}^H\,\mathbf{c}_d = \begin{bmatrix} -0.0320 \\ -0.0148 \\ +0.0292 \\ +0.2989 \end{bmatrix},\ \mathbf{c}_d^{MMSE} = \begin{bmatrix} -0.0107 \\ +0.0110 \\ -0.0148 \\ +0.0542 \\ -0.1276 \\ +0.3135 \\ +0.1494 \end{bmatrix}$$

$$\sigma_{ISI}^2 = E_s \times 0.1180$$

$$\sigma_{z'}^2 = \sigma_z^2 \times 0.0914$$

$$\frac{|c[d]|}{2\sqrt{\sigma_{ISI}^2 + \sigma_{z'}^2}} = 0.2096$$

(ZF: 2096)

Channel D - Regularized pseudoinverse matrix

- Channel $p[n] = \frac{1}{3}\,\delta[n] - \frac{1}{2}\,\delta[n-1] + \delta[n-2] + \frac{1}{2}\,\delta[n-3]$
- Channel matrix

$$\mathbf{P} = \underbrace{\begin{bmatrix} +\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{2} & +\frac{1}{3} & 0 & 0 \\ +1 & -\frac{1}{2} & +\frac{1}{3} & 0 \\ +\frac{1}{2} & +1 & -\frac{1}{2} & +\frac{1}{3} \\ 0 & +\frac{1}{2} & +1 & -\frac{1}{2} \\ 0 & 0 & +\frac{1}{2} & +1 \\ 0 & 0 & 0 & +\frac{1}{2} \end{bmatrix}}_{7\times 4 \equiv (K_p+K_w+1)\times(K_w+1)}$$

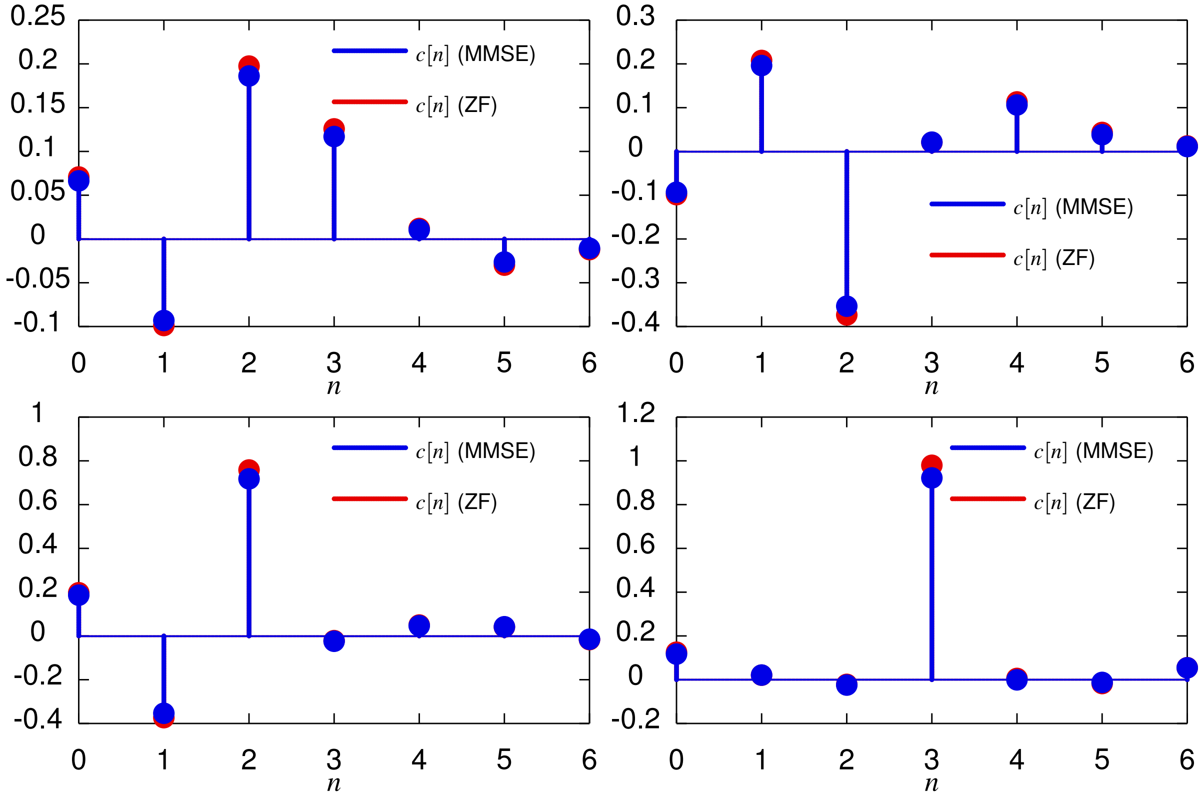
- Regularized pseudo-inverse, with $\lambda = \frac{\sigma_z^2}{E_s} = 0.1$

$$\mathbf{P}_\lambda^{\#} = (\mathbf{P}^H\,\mathbf{P} + \lambda\,\mathbf{I})^{-1}\,\mathbf{P}^H$$

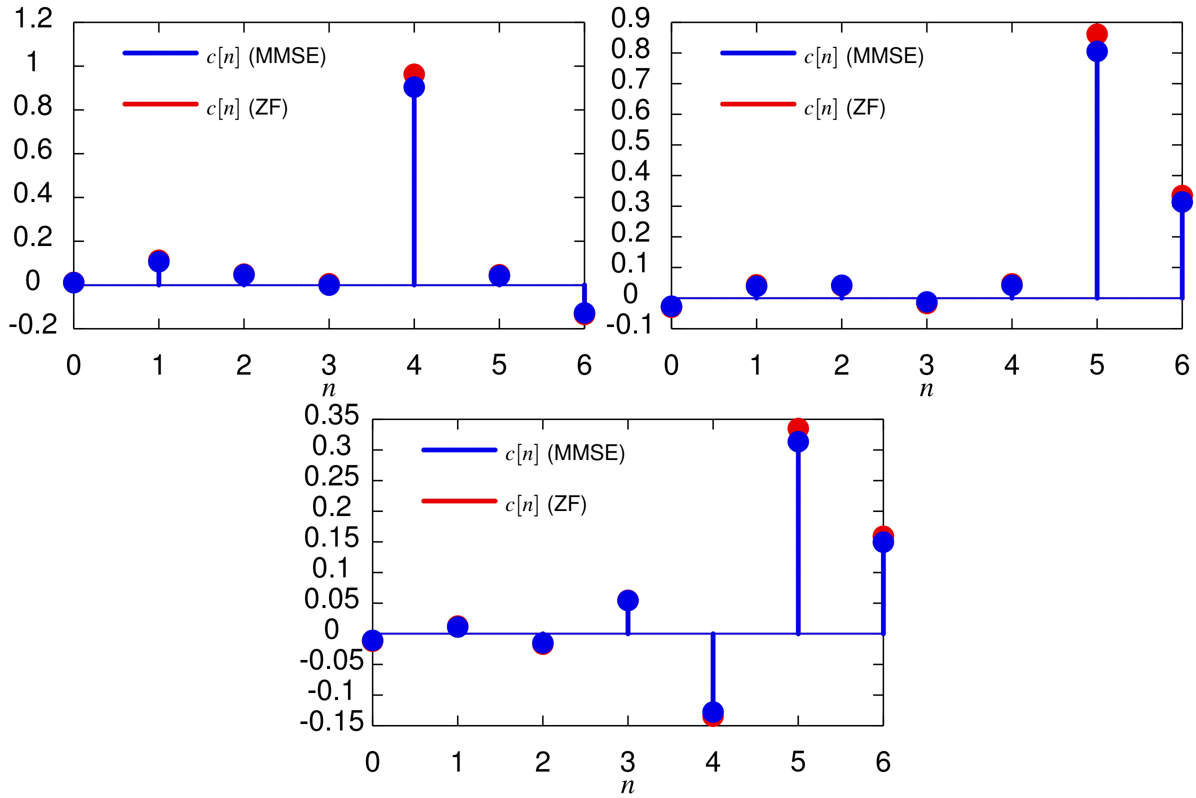
$$= \underbrace{\begin{bmatrix} +0.1993 & -0.2794 & +0.5587 & +0.3508 & +0.0316 & -0.0788 & -0.0320 \\ +0.0195 & +0.1697 & -0.2224 & +0.5898 & +0.3655 & -0.0035 & -0.0148 \\ -0.0099 & +0.0323 & +0.1430 & -0.2413 & +0.5936 & +0.3568 & +0.0292 \\ -0.0213 & +0.0221 & -0.0297 & +0.1084 & -0.2553 & +0.6270 & +0.2989 \end{bmatrix}}_{4\times 7 \equiv (K_w+1)\times(K_p+K_w+1)}$$

  - MMSE solutions are given by columms of this matrix (according to delay d)

MMSE - Joint responses for delays $d \in \{0, 1, 2, 3\}$



MMSE - Joint responses for delays $d \in \{4, 5, 6\}$



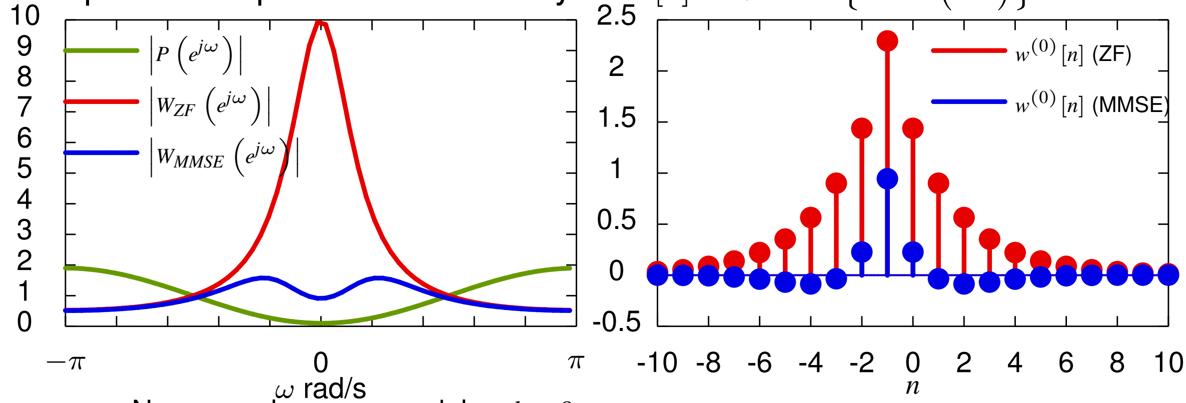
Equalizers without limits - Example - Channel E

- Channel $p[n] = -0.45\ \delta[n] + \delta[n-1] - 0.45\ \delta[n-2]$
- Channel frequency response

$$P\left(e^{j\omega}\right) = -0.45 + e^{-j\omega} + e^{-j\omega} - 0.45\ e^{-j\omega 2}$$

$$W_{ZF}^{(0)}\left(e^{j\omega}\right) = \frac{1}{P\left(e^{j\omega}\right)},\ W_{MMSE}^{(0)}\left(e^{j\omega}\right) = \frac{P^*\left(e^{j\omega}\right)}{\left|P\left(e^{j\omega}\right)\right|^2 + \frac{\sigma_z^2}{E_s}}$$

- Equalizer response without delay: $w^{(0)}[n] = \mathcal{TF}^{-1}\left\{W^{(0)}\left(e^{j\omega}\right)\right\}$



  - Non-causal response - delay $d \approx 9$
  - Relevant coefficients ≈ 19 ($K_w \approx 18$)

Equalizer ZF with 4 coefficients ($K_w = 3$) - Channel E - $d = 2$

- Channel $p[n] = -0.45\ \delta[n] + \delta[n-1] - 0.45\ \delta[n-2]$
- Involved matrices and vectors - delay $d = 2$

$$\mathbf{c}_d = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{6\times 1},\ \mathbf{P} = \underbrace{\begin{bmatrix} -0.45 & 0 & 0 & 0 \\ 1 & -0.45 & 0 & 0 \\ -0.45 & 1 & -0.45 & 0 \\ 0 & -0.45 & 1 & -0.45 \\ 0 & 0 & -0.45 & 1 \\ 0 & 0 & 0 & -0.45 \end{bmatrix}}_{6\times 4}$$

- ZF solution

$$\mathbf{w}_d^{ZF} = \mathbf{P}^{\#}\ \mathbf{c}_d = \begin{bmatrix} +0.4893 \\ +1.4097 \\ +0.6479 \\ +0.2118 \end{bmatrix},\ \mathbf{c}_d^{ZF} = \mathbf{P}\ \mathbf{w} = \begin{bmatrix} -0.2202 \\ -0.1450 \\ +0.8979 \\ -0.0818 \\ -0.0797 \\ -0.0953 \end{bmatrix}$$

$$\sigma_{ISI}^2 = E_s \times 0.0917$$

$$\sigma_{z'}^2 = \sigma_z^2 \times 2.6913$$

$$\frac{|c[d]|}{2\sqrt{\sigma_{ISI}^2 + \sigma_{z'}^2}} = 0.7475$$

Equalizer MMSE with 4 coefficients ($K_w = 3$) - Channel E - $d = 2$

- Noise variance $\sigma_z^2 = 0.1$, 2-PAM con $E_s = 1$
- Channel $p[n] = -0.45\ \delta[n] + \delta[n-1] - 0.45\ \delta[n-2]$
- Involved matrices and vectors - delay $d = 2$

$$\mathbf{c}_d = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{6\times 1},\ \mathbf{P} = \underbrace{\begin{bmatrix} -0.45 & 0 & 0 & 0 \\ 1 & -0.45 & 0 & 0 \\ -0.45 & 1 & -0.45 & 0 \\ 0 & -0.45 & 1 & -0.45 \\ 0 & 0 & -0.45 & 1 \\ 0 & 0 & 0 & -0.45 \end{bmatrix}}_{6\times 4},\ \lambda = \frac{\sigma_z^2}{E_s} = 0.1,\ \mathbf{I} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{4\times 4}$$

- MMSE solution

$$\mathbf{w}_d^{MMSE} = (\mathbf{P}^H\ \mathbf{P} + \lambda\ \mathbf{I})^{-1}\ \mathbf{P}^H\ \mathbf{c}_d = \begin{bmatrix} +0.2387 \\ +0.9566 \\ +0.2553 \\ +0.0239 \end{bmatrix},\ \mathbf{c}_d^{MMSE} = \begin{bmatrix} -0.2202 \\ -0.1450 \\ +0.8979 \\ -0.0818 \\ -0.0797 \\ -0.0953 \end{bmatrix}$$

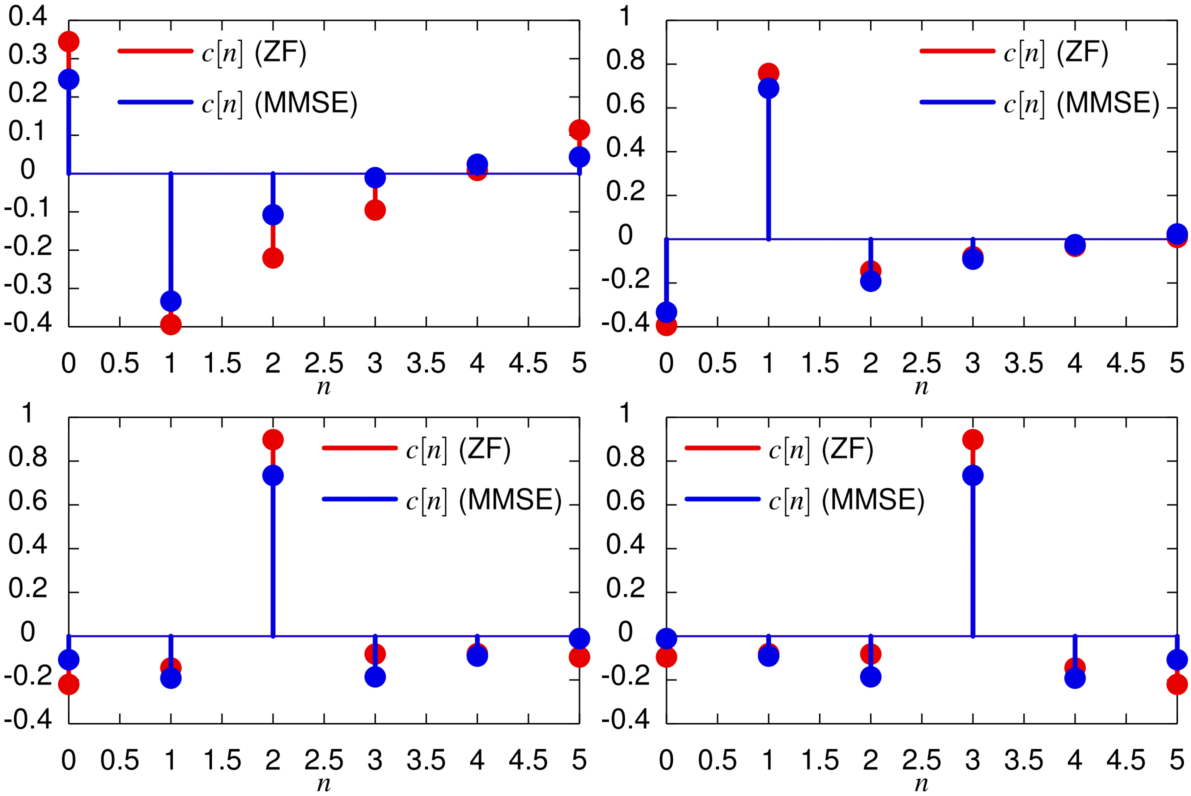
$$\sigma_{ISI}^2 = E_s \times 0.0913$$

$$\sigma_{z'}^2 = \sigma_z^2 \times 1.0379$$

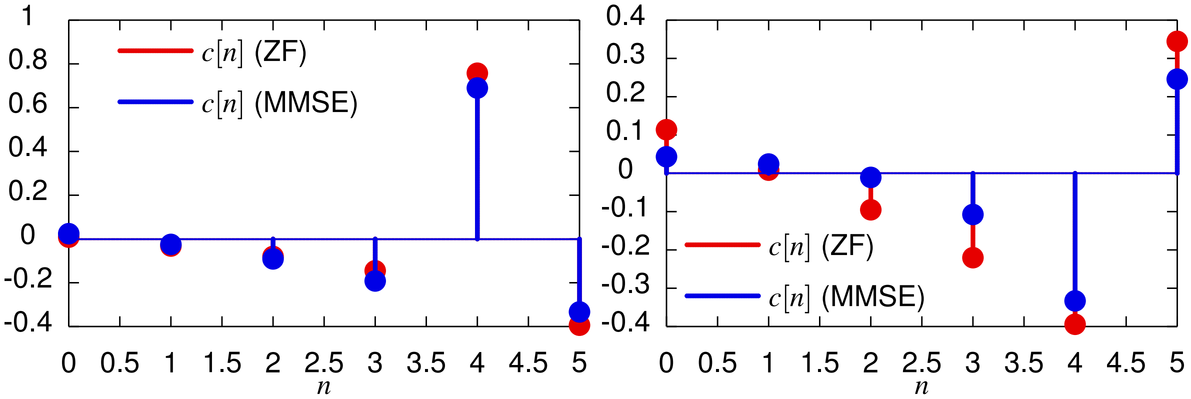
$$\frac{|c[d]|}{2\sqrt{\sigma_{ISI}^2 + \sigma_{z'}^2}} = 0.8313$$

(ZF: 0.7475)

ZF/MMSE - Joint responses for delays $d \in \{0, 1, 2, 3\}$



ZF/MMSE - Joint responses for delays $d \in \{4, 5\}$



Equalizers con 17 coefficients ($K_w = 16$) - Channel E - $d = 9$

- In this case solution is closer to the one without limitations in K_w
- ZF solution forces an almost zero ISI joint response
  - Noise can be seriously amplified

ZF	MMSE
$\sigma^2_{ISI} = E_s \times 0.0003$	$\sigma^2_{ISI} = E_s \times 0.0895$
$\sigma^2_{z'} = \sigma^2_z \times 11.8850$	$\sigma^2_{z'} = \sigma^2_z \times 1.0255$
$\frac{\lvert c[d]\rvert}{2\sqrt{\sigma^2_{ISI}+\sigma^2_{z'}}} = 0.4585$	$\frac{\lvert c[d]\rvert}{2\sqrt{\sigma^2_{ISI}+\sigma^2_{z'}}} = 0.8450$

Joint responses with 17 coefficients - Channel E