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Digital Communications
Grades in English

Chapter 3

Angle modulations (phase and frequency)

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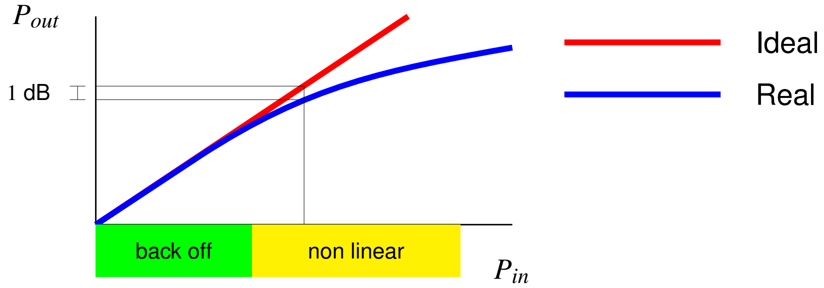
General characteristics of angle modulations

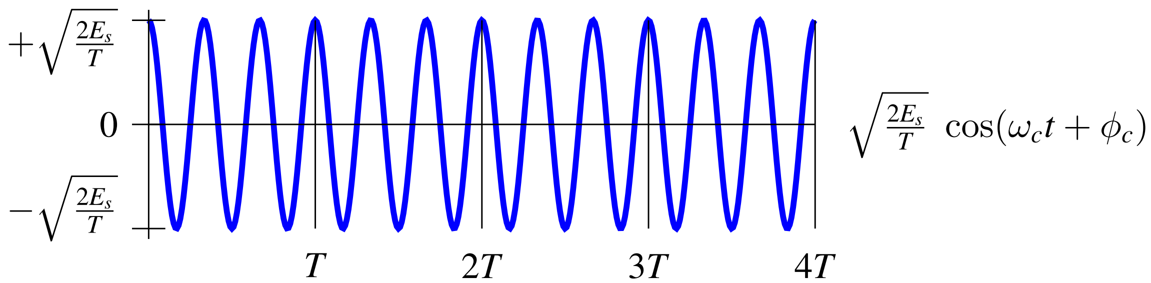
- Transmitted information ($A[n]$) is not impressed in the amplitude of modulated signal but in its angle information
  - Phase of the modulated signal in the symbol interval
  - Frequency of the modulated signal in the symbol interval
- Appropriate for transmission under strong amplitude distortion
  - Example: use of RF amplifiers in the nonlinear region



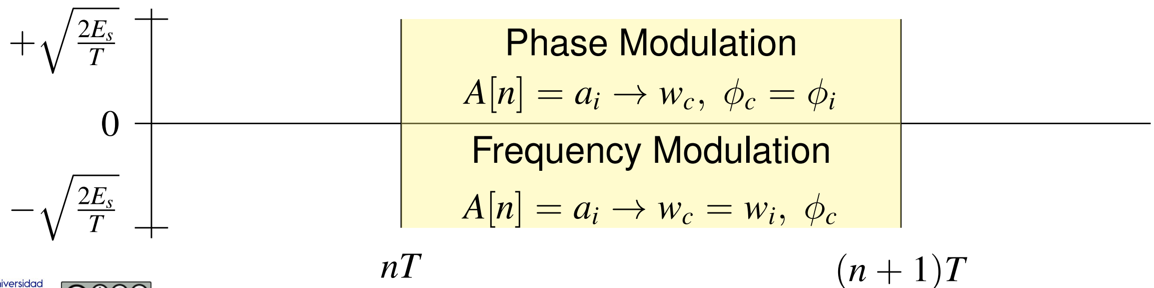
- Drawback: bandwidth is higher than that of linear amplitude modulations

Angle modulations

- Carrier with a constant amplitude



  - Two parameters: phase ϕ_c / frequency ω_c
- Angle modulations: phase / frequency inside a symbol interval depends on the symbol that is transmitted

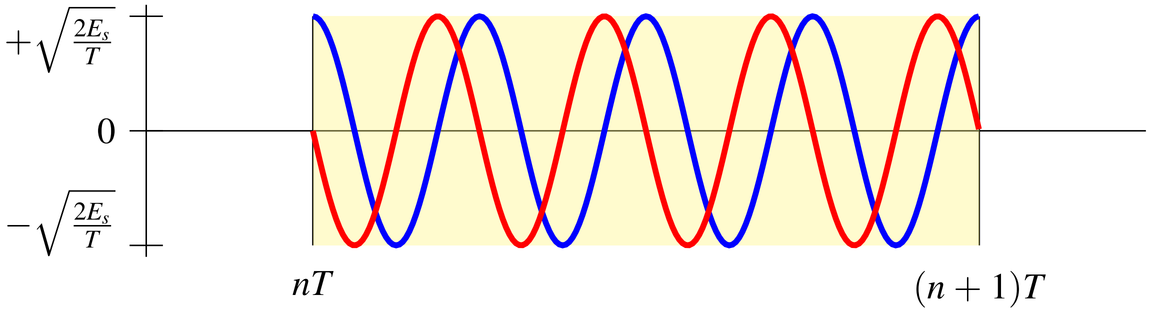


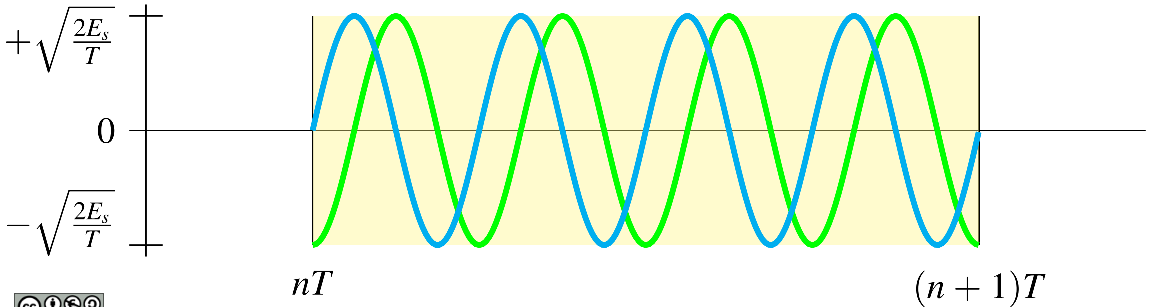
Angle modulations: example 4-ary constellations

$$A[n] \in \{a_0 \text{ (blue)}, a_1 \text{ (red)}, a_2 \text{ (green)}, a_3 \text{ (cyan)}\}$$

$$\boldsymbol{a}_0 \equiv \phi_c = 0 \quad \boldsymbol{a}_1 \equiv \phi_c = 90 \quad \boldsymbol{a}_2 \equiv \phi_c = 180 \quad \boldsymbol{a}_3 \equiv \phi_c = 270$$

- Phase modulation





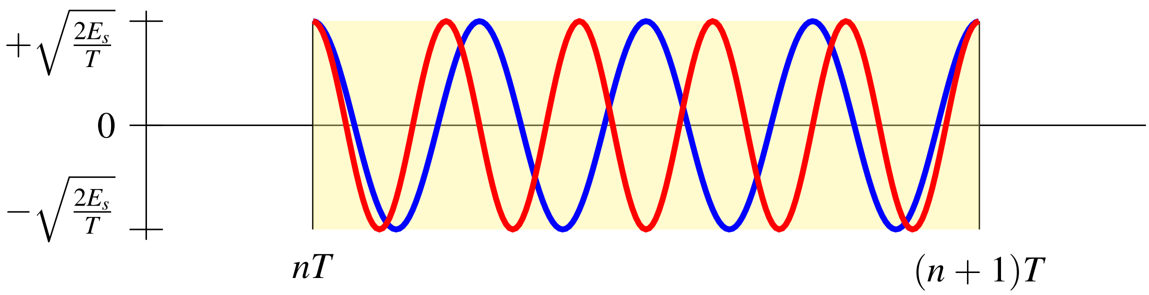
Angle modulations: example 4-ary constellations

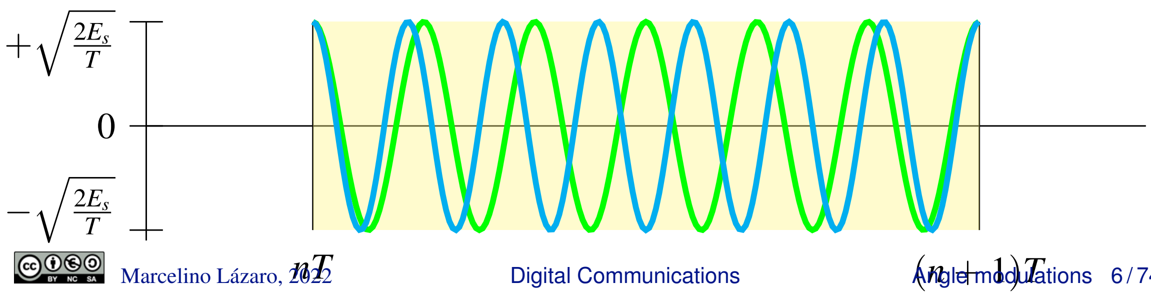
$$A[n] \in \{a_0 \text{ (blue)}, a_1 \text{ (red)}, a_2 \text{ (green)}, a_3 \text{ (cyan)}\}$$

$$\boldsymbol{a}_0 \equiv \omega_c = \frac{8\pi}{T} \quad \boldsymbol{a}_1 \equiv \omega_c = \frac{10\pi}{T} \quad \boldsymbol{a}_2 \equiv \omega_c = \frac{12\pi}{T} \quad \boldsymbol{a}_3 \equiv \omega_c = \frac{14\pi}{T} \text{ rad/s}$$

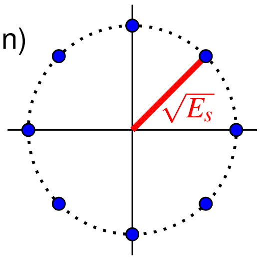
$$\boldsymbol{a}_0 \equiv f_c = 4R_s \quad \boldsymbol{a}_1 \equiv f_c = 5R_s \quad \boldsymbol{a}_2 \equiv f_c = 6R_s \quad \boldsymbol{a}_3 \equiv f_c = 7R_s \text{ Hz}$$

- Frequency modulation





Phase modulations



- Phase shift keying (PSK) modulation: linear (bandpass PAM formulation)
  - Constellations: Constant modulus - Information in the phase
    - ★ Symbols

$$\boxed{A[n] = \sqrt{E_s}\; e^{j\phi[n]}}$$

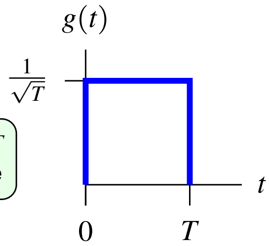
    - ★ Complex baseband signal

$$s(t) = \sum_n A[n]\; g(t - nT) = \sqrt{E_s} \sum_n e^{j\phi[n]}\; g(t - nT)$$

    - ★ Bandpass modulated signal

$$x(t) = \sqrt{2}\mathcal{R}e\left\{s(t)\; e^{j\omega_c t}\right\} = \sqrt{2E_s}\mathcal{R}e\left\{\sum_n g(t - nT)\; e^{j(\omega_c t + \phi[n])}\right\}$$

$$= \underbrace{\sqrt{2E_s}\sum_n g(t - nT)}_{\text{envelope}}\; \cos(\omega_c t + \phi[n])$$



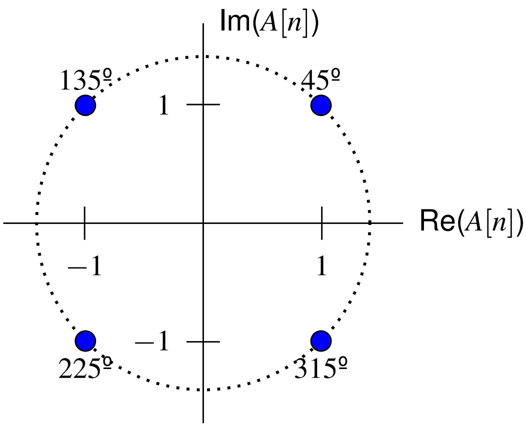
  - Transmitter filter : Constant envelope can be achieved with

$$\boxed{g(t) = \frac{1}{\sqrt{T}}\, w_T(t), \qquad w_T(t) = \begin{cases} 1, & 0 \le t < T \\ 0, & \text{other case} \end{cases}}$$

- Drawback: high bandwidth (phase shifts at $t = nT$)

$$\boxed{S_X(j\omega) = \frac{E_s}{2T}\left[|G(j\omega - j\omega_c)|^2 + |G(j\omega + j\omega_c)|^2\right],\; |G(j\omega)| = \sqrt{T}\;\text{sinc}\left(\frac{\omega T}{2\pi}\right)}$$

QPSK Modulation - PSK for $M = 4$ - Constellation



$$\phi[n] = 45: A[n] = +1 + j \equiv \begin{bmatrix} +1 \\ +1 \end{bmatrix}$$

$$\phi[n] = 135: A[n] = -1 + j \equiv \begin{bmatrix} -1 \\ +1 \end{bmatrix}$$

$$\phi[n] = 225: A[n] = -1 - j \equiv \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\phi[n] = 315: A[n] = +1 - j \equiv \begin{bmatrix} +1 \\ -1 \end{bmatrix}$$

Phase shifts in a QPSK modulation

- PSK signal

$$x(t) = \sqrt{2}\, s_I(t)\ \cos(\omega_c t) - \sqrt{2}\, s_Q(t)\ \sin(\omega_c t) = \sqrt{2E_s} \sum_n g(t-nT)\ \cos(\omega_c t + \phi[n])$$

with

$$s_I(t) = \sum_n \mathcal{R}e\{A[n]\}\ g(t-nT) = \sum_n A_I[n]\ g(t-nT)$$

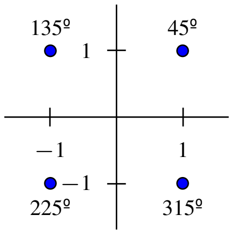
$$s_Q(t) = \sum_n \mathcal{I}m\{A[n]\}\ g(t-nT) = \sum_n A_Q[n]\ g(t-nT)$$

- Phase shifts
  - $\pm 90^{\text{o}}$: a change in $s_I(t)$ or in $s_Q(t)$
  - 180^{o}: simultaneous change in both $s_I(t)$ and $s_Q(t)$

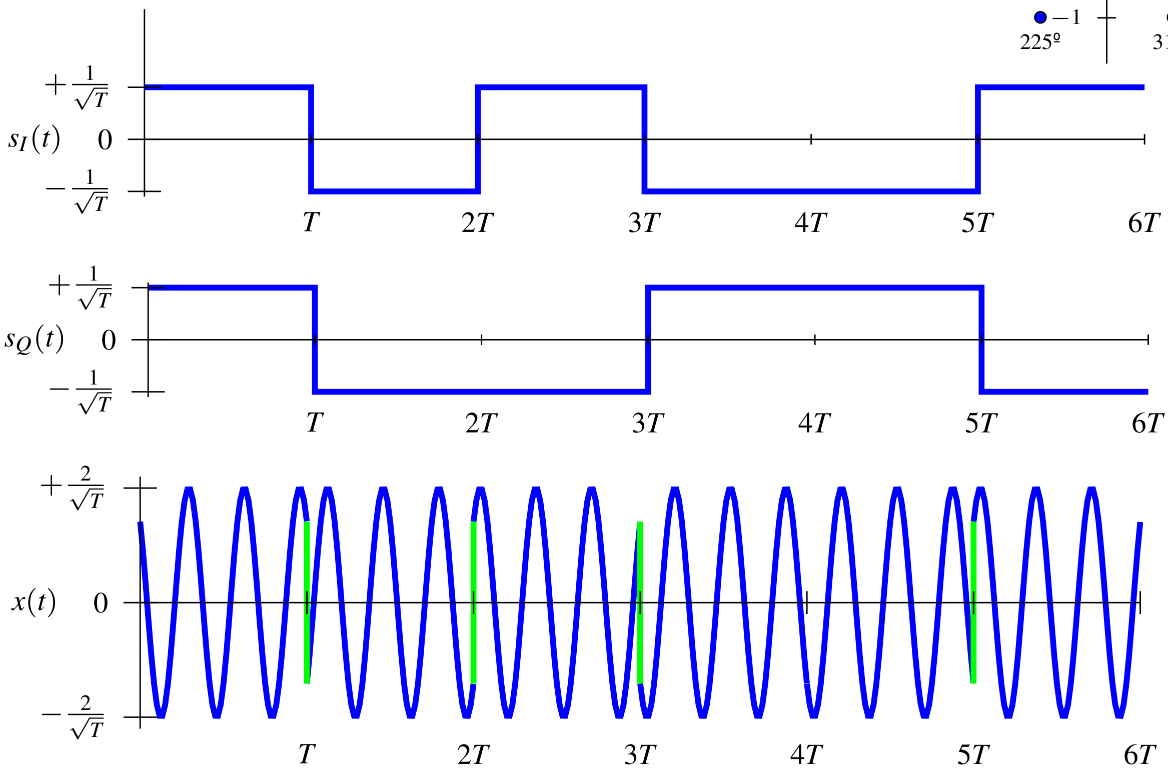
Some trigonometric identities

$$+\cos(\omega_c t) - \sin(\omega_c t) = \sqrt{2}\cos(\omega_c t + 45^{\text{o}})$$
$$-\cos(\omega_c t) - \sin(\omega_c t) = \sqrt{2}\cos(\omega_c t + 135^{\text{o}})$$
$$-\cos(\omega_c t) + \sin(\omega_c t) = \sqrt{2}\cos(\omega_c t + 225^{\text{o}})$$
$$+\cos(\omega_c t) + \sin(\omega_c t) = \sqrt{2}\cos(\omega_c t + 135^{\text{o}})$$

QPSK Modulation - Waveforms



$$A[0] = \begin{bmatrix} +1 \\ +1 \end{bmatrix}, A[1] = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, A[2] = \begin{bmatrix} +1 \\ -1 \end{bmatrix}, A[3] = \begin{bmatrix} -1 \\ +1 \end{bmatrix}, A[4] = \begin{bmatrix} -1 \\ +1 \end{bmatrix}, A[5] = \begin{bmatrix} +1 \\ -1 \end{bmatrix}$$



Offset QPSK modulation (OQPSK)

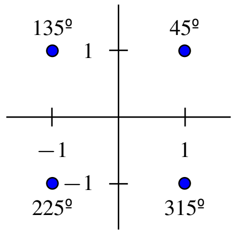
- Goal: to avoid 180º phase shifts
  - Avoidance of simultaneous transitions of $s_I(t)$ and $s_Q(t)$
- OQPSK signal
  - Quadrature component is delayed $T/2$ seconds
  - Phase shifts are limited to $\pm 90º$
  - Phase shifts happen often (can occur each $T/2$ seconds)

$$x(t) = \sqrt{2}\, s_I(t)\ \cos(\omega_c t) - \sqrt{2}\, s_Q(t)\ \sin(\omega_c t)$$

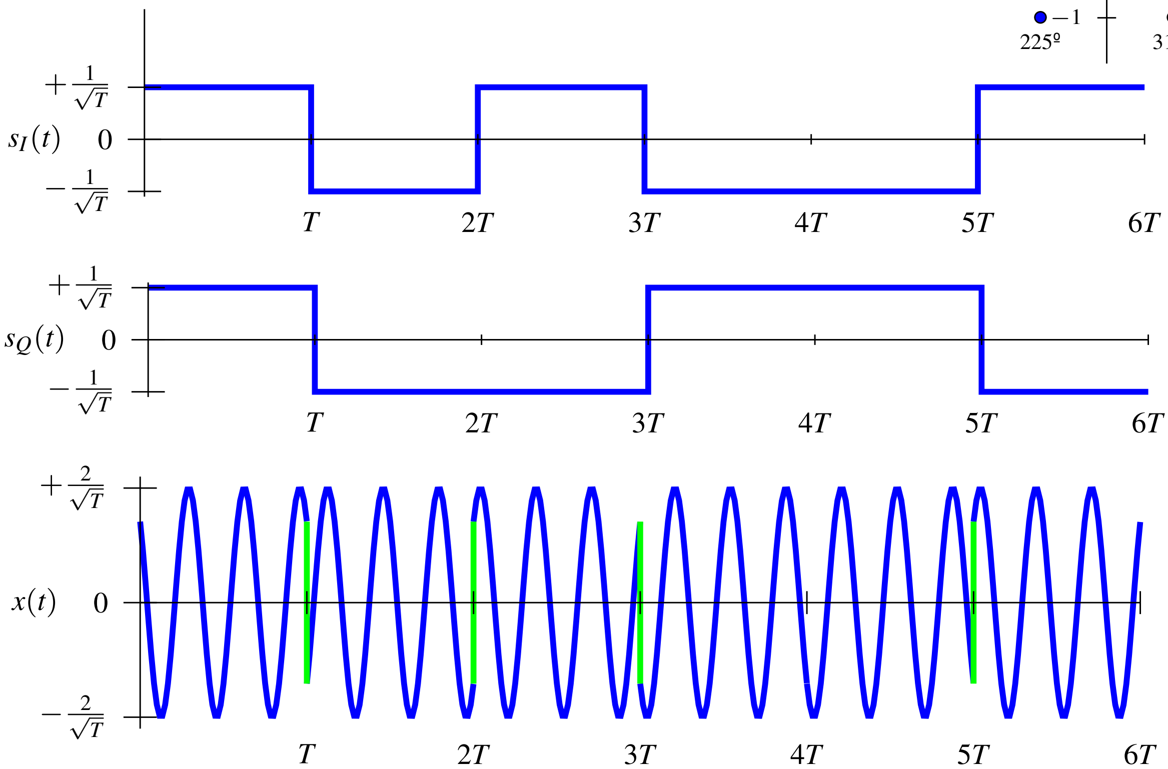
$$s_I(t) = \sum_n A_I[n]\ g(t - nT)$$

$$s_Q(t) = \sum_n A_Q[n]\ g(t - nT - T/2)$$

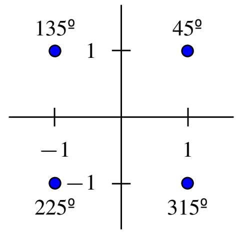
QPSK Modulation - Waveforms



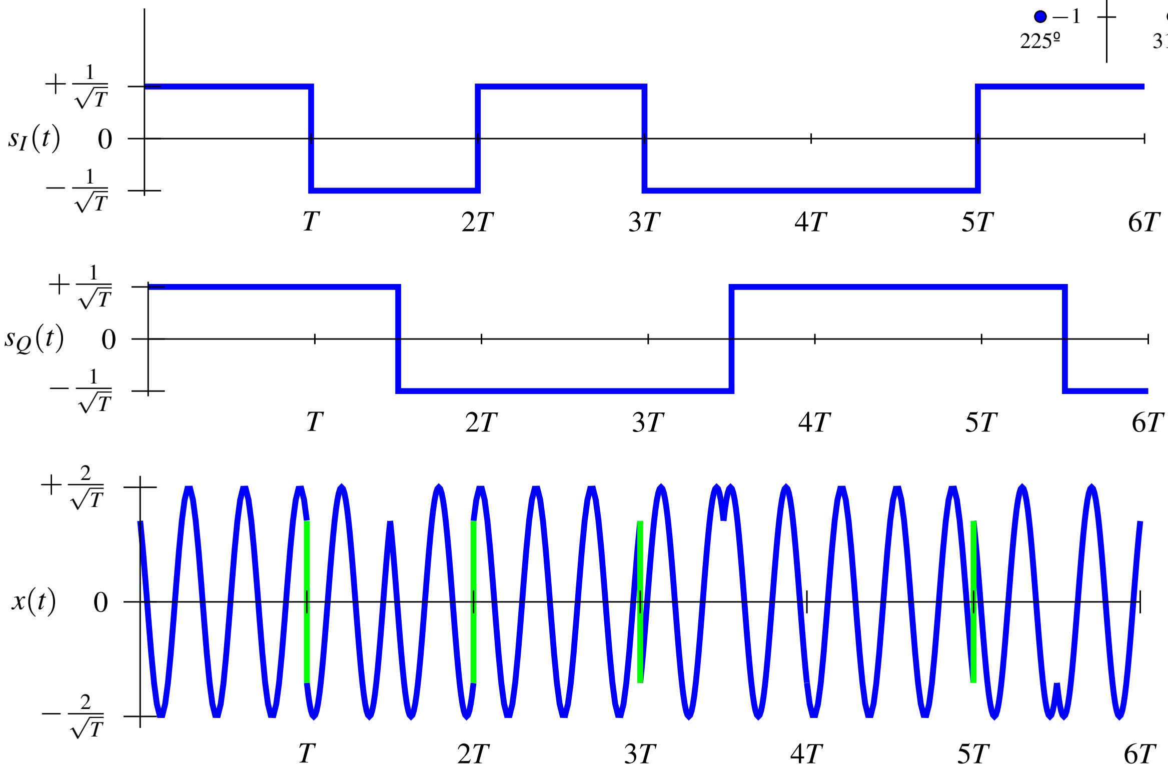
$A[0] = \begin{bmatrix} +1 \\ +1 \end{bmatrix}, A[1] = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, A[2] = \begin{bmatrix} +1 \\ -1 \end{bmatrix}, A[3] = \begin{bmatrix} -1 \\ +1 \end{bmatrix}, A[4] = \begin{bmatrix} -1 \\ +1 \end{bmatrix}, A[5] = \begin{bmatrix} +1 \\ -1 \end{bmatrix}$



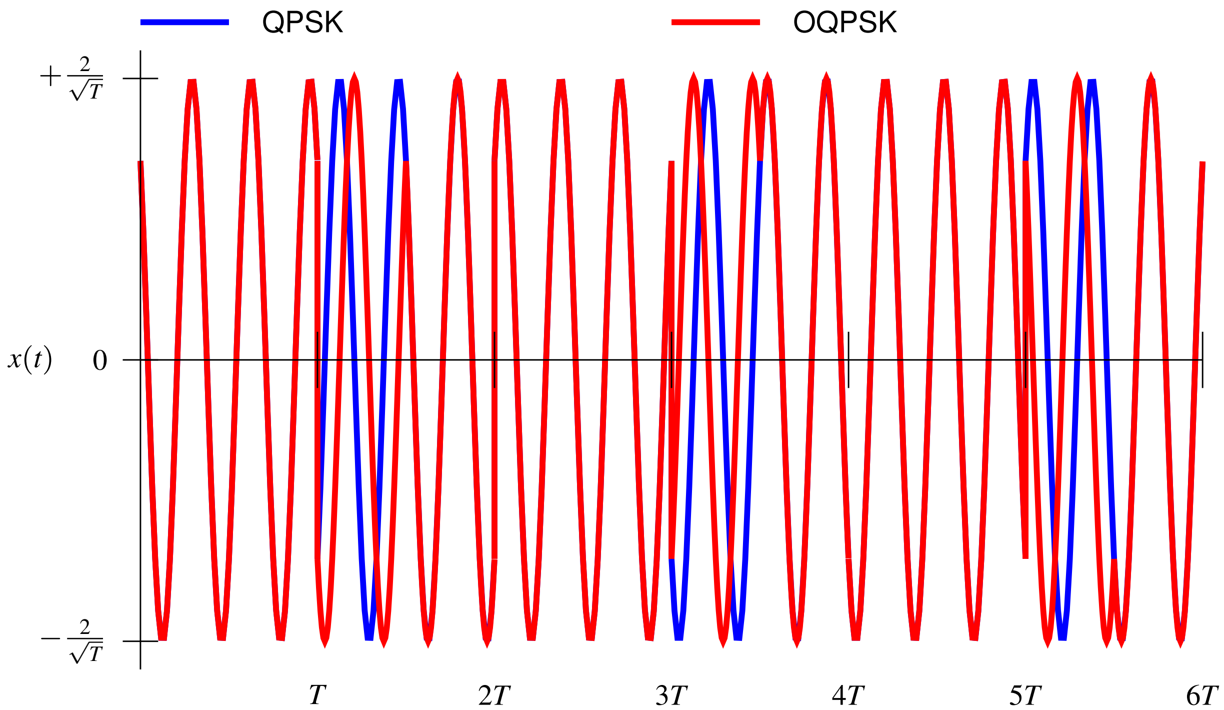
OQPSK Modulation - Waveforms - Delay of $s_Q(t)$



$A[0] = \begin{bmatrix} +1 \\ +1 \end{bmatrix}, A[1] = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, A[2] = \begin{bmatrix} +1 \\ -1 \end{bmatrix}, A[3] = \begin{bmatrix} -1 \\ +1 \end{bmatrix}, A[4] = \begin{bmatrix} -1 \\ +1 \end{bmatrix}, A[5] = \begin{bmatrix} +1 \\ -1 \end{bmatrix}$



Modulation waveforms - QPSK vs OQPSK



Spectrum of OQPSK modulation

- Definitions

$$x_I(t) = \sqrt{2}\, s_I(t)\ \cos(\omega_c t),\ x_Q(t) = \sqrt{2}\, s_Q(t)\ \sin(\omega_c t)$$

- Spectrum for each component (s_k, $k \in \{I, Q\}$)

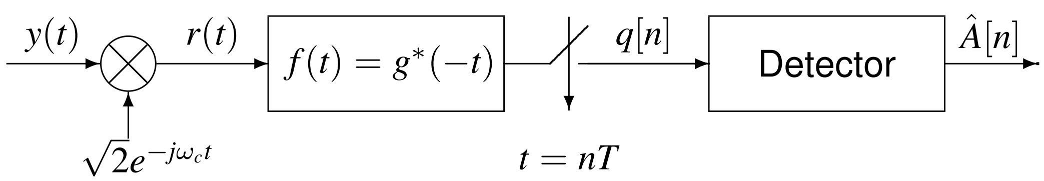
$$S_{x_k}(j\omega) = \frac{1}{2}\left[S_{s_k}(j\omega - j\omega_c) + S^*_{s_k}(-j\omega - j\omega_c)\right]$$

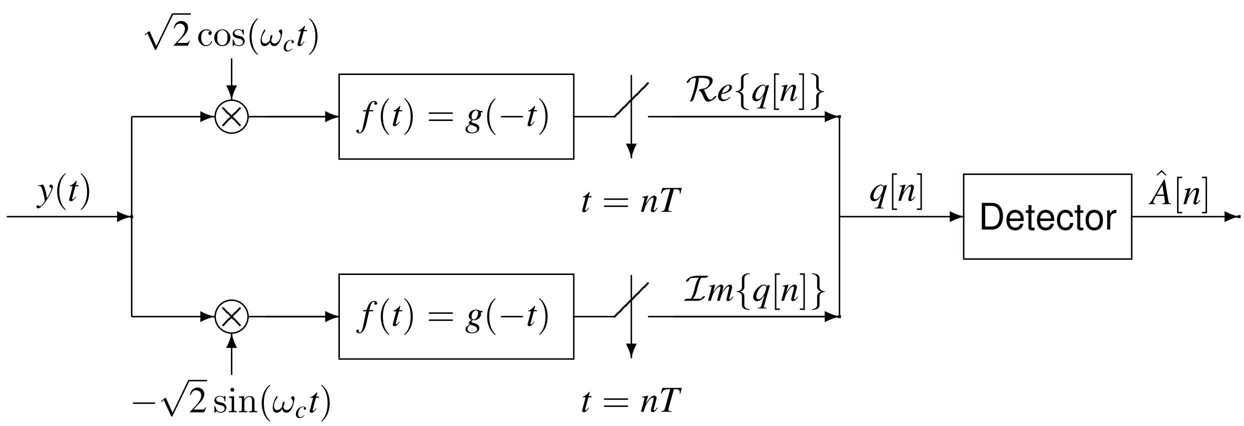
$$S_{s_I}(j\omega) = \frac{\mathcal{E}\{\mathcal{Re}\{A[n]\}\}}{T}\,|G(j\omega)|^2\,, S_{s_Q}(j\omega) = \frac{\mathcal{E}\{\mathcal{Im}\{A[n]\}\}}{T}\,|G(j\omega)|^2$$

- Spectrum of OQPSK modulation

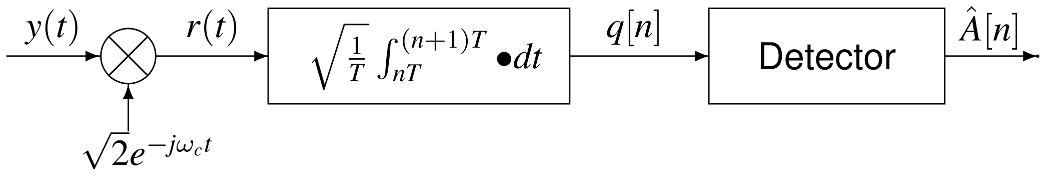
$$S_X(j\omega) = \frac{E_s}{2T}\left[|G(j\omega - j\omega_c)|^2 + |G(-j\omega - j\omega_c)|^2\right]$$

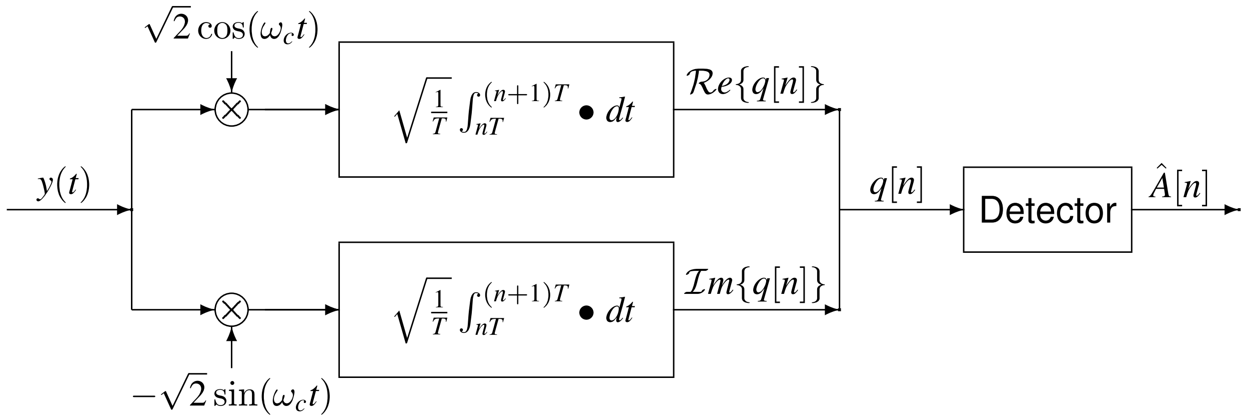
Receiver for PSK modulations





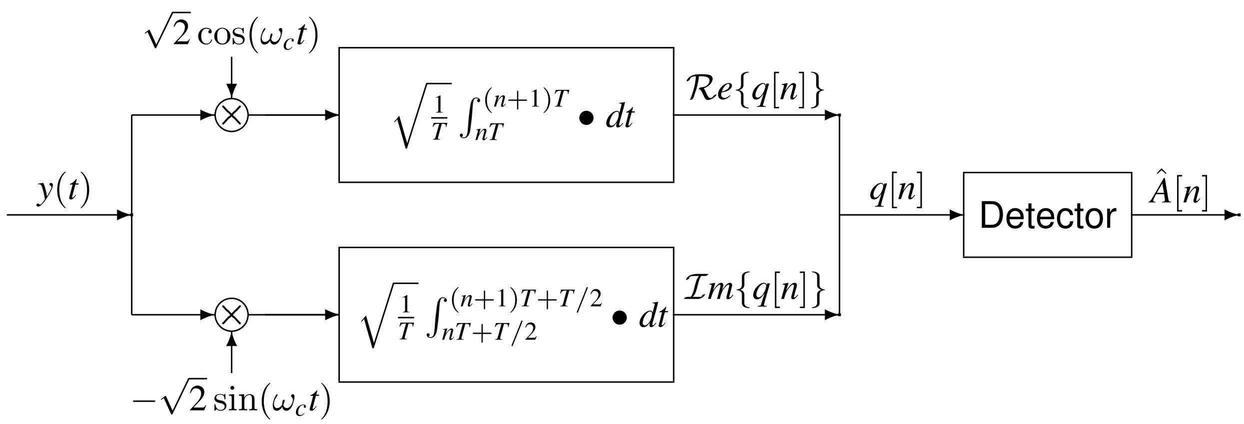
Receiver for PSK modulations (II)





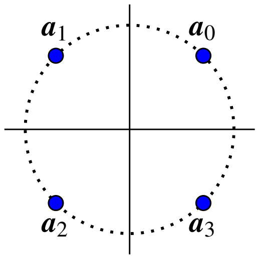
Receiver for OQPSK

The $T/2$ delay in the quadrature component is taken into account (delay in the correlator)

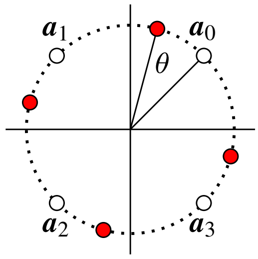


Effect of non-coherent receiver in PSK modulations

- In a non-coherent receiver, phase of carriers used at receiver to demodulate (θ_R) is different from phase of carriers used at the transmitter to modulate (θ_T)
  - Difference of $\theta = \theta_R - \theta_T$ radians
- The effect of this phase difference is that received constellation is rotated θ radians



Transmitted phase $\phi[n]$



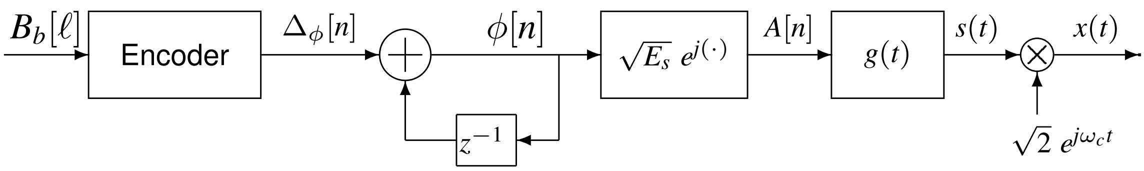
Received phase $\phi_R[n] = \phi[n] + \theta$

  - This effect can seriously affect performance
  - However, non-coherent receivers have a lower cost
    - ★ Differential PSK modulation allows the use of non-coherent receivers

Differential phase modulations (DPSK)

- They do not require a coherent demodulation
- PSK with differential phase enconding

$$\phi[n] = \phi[n-1] + \Delta_\phi[n]$$



- Encoder for M-ary modulation

$$\Delta_\phi[n] \in \left\{0, \frac{2\pi}{M}, \cdots, \frac{2\pi(M-1)}{M}\right\} \text{ rad}$$

Bit assignment is performed through $\Delta_\phi[n]$

Example: 4-PSK (Gray encoding)

$\Delta_\phi[n]$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
Bits	00	01	11	10

- Initialization: selection of an arbitrary (known) value for $\phi[-1]$
  - No error propagation because of initialization

Differential phase demodulation

- Decoding of the m bits associatted to discrete instant n
  - PSK modulation
    - ★ From $\hat{\phi}[n]$ (estimate of the phase of received symbol)

$$\phi_R[n] = \phi[n] + \theta \quad \Rightarrow \quad \hat{\phi}[n] \quad \Rightarrow \quad m \text{ bits}$$

Phase shift θ in the phase of the received symbol can be catastrophic!!!

  - DPSK modulation
    - ★ From $\hat{\Delta}_\phi[n]$ (estimate of the difference between phases)

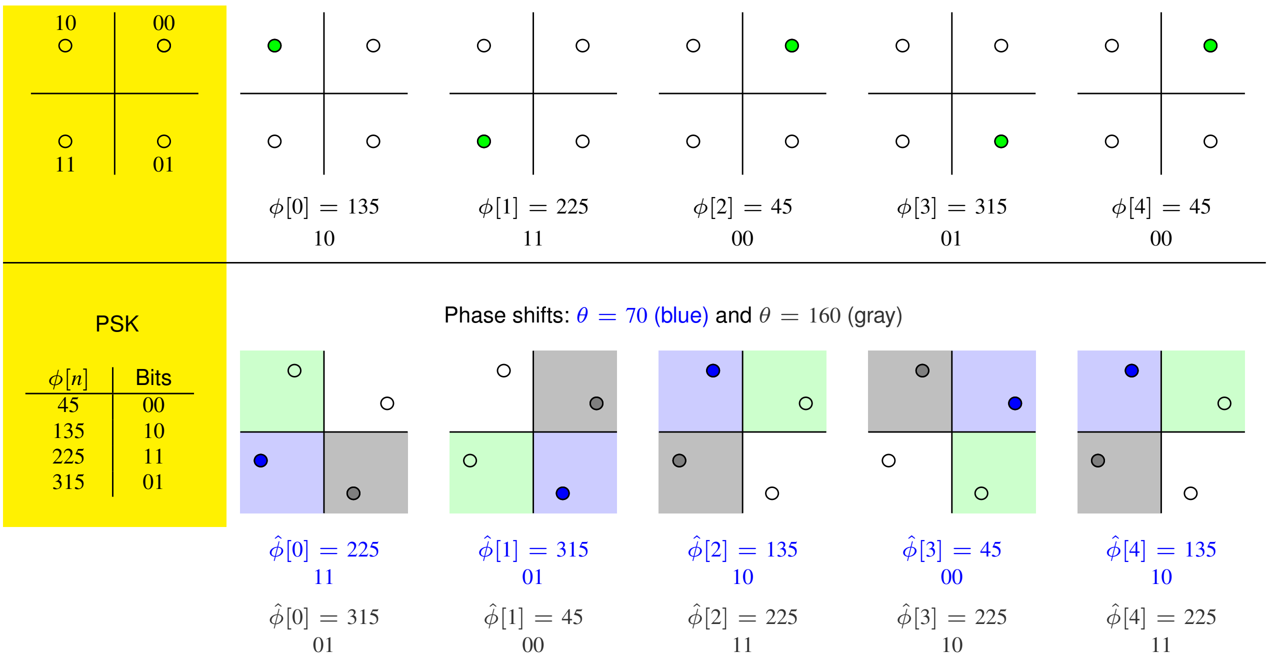
$$\phi_R[n] = \phi[n] + \theta, \qquad \phi_R[n-1] = \phi[n-1] + \theta$$

$$\Delta_{\phi_R}[n] = \phi_R[n] - \phi_R[n-1] = \phi[n] - \phi[n-1]$$

$$\Delta_{\phi_R}[n] \quad \Rightarrow \quad \hat{\Delta}_\phi[n] \quad \Rightarrow \quad m \text{ bits}$$

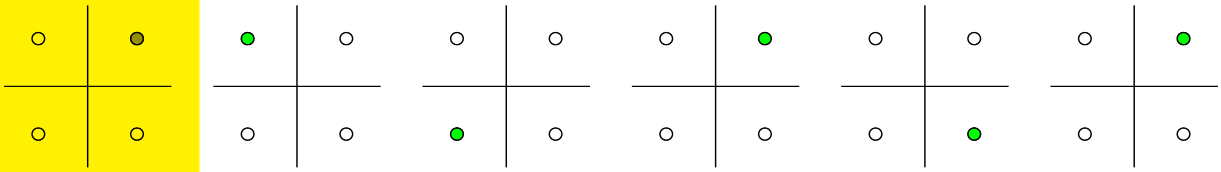
Phase shift θ is irrelevant (has no effect)

Effect of non-coherent receiver in PSK modulations



ℓ	0	1	2	3	4	5	6	7	8	9
$B_b[\ell]$	1	0	1	1	0	0	0	1	0	0
$\hat{B}_b[\ell]$	1	1	0	1	1	0	0	0	1	0
$\hat{B}_b[\ell]$	0	1	0	0	1	1	1	0	1	1

Effect of non-coherent receiver in DPSK modulations

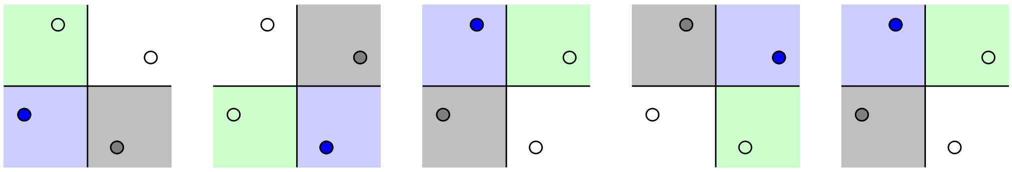


$\phi[-1]=45$ REF (NO Tx)	$\phi[0]=135$	$\phi[1]=225$	$\phi[2]=45$	$\phi[3]=315$	$\phi[4]=45$
	$\Delta_\phi[0]=90\equiv 10$	$\Delta_\phi[1]=90\equiv 10$	$\Delta_\phi[2]=180\equiv 11$	$\Delta_\phi[3]=270\equiv 01$	$\Delta_\phi[4]=90\equiv 10$

DPSK

$\Delta_\phi[n]$	Bits
0	00
90	10
180	11
270	01

Phase shifts: $\theta=70$ (blue) and $\theta=160$ (gray)



	$\phi_R[0]=205$	$\phi_R[1]=295$	$\phi_R[2]=115$	$\phi_R[3]=385(25)$	$\phi_R[4]=115$
$\Delta_{\phi_R}[0]=160\rightarrow$	$\hat{\Delta}_\phi[0]=180\equiv 11$	$\hat{\Delta}_\phi[1]=90\equiv 10$	$\hat{\Delta}_\phi[2]=180\equiv 11$	$\hat{\Delta}_\phi[3]=270\equiv 01$	$\hat{\Delta}_\phi[4]=90\equiv 10$
	$\hat{\phi}[0]=295$	$\phi_R[1]=385(25)$	$\phi_R[2]=205$	$\phi_R[3]=225$	$\phi_R[4]=205$
$\Delta_{\phi_R}[0]=250\rightarrow$	$\hat{\Delta}_\phi[0]=270\equiv 01$	$\hat{\Delta}_\phi[1]=90\equiv 10$	$\hat{\Delta}_\phi[2]=180\equiv 11$	$\hat{\Delta}_\phi[3]=270\equiv 01$	$\hat{\Delta}_\phi[4]=90\equiv 10$

ℓ	0	1	2	3	4	5	6	7	8	9
$B_b[\ell]$	1	0	1	0	1	1	0	1	1	0
$\hat{B}_b[\ell]$	1	1	1	0	1	1	0	1	1	0
$\hat{B}_b[\ell]$	0	1	1	0	1	1	0	1	1	0

DPSK modulation / demodulation - Example

- 4-PSK constellation

$$\phi[n]\in\left\{\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}\right\}\qquad \Delta_\phi[n]\in\left\{0,\frac{\pi}{2},\pi,\frac{3\pi}{2}\right\}\qquad \phi[-1]=\frac{\pi}{4}$$

- Binary assignment

PSK: Binary assignment is done over $\phi[n]$

$\phi[n]$	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$
Bits	00	01	11	10

(Gray encoding)

DPSK: Binary assignment is done over $\Delta_\phi[n]$

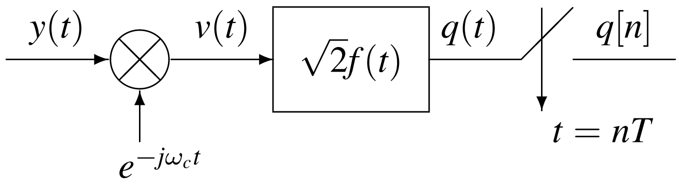
$\Delta_\phi[n]$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
Bits	00	01	11	10

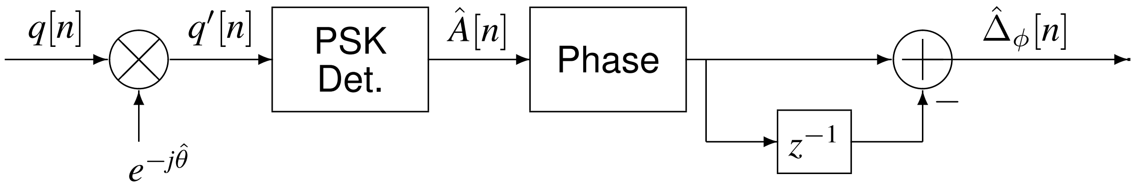
(Gray encoding)

- Bits to be transmitted $B_b[\ell]=00\ 10\ 01\ 11\ 10\cdots$

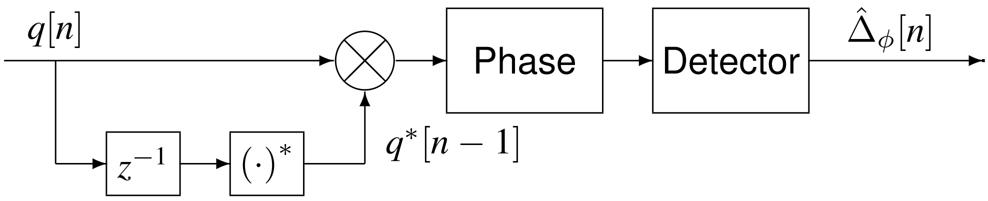
n	0	1	2	3	4
$B[n]$	00	10	01	11	10
PSK: $\phi[n]$	$\frac{\pi}{4}$	$\frac{7\pi}{4}$	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$
PSK: $\phi_R[n]$	$\frac{\pi}{4}+\theta$	$\frac{7\pi}{4}+\theta$	$\frac{3\pi}{4}+\theta$	$\frac{5\pi}{4}+\theta$	$\frac{7\pi}{4}+\theta$
DPSK: $\Delta_\phi[n]$	0	$\frac{3\pi}{2}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
DPSK: $\phi[n]$	$\frac{\pi}{4}$	$\frac{7\pi}{4}$	$\frac{\pi}{4}$	$\frac{5\pi}{4}$	$\frac{3\pi}{4}$
DPSK: $\phi_R[n]$	$\frac{\pi}{4}+\theta$	$\frac{7\pi}{4}+\theta$	$\frac{\pi}{4}+\theta$	$\frac{5\pi}{4}+\theta$	$\frac{3\pi}{4}+\theta$
DPSK: $\Delta_{\phi_R}[n]$	θ	$\frac{3\pi}{2}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$

Differential PSK demodulators





Coherent Receiver



Non-coherent Receiver

Non-coherent DPSK demodulator

- Observation

$$q[n] = \sqrt{E_s}\; e^{j(\phi[n]+\theta)} + z[n]$$

  - Previous observation (complex conjugated)

$$q^*[n-1] = \sqrt{E_s}\; e^{-j(\phi[n-1]+\theta)} + z^*[n-1]$$

- Multiplier

$$\begin{aligned} q[n]\; q^*[n-1] = & E_s\; e^{j(\phi[n]-\phi[n-1])} + \sqrt{E_s}\; e^{j(\phi[n]+\theta)}\; z^*[n-1] \\ & + \sqrt{E_s}\; e^{-j(\phi[n-1]+\theta)}\; z[n] + z[n]\; z^*[n-1] \end{aligned}$$

- Detector

$$\hat{\Delta}_{\phi_R}[n] = \angle\{q[n] \times q^*[n-1]\}$$

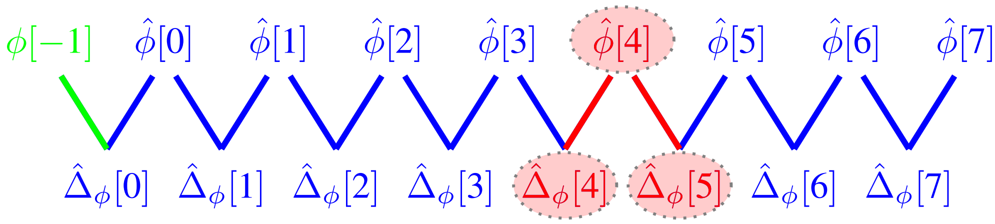
$$\hat{\Delta}_\phi[n] = \arg \min_{\Delta_\phi[n]} d\left(\hat{\Delta}_{\phi_R}[n], \Delta_\phi[n]\right)$$

Probability of error for DPSK

- Probability of error using coherent receivers

$$P_e \approx 2\, P_e^{PSK}$$

  - An erroneous decided symbol $\hat{A}[n]$ (phase $\hat{\phi}[n]$) affects two increments $\Delta_\phi[n]$ and $\Delta_\phi[n+1]$



- To use a differential PSK with coherent receivers is useless
  - No advantages (in the cost of the receiver)
  - Performance is worse than using conventional PSK

Probability of error for DPSK (II)

- Probability of error with non-coherent receivers
  - Decoding: two observations are processed $(q[n],\ q[n-1])$
    - ★ Effect of two noise samples

$$z[n],\ z[n-1]$$

  - Statistic used for detection

Coherent Receiver PSK

$$q[n] = \underbrace{\sqrt{E_s}\, e^{j\phi[n]}}_{\text{Phase } \phi[n]} + z[n]$$

$$\frac{q[n]\, q^*[n-1]}{\sqrt{E_s}} = \sqrt{E_s}\, \underbrace{e^{j(\phi[n]-\phi[n-1])}}_{\text{Phase } \Delta_\phi[n]} + e^{j(\phi[n]+\theta)}\, z^*[n-1] + e^{-j(\phi[n-1]+\theta)}\, z[n] + \frac{z[n]\, z^*[n-1]}{\sqrt{E_s}}$$

  - Three terms of noise
    - ★ Last one can be negligible for high E_s/σ_z^2
    - ★ The other two terms: independent, circularly symmetric
  - Signal to noise ratio: 3 dB loss (approx. double noise energy)
    - ★ Signal energy: E_s
    - ★ Noise energy: $\approx 2\sigma_z^2$

Frequency shift keying (FSK) modulation

- Information: discrete frequency changes in the frequency of a carrier
- Definition: M pulses (to map M symbols)

$$g_i(t) = \sin(\omega_i t)\; w_T(t), \quad i = 0, 1, \cdots, M-1$$

- Encoder: defines index of transmitted pulse at discrete time n

$$A[n] \in \{i = 0, 1, \cdots, M-1\}$$

- FSK signal in the time domain

$$x(t) = K \sum_n g_{A[n]}(t - nT)$$

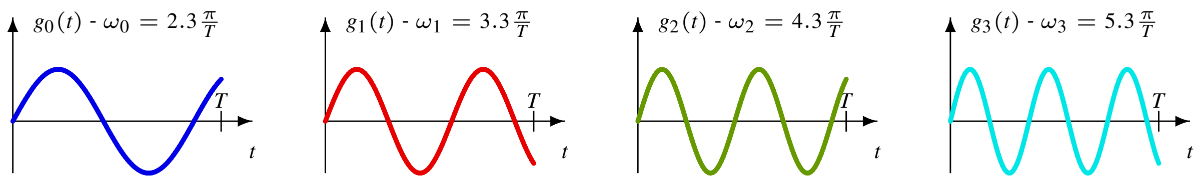
- Continuous phase FSK (CPFSK)
  - Phase continuity: pulses with an integer number of periods in T seconds

Frequencies: $\omega_i = \frac{2\pi}{T} N_i$ rad/s, $\quad f_i = R_s \times N_i$ Hz, $\quad N_i \in \mathbb{Z}, \quad i = 0, \cdots, M-1$

  - Minimum bandwidth: consecutive N_i (spectrum of $g_i(t)$ is centered at ω_i)

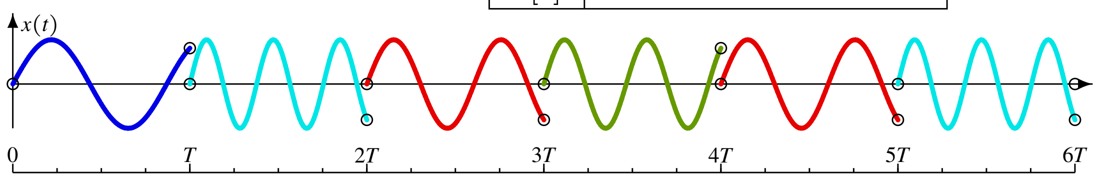
FSK waveforms - Example for $M = 4$ - Phase shifts

- Example with frequencies not being integer multiples of $\frac{2\pi}{T}$ rad/s



- Waveform for data sequence

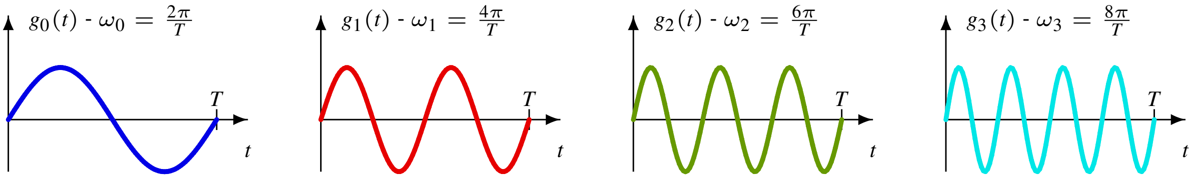
n	0	1	2	3	4	5
$A[n]$	0	3	1	2	1	3



Phase shifts happen at multiples of T (when $A[n] \neq A[n-1]$)

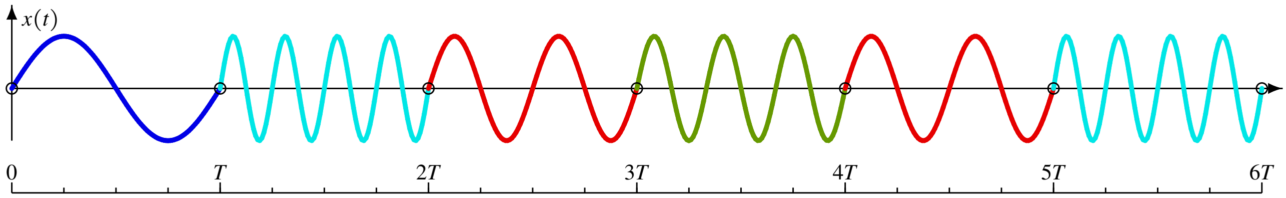
CPFSK waveforms - Example for $M = 4$

- CPFSK pulses for $M = 4$ (just one example)



- Waveform for data sequence

n	0	1	2	3	4	5
$A[n]$	0	3	1	2	1	3



Orthogonality of CPFSK

- Inner product of two pulses

$$\begin{aligned}\langle g_i(t), g_\ell(t)\rangle &= \int_0^T \sin(\omega_i t)\ \sin(\omega_\ell t)\ dt \\ &= \frac{1}{2}\int_0^T \cos(\underbrace{(\omega_i-\omega_\ell)}_{(N_i-N_\ell)\frac{2\pi}{T}}\ t)\ dt - \frac{1}{2}\int_0^T \cos(\underbrace{(\omega_i+\omega_\ell)}_{(N_i+N_\ell)\frac{2\pi}{T}}\ t)\ dt \\ &= \frac{T}{2}\ \delta[i-\ell]\end{aligned}$$

Pulses of CPFSK are orthogonal!!!

- Definition of an orthonormal base of dimension M

$$\phi_i(t) = \sqrt{\frac{2}{T}}\sin(\omega_i t)\ w_T(t), \quad i = 0, 1, \cdots, M-1$$

- CPFKS signal as an expansion in the orthonormal base

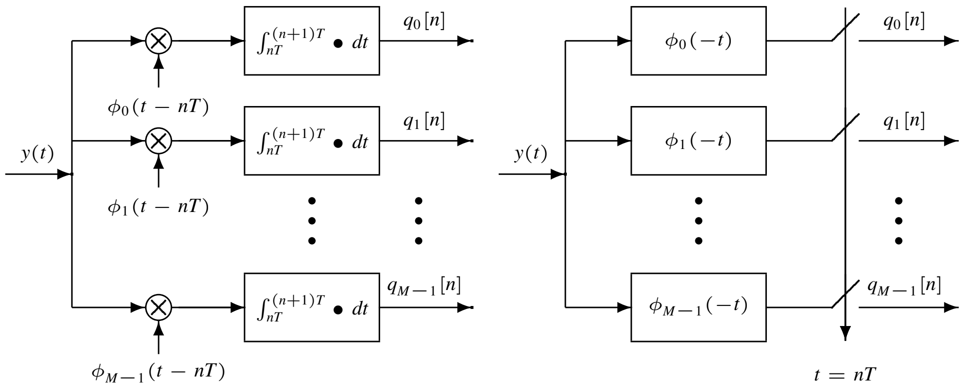
$$x(t) = \sqrt{E_s}\sum_n \phi_{A[n]}(t - nT)$$

Receiver for CPFSK (coherent)

- Orthonormal base (M-dimensional space)
  - Orthogonal constellation (dimension M)

$$A[n]=0\equiv \boldsymbol{a}_0=[\sqrt{E_s},0,0,0,\cdots,0]^T$$
$$A[n]=1\equiv \boldsymbol{a}_1=[0,\sqrt{E_s},0,0,\cdots,0]^T$$
$$A[n]=k\equiv \boldsymbol{a}_k=[\underbrace{0,0,\cdots,0}_{k\text{ zeros}}\sqrt{E_s},0,\cdots,0]^T$$
$$A[n]=M-1\equiv \boldsymbol{a}_{M-1}=[0,0,\cdots,0,\sqrt{E_s}]^T$$

  - Structure for the receiver



CORRELATORS MATCHED FILTERS

Receivers for CPFSK modulation

- Coherent receiver with matched filters or correlators

$$P_e=Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

- Efect of phase error - Example: $n=0$, $A[n]=i$, phase error θ

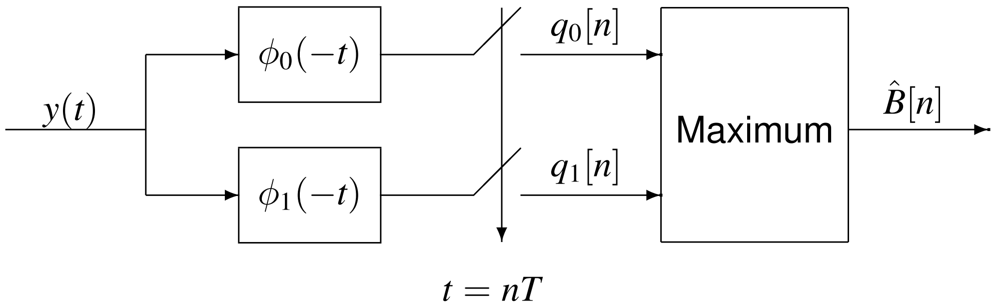
$$y(t)=\sqrt{\frac{2E_s}{T}}\ \sin(\omega_i t+\theta)\ w_T(t)$$

Output of demodulator for index ℓ (inner product of $y(t)$ and $\phi_\ell(t)$)

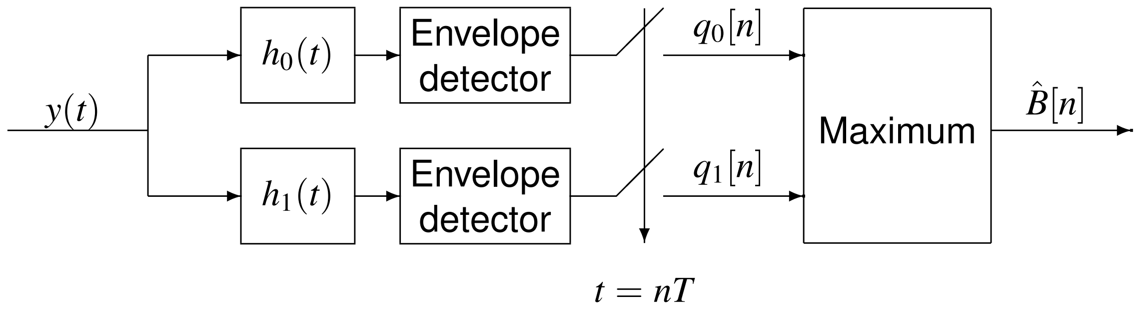
$$q_\ell[0]=\int_0^T y(t)\ \phi_\ell(t)\ dt=\int_0^T\sqrt{\frac{2E_s}{T}}\ \sin(\omega_i t+\theta)\ \sqrt{\frac{2}{T}}\ \sin(\omega_\ell t)\ dt$$
$$=\frac{\sqrt{E_s}}{T}\int_0^T[\cos((\omega_i-\omega_\ell)t+\theta)-\cos((\omega_i+\omega_\ell)t+\theta)]\,dt$$
$$=\sqrt{E_s}\ \cos(\theta)\ \delta[i-\ell]$$

  - Ideal value: $\sqrt{E_s}$
  - Attenuation term: $\cos(\theta)$
  - Need for coherent (synchronous) receivers

Coherent/non-coherent receiver for binary FSK ($M = 2$)



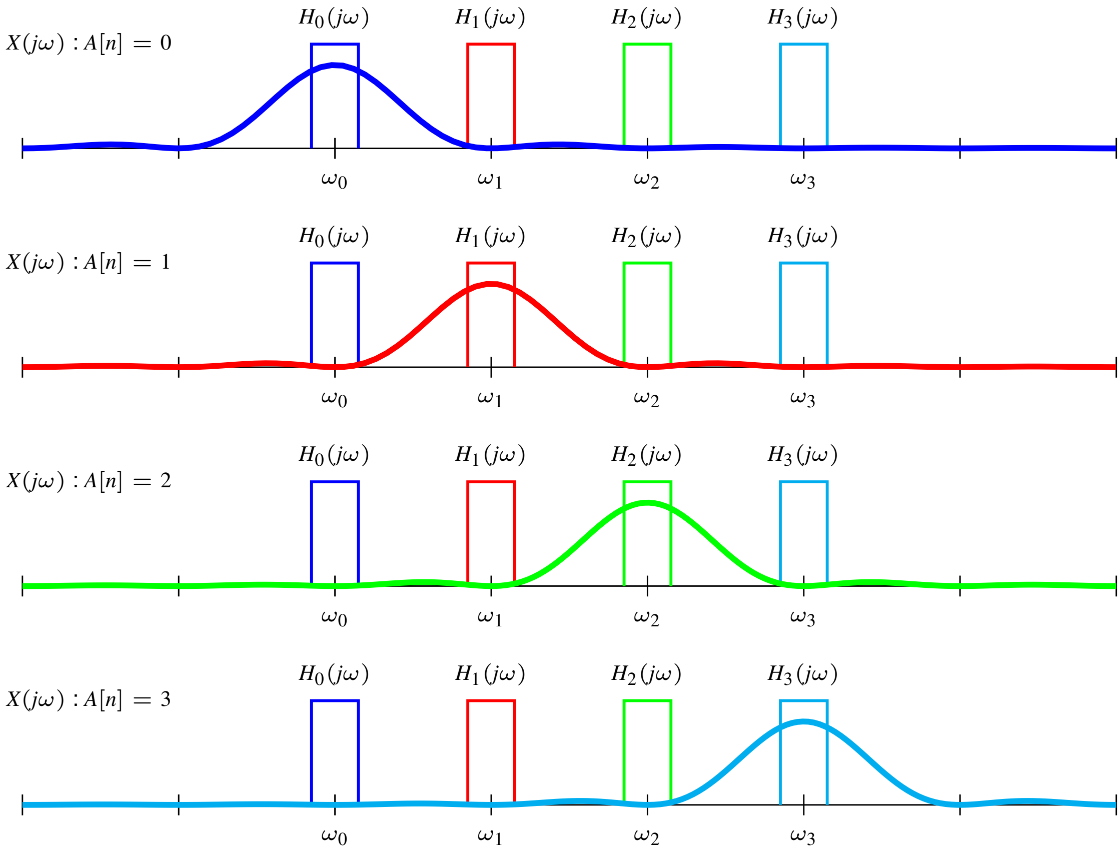
Coherent Receiver



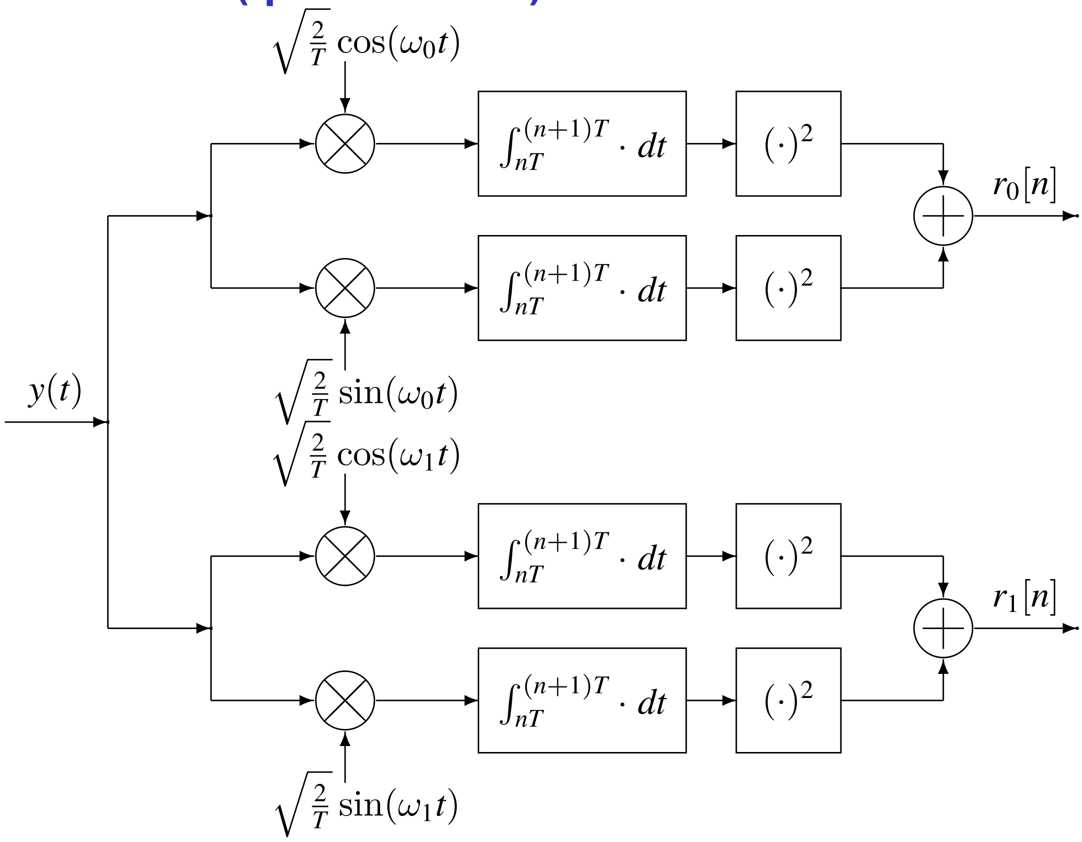
$h_k(t)$: narrow band filter centered at ω_k

Non-coherent Receiver

CPFSK receiver with narrow band filters ($M = 4$)



Non-coherent (quadratic law) receiver for FSK



Non-coherent (quadratic law) receiver for FSK

- Example: $n=0$, $A[n]=i$, phase shift θ

$$y(t)=\sqrt{\frac{2E_s}{T}}\ \sin(\omega_i t+\theta)\ w_T(t)$$

Output of correlators of index ℓ

$$\begin{aligned} q_{\ell,0}[0] =& \int_0^T \sqrt{\frac{2E_s}{T}}\ \sin(\omega_i t+\theta)\ \sqrt{\frac{2}{T}}\ \cos(\omega_\ell t)\ dt \\ =& \frac{\sqrt{E_s}}{T}\int_0^T [\sin((\omega_i-\omega_\ell)t+\theta)+\sin((\omega_i+\omega_\ell)t+\theta)]\, dt \\ =& \sqrt{E_s}\ \sin(\theta)\ \delta[i-\ell] \end{aligned}$$

$$\begin{aligned} q_{\ell,1}[0] =& \int_0^T \sqrt{\frac{2E_s}{T}}\ \sin(\omega_i t+\theta)\ \sqrt{\frac{2}{T}}\ \sin(\omega_\ell t)\ dt \\ =& \frac{\sqrt{E_s}}{T}\int_0^T [\cos((\omega_i-\omega_\ell)t+\theta)-\cos((\omega_i+\omega_\ell)t+\theta)]\, dt \\ =& \sqrt{E_s}\ \cos(\theta)\ \delta[i-\ell] \end{aligned}$$

Output of demodulator branch for index ℓ

$$\boxed{r_\ell[0]=\left(q_{\ell,0}[0]\right)^2+\left(q_{\ell,1}[0]\right)^2=E_s\ \delta[i-\ell]}$$

  - Noise includes the contribution of two correlators per branch

FSK seen as frequency shift from a central frequency

- Definition of central frequency

$$\omega_c = \frac{\omega_0 + \omega_{M-1}}{2} = \frac{\pi}{T} \times C, \; C \in \mathbb{Z}, \; C \text{ odd}$$

  - Value of central frequency: $\omega_c = \frac{\pi}{T} \times$ odd integer
  - Frecuencies of the pulse for symbol of discrete index n

$$\omega_c + I[n] \, \frac{\pi}{T}$$

- Encoder

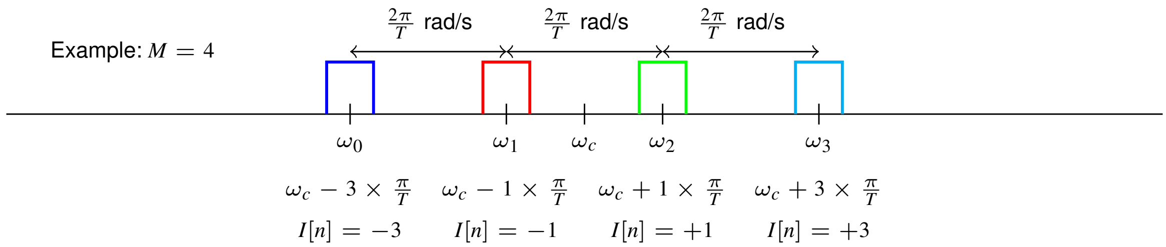
$$I[n] \in \{\pm 1, \pm 3, \cdots, \pm(M-1)\}$$

- FSK analytic expression as shift from ω_c

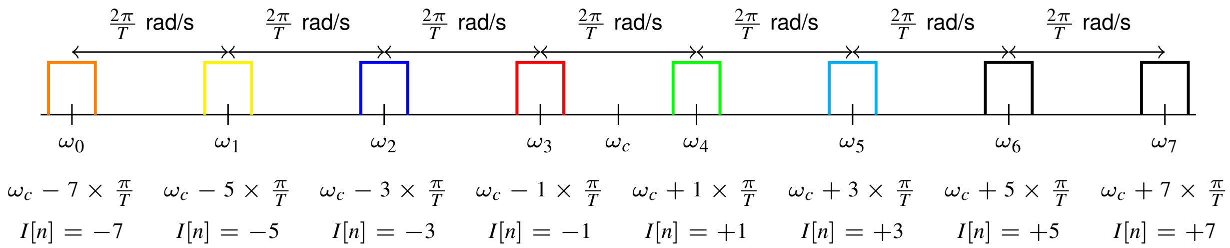
$$x(t) = \sqrt{\frac{2E_s}{T}} \sum_n \sin\left(\omega_c t + I[n] \, \frac{\pi t}{T}\right) \, w_T(t - nT)$$

FSK seen as frequency shift from a central frequency (II)

Example: $M = 4$



Example: $M = 8$



FSK spectrum

- Mean of the signal is periodic
- Discrete spectrum (spectrum of the periodic mean)

$$S_{Xd}(j\omega) = \frac{2E_s}{T}\frac{1}{(MT)^2}\left|\sum_{i=0}^{M-1} G_i(j\omega)\right|^2 \sum_k \delta\left(\omega - \frac{2\pi k}{T}\right)$$

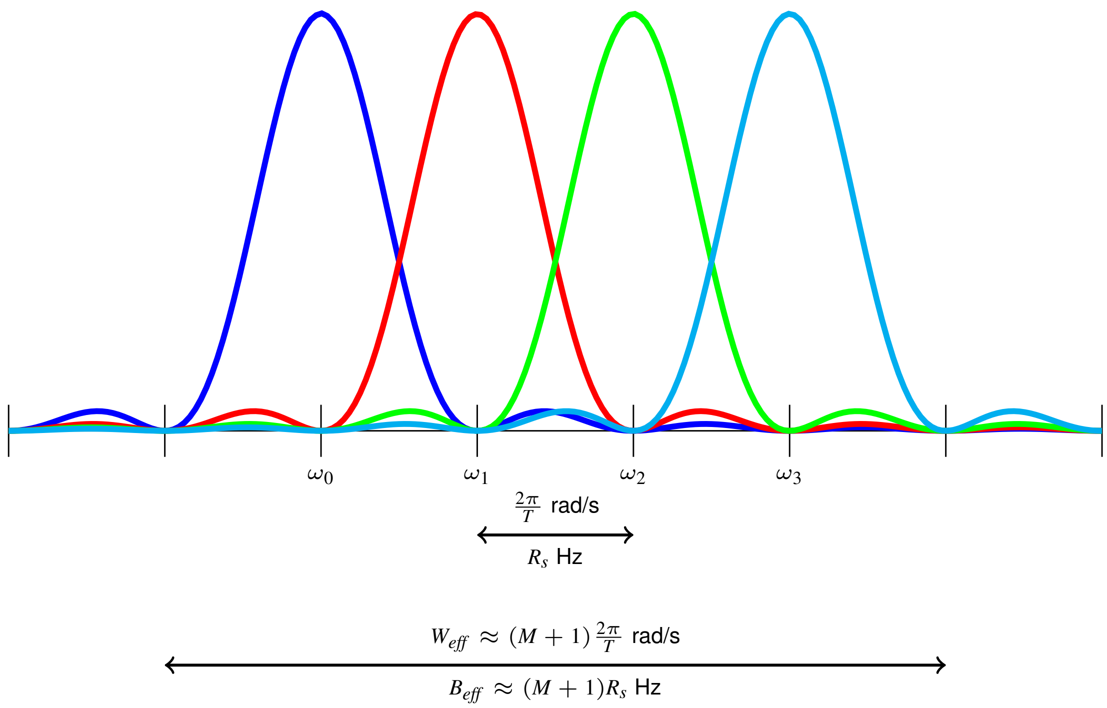
- Continuous spectrum (spectrum of the signal without the mean)

$$S_{Xc}(j\omega) = \frac{2E_s}{T}\frac{1}{MT}\left\{\sum_{i=0}^{M-1} |G_i(j\omega)|^2 - \frac{1}{M}\left|\sum_{i=0}^{M-1} G_i(j\omega)\right|^2\right\}$$

- FSK - Power spectral density

$$S_X(j\omega) = S_{Xc}(j\omega) + S_{Xd}(j\omega)$$

CPFSK spectrum, example $M = 4$



Minimum shift keying (MSK) modulation

- Information: discrete frequency changes in the frequency of a carrier
- (Quasi) Orthogonal carriers with minimum frequency separation
- Inner product of pulses $g_i(t)$

$$\begin{aligned}\langle \mathbf{g}_i, \mathbf{g}_\ell \rangle &= \int_0^T \sin(\omega_i t)\ \sin(\omega_\ell t)\ dt \\ &= \frac{1}{2}\int_0^T \cos[(\omega_i - \omega_\ell)\ t]\ dt - \frac{1}{2}\cos[(\omega_i + \omega_\ell)\ t]\ dt \\ &= \frac{1}{2}\frac{\sin[(\omega_i - \omega_\ell)\ T]}{(\omega_i - \omega_\ell)} - \frac{1}{2}\frac{\sin[(\omega_i + \omega_\ell)\ T]}{(\omega_i + \omega_\ell)}\end{aligned}$$

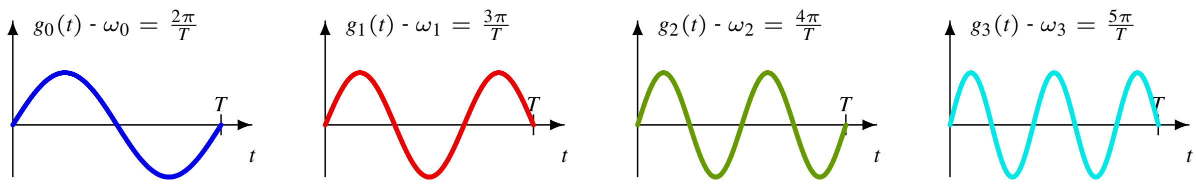
- Minimum required frequency separation (in narrow band systems)
  - Assumption: $\frac{\sin[(\omega_i+\omega_\ell)\ T]}{(\omega_i+\omega_\ell)}$ can be neglected (high denominator)

$$\omega_i - \omega_\ell = \frac{\pi}{T} N_{i,\ell} \text{ rad/s}, \quad \text{with } N_{i,\ell} \in \mathbb{Z}, \quad i,\ell = 0,1,\cdots,M-1, \quad i \neq \ell$$

$$f_i - f_\ell = \frac{R_s}{2} \times N_{i,\ell} \text{ Hz}$$

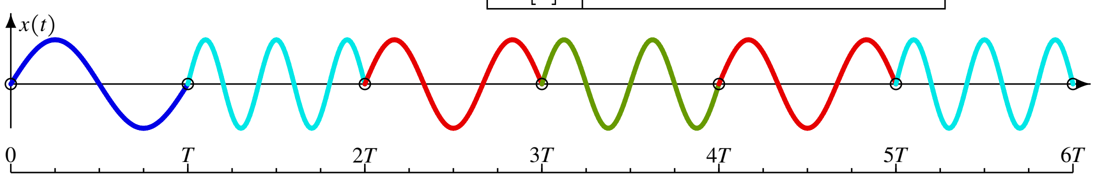
MSK waveforms - Example for $M = 4$

- Pulses for $M = 4$ (just one example)

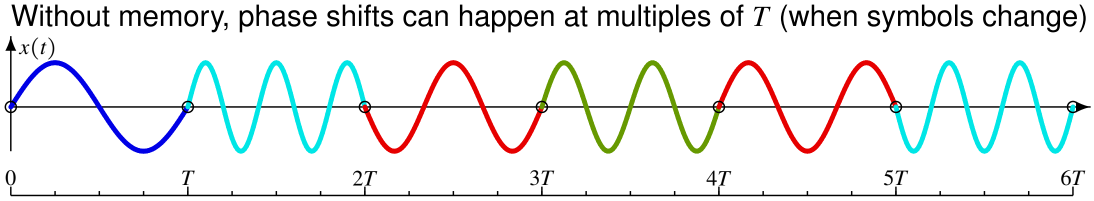


- Waveform for data sequence

n	0	1	2	3	4	5
$A[n]$	0	3	1	2	1	3



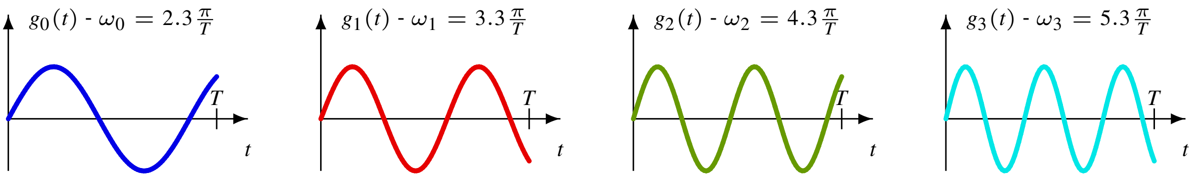
Without memory, phase shifts can happen at multiples of T (when symbols change)



Idenfification of phase at the end of each symbol interval ($\theta[n]$) allows phase continuity

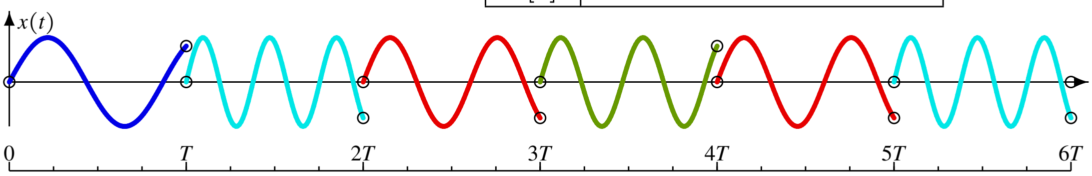
MSK waveforms - Example for $M = 4$ (II)

- Another example with frequencies not being integer multiples of $\frac{\pi}{T}$

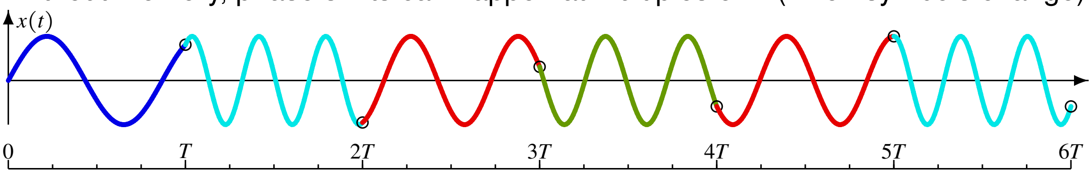


- Waveform for data sequence

n	0	1	2	3	4	5
$A[n]$	0	3	1	2	1	3



Without memory, phase shifts can happen at multiples of T (when symbols change)



Idenfification of phase at the end of each symbol interval ($\theta[n]$) allows phase continuity

Minimum shift keying (MSK) modulation (II)

- Key differences with CPFSK modulation
  - Separation between consecutive frequencies is half for MSK
    - MSK: $\Delta_\omega = \omega_i - \omega_{i-1} = \frac{\pi}{T}$
    - CPFSK: $\Delta_\omega = \omega_i - \omega_{i-1} = \frac{2\pi}{T}$
  - Values for ω_i are not constrained to be integer multiples of $\frac{2\pi}{T}$ as in CPFSK (neither integer multiples of $\frac{\pi}{T}$)
    - Frequency selection does not automatically provides phase continuity
    - Memory must be introduced to provide phase continuity
- MSK signal using central frequency notation

$$x(t) = \sqrt{\frac{2E_s}{T}} \sum_n \sin\left(\omega_c t + I[n] \times \frac{\pi}{2T} t + \theta[n]\right) \, w_T(t - nT)$$

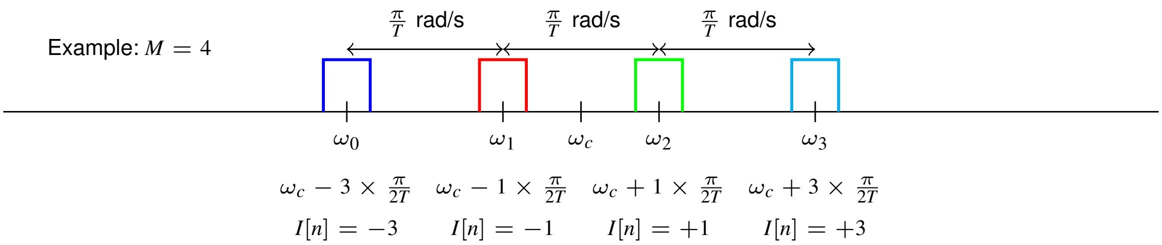
  - Encoder: $I[n] \in \{\pm 1, \pm 3, \cdots, \pm(M-1)\}$
  - Phase continuity is achieved by introducing memory term $\theta[n]$

$$\theta[n] = \theta[n-1] + \frac{\pi n}{2} \, (I[n-1] - I[n]), \quad \text{mod } 2\pi$$

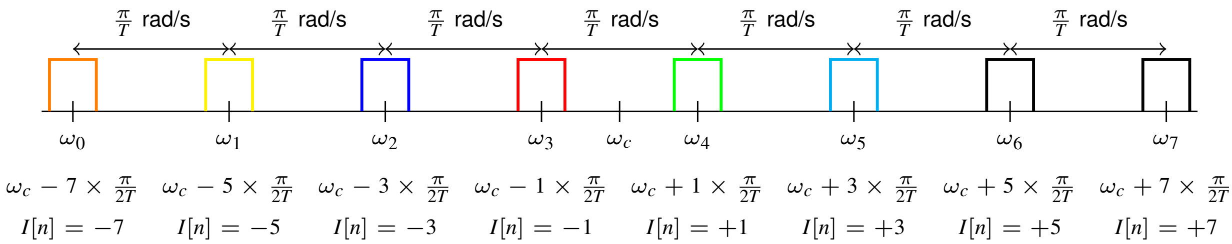
Recursive estimation of accumulated phase at the end of symbol intervals

MSK seen as frequency shift from a central frequency

Example: $M = 4$

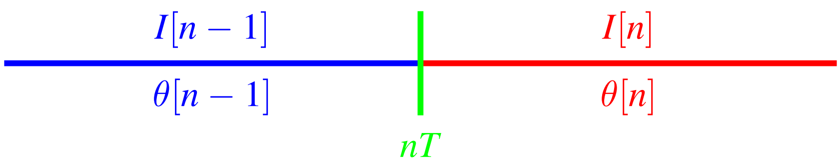


Example: $M = 8$



Memory $\theta[n]$: recursive computation

$$x(t) = \sqrt{\frac{2E_s}{T}} \sin\left(\omega_c t + I[n]\frac{\pi}{2T} t + \theta[n]\right) \; w_T(t - nT)$$



- Phase continuity at $t = nT$

$$\omega_c nT + I[n-1] \underbrace{\frac{\pi}{2T} nT}_{\frac{\pi n}{2}} + \theta[n-1] = \omega_c nT + I[n] \underbrace{\frac{\pi}{2T} nT}_{\frac{\pi n}{2}} + \theta[n]$$

$$\theta[n] = \theta[n-1] + \frac{\pi n}{2} \left(I[n-1] - I[n]\right)$$

MSK spectrum

- Alternative expression for MSK

$$x(t) = \sqrt{2E_s}\cos(\omega_c t)\sum_{\text{even } n} I[n]\cos(\theta[n])(-1)^{n/2} g(t-nT)$$
$$+\sqrt{2E_s}\sin(\omega_c t)\sum_{\text{even } n}\cos(\theta[n])(-1)^{n/2} g(t-nT+T)$$

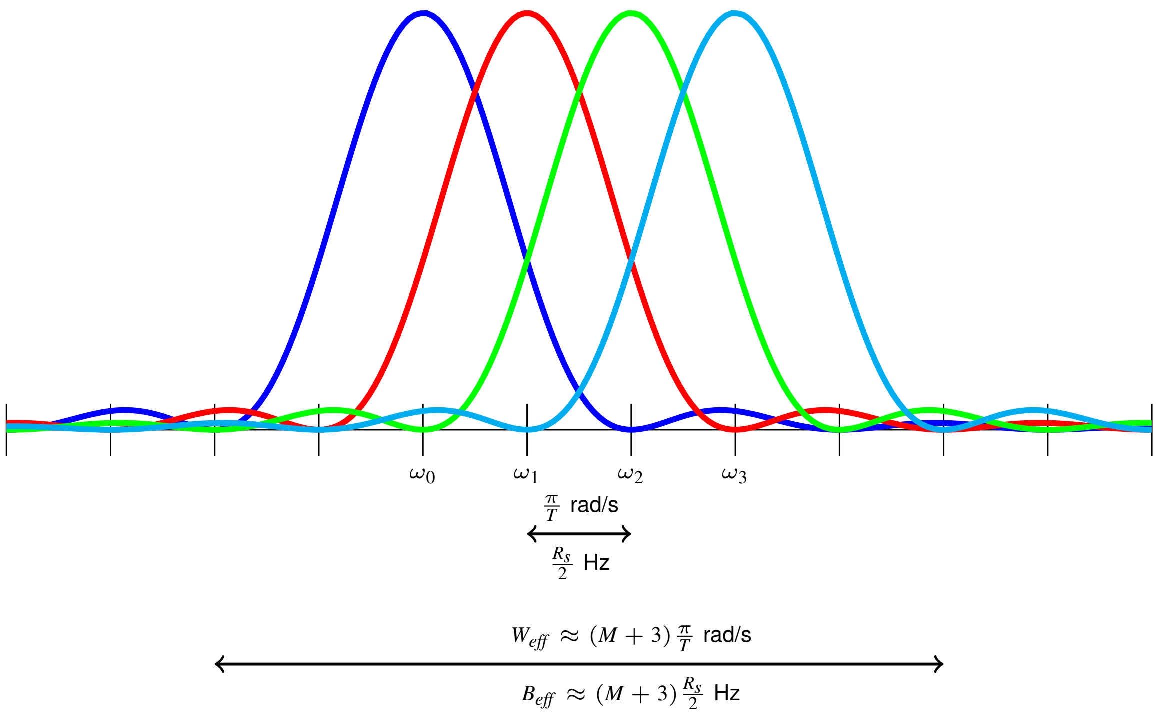
- Similar to OQPSK
  - Modified symbols
  - Pulse:

$$g(t) = \sqrt{\frac{1}{T}}\sin\left(\frac{\pi t}{2T}\right) w_{2T}(t),\ |G(j\omega)|^2 = 16T\pi^2\left(\frac{\cos(\omega T)}{\pi^2-4\omega^2T^2}\right)^2$$

- MSK spectrum

$$S_X(j\omega) = 8E_s\pi^2\left(\frac{\cos[(\omega-\omega_c)T]}{\pi^2-4(\omega-\omega_c)^2T^2}\right)^2 + 8E_s\pi^2\left(\frac{\cos[(\omega+\omega_c)T]}{\pi^2-4(\omega+\omega_c)^2T^2}\right)^2$$

MSK spectrum, example $M = 4$



Receiver for MSK modulation

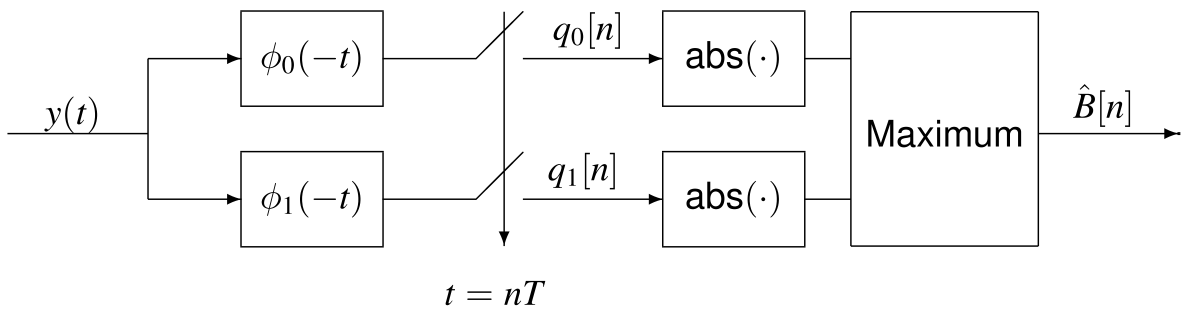
- Demodulator based on the ML receiver for FSK
- Demodulator based on the ML receiver for OQPSK
- Probability of error

$$P_e = 2\,Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

  - Memory is not taken into account at the receiver
  - Optimum receiver has a higher complexity

Receiver for binary MSK

- Sub-optimal MSK binary receiver based on a FSK receiver where the absolute value evaluation for each possible frequency is introduced to consider different possible initial phases



Continuous phase (CPM) modulations

- Family of modulations including CPFSK and MSK modulations
- Basic characteristics
  - Constant envelope
  - Phase continuity
  - Bandwidth reduction:
    - Smoothing the evolution of the instantaneous phase
- CPM signal: analytic expression in the time domain

$$x(t) = \sqrt{\frac{2E_s}{T}}\ \sin\left(\omega_c t + \theta_0 + \theta(t, \mathbf{I})\right)$$

  - $\mathbf{I}$: Sequence of transmitted symbols
  - ω_c: nominal carrier frequency
  - θ_0: initial carrier phase
  - E_s: mean energy transmitted in a symbol interval

Instantaneous frequency of a sinusoid

- Sinudoidal signal with variable argument

$$\sin\left(\phi(t)\right)$$

  - Instantaneous frequency

$$\omega_i(t) = \frac{d\phi(t)}{dt} \text{ rad/s} \qquad f_i(t) = \frac{1}{2\pi}\frac{d\phi(t)}{dt} \text{ Hz}$$

- Signal with frequency $\omega_c = 2\pi f_c$ rad/s + variable argument

$$\sin\left(\omega_c t + \theta_0 + \theta(t)\right)$$

  - Instantaneous frequency

$$\omega_i(t) = \omega_c + \frac{d\theta(t)}{dt} \text{ rad/s}$$

$$f_i(t) = f_c + \frac{1}{2\pi}\frac{d\theta(t)}{dt} \text{ Hz}$$

Generation of the full-response CPM signal

- Encoder: $I[n] \in \{\pm 1, \pm 3, \cdots, \pm(M-1)\}$
- Base band PAM signal

$$s(t) = \sum_n I[n]\; g(t-nT)$$

- Full response:
  - Pulse $g(t)$ is causal, of maximum length T sec.
    - Usually, it is normalized

$$g(t) = 0 \text{ if } t > T$$

$$\text{Normalización: } \int_{-\infty}^{\infty} g(t)\; dt = \frac{1}{2}$$

- CPM signal: instantaneous frequency $\omega_c + 2\,\omega_d\, T\, s(t)$ rad/s
- Instantaneous phase is obtained by integrating this frequency

$$\theta(t, \mathbf{I}) = 2\,\omega_d\, T \int_{-\infty}^{t} s(\tau)\; d\tau \text{ rad}$$

  - ω_d: peak frequency deviation

Time domain expression for CPM

$$x(t) = \sqrt{\frac{2E_s}{T}}\; \sin\left[\omega_c t + \theta_0 + \overbrace{2\,\omega_d\, T \int_{-\infty}^{t} \underbrace{\sum_n I[n]\; g(\tau - nT)}_{s(\tau)}\; d\tau}^{\theta(t,\mathbf{I})}\right]$$

- Phase value $\theta(t, \mathbf{I})$ inside interval $[nT, (n+1)T]$ (symbol interval for $I[n]$)

$$\theta(t, \mathbf{I}) = 2\,\omega_d\, T \int_{-\infty}^{t} s(\tau)\; d\tau = \theta[n] + \theta(t, n) \text{ rad}$$

  - $\theta[n]$: phase that has been accumulated up to $t = nT$:
    - Due to previous transmitted symbols (up to $I[n-1]$)

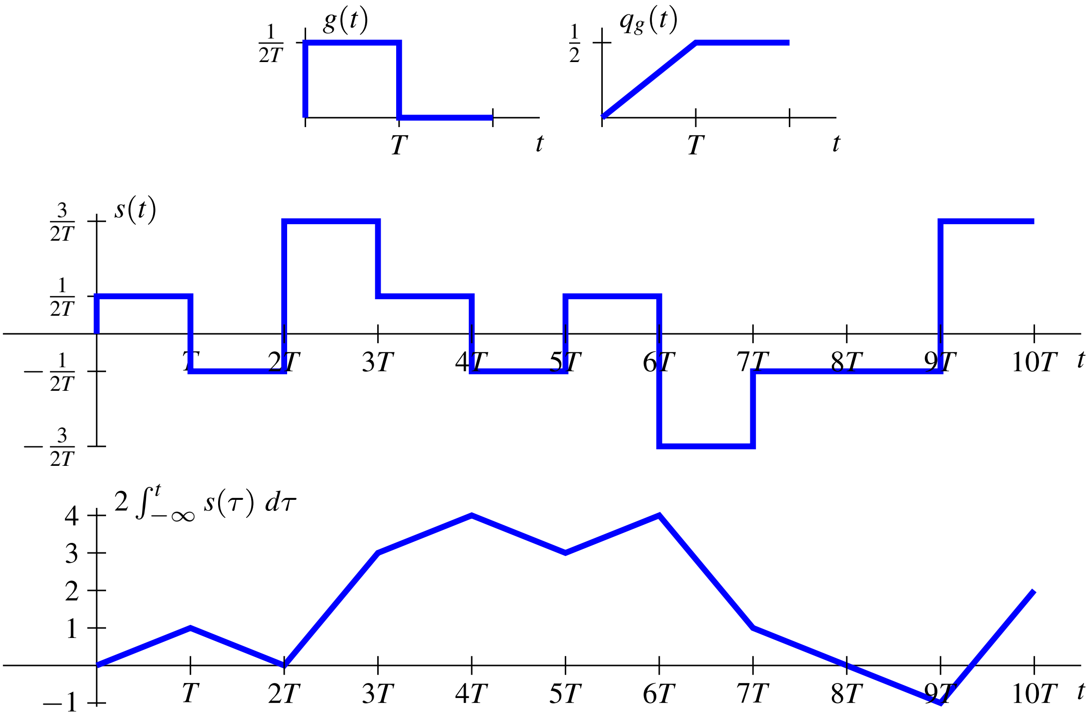
$$\theta[n] = \omega_d\, T \sum_{m=-\infty}^{n-1} I[m] \text{ rad}$$

  - $\theta(t, n)$: incremental phase starting from $t = nT$:
    - Due only to current symbol $I[n]$

$$\theta(t, n) = 2\,\omega_d\, T\, I[n]\; q_g(t - nT) \text{ rad, with } q_g(t) = \int_{-\infty}^{t} g(\tau)\; d\tau$$

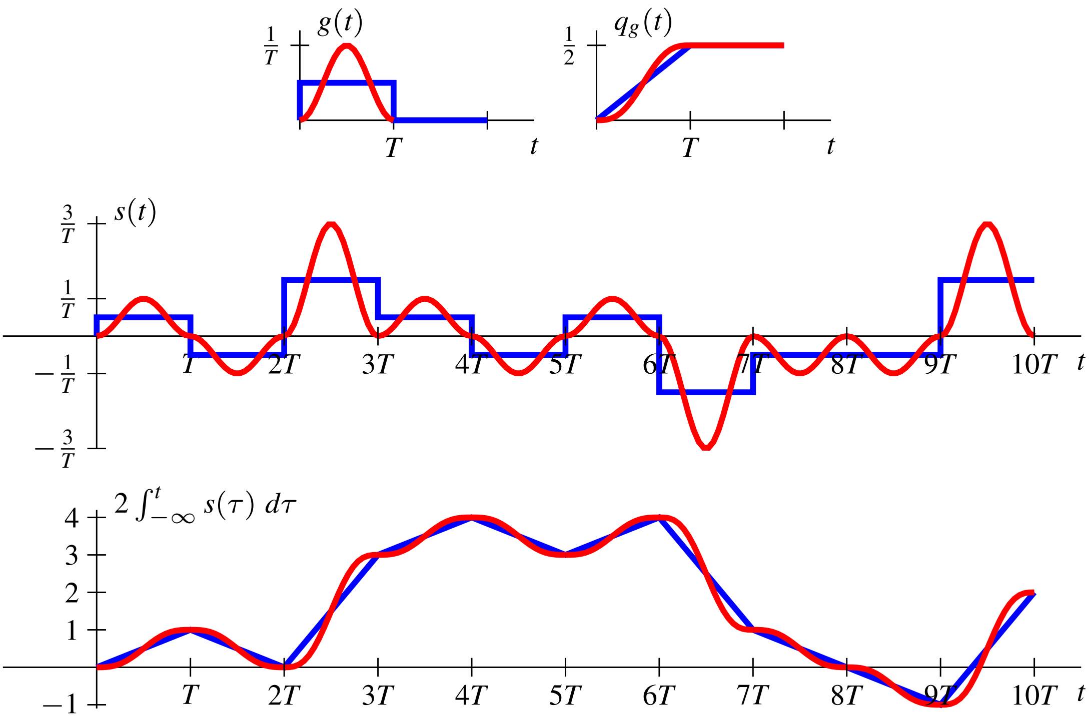
CPM - An example of phase evolution

n	0	1	2	3	4	5	6	7	8	9
$I[n]$	$+1$	-1	$+3$	$+1$	-1	$+1$	-3	-1	-1	$+3$



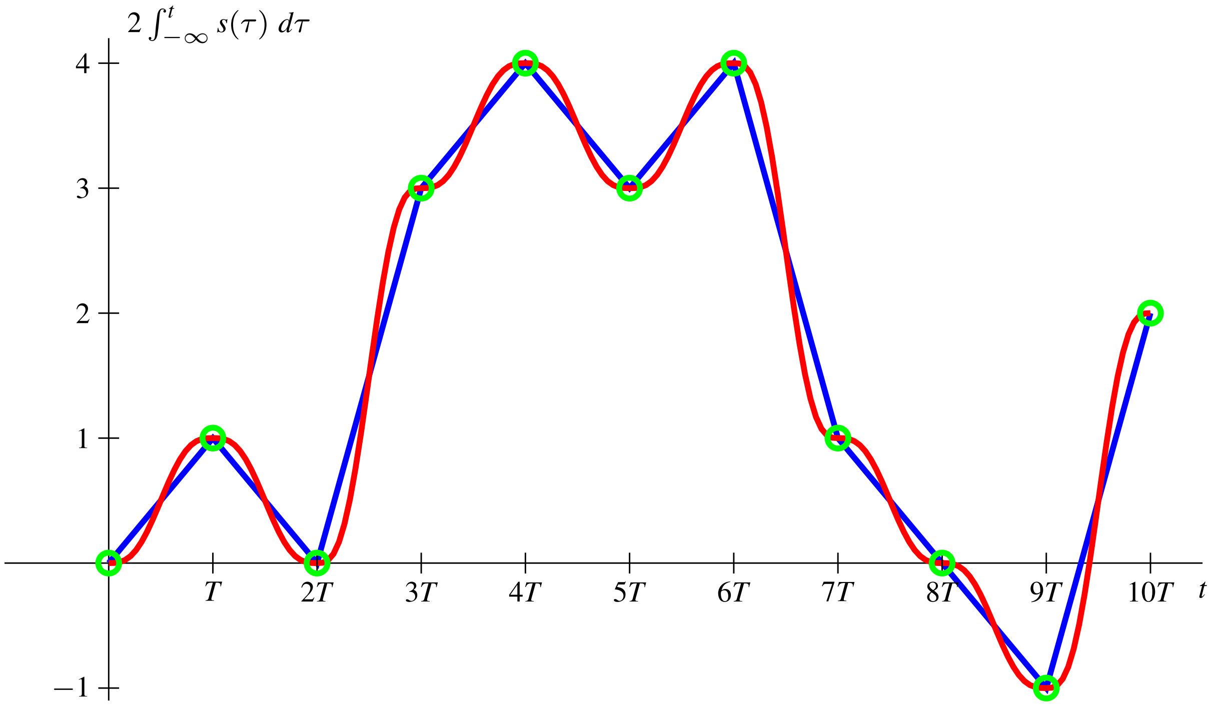
CPM - An example of phase evolution (II)

n	0	1	2	3	4	5	6	7	8	9
$I[n]$	$+1$	-1	$+3$	$+1$	-1	$+1$	-3	-1	-1	$+3$



CPM - An example of phase evolution (III)

n	0	1	2	3	4	5	6	7	8	9
$I[n]$	$+1$	-1	$+3$	$+1$	-1	$+1$	-3	-1	-1	$+3$



Time domain expression for CPM - Modulation index

- Alternative time domain expression introducing a different parameter (replacing peak frequency deviation)
- Definition of modulation index h:

$$h = \omega_d \frac{T}{\pi}$$

- Phase value in the symbol interval associated to $I[n]$:
  - $\theta[n]$: accumulated phase up to $t = nT$:

$$\theta[n] = \pi\, h \sum_{m=-\infty}^{n-1} I[m] \text{ rad}$$

  - $\theta(t,n)$: incremental phase from $t = nT$:

$$\theta(t,n) = 2\,\pi\, h\, I[n]\, q_g(t-nT) \text{ rad}$$

Identification of binary CPFSK modulation as a CPM

- Analytic expression for a CPFSK modulation

$$x(t) = \sqrt{\frac{2E_s}{T}} \sum_n \sin\left(\omega_c t + I[n] \frac{\pi t}{T}\right) w_T(t-nT)$$

- Binary CPFSK as a CPM: $\omega_d = \frac{\pi}{T}, \quad h = 1$
- Considering $\theta[0] = 0$

$$\theta(t,\mathbf{I}) = \pi \sum_{m=0}^{n-1} I[m] + 2\pi\, I[n] \frac{(t-nT)}{2T} = \pi \sum_{m=0}^{n-1} I[m] - n\,\pi\, I[n] + \frac{\pi t}{T}\, I[n]$$

  - Taking into account that

$$\pi \sum_{m=0}^{n-1} I[m] - n\,\pi\, I[n] = K\, 2\pi, \quad K \in \mathbb{Z}$$

  - Phase $\theta(t,\mathbf{I})$ is, 2π modulus

$$\theta(t,\mathbf{I}) = \frac{\pi t}{T}\, I[n] = \pm \frac{\pi t}{T}$$

Identification of MSK modulation as a CPM

- MSK signal in the time domain

$$x(t) = \sqrt{\frac{2E_s}{T}} \sum_n \sin\left(\omega_c t + I[n] \frac{\pi t}{2T} + \theta[n]\right) w_T(t-nT)$$

- Parameters identifying MSK as a CPM

$$\omega_d = \frac{\pi}{2T}, \qquad h = \frac{1}{2}$$

Phase trees in CPM modulations

- Drawing of every possible phase evolution
  - A path for every possible symbol sequence
- Transitions in a symbol interval
  - Phase increment in the symbol interval of index n

$$\theta((n+1)T) - \theta(nT) = \theta[n+1] - \theta[n] = \pi\ h\ I[n] \text{ rad}$$

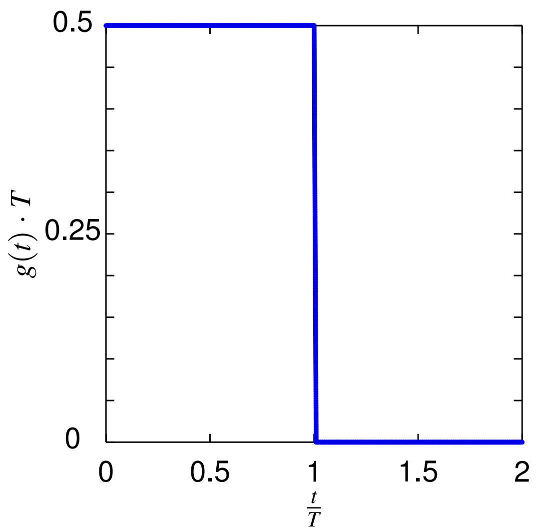
  - Shape for moving from the value of phase at the begining of symbol interval to the value of phase at the end of symbol interval
    - ⋆ Proportional to the integral of pulse $g(t)$, i.e., $q_g(t)$

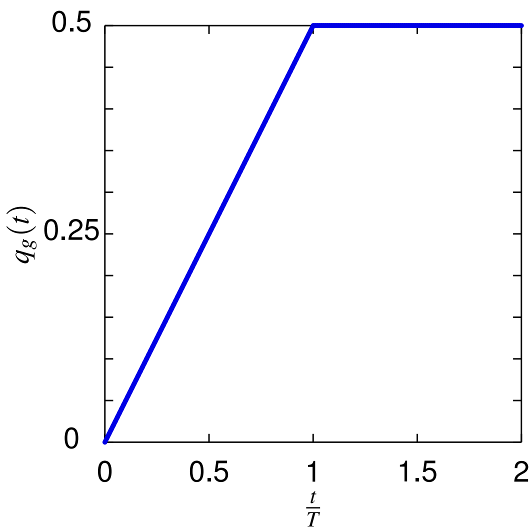
$$\theta(t,n) = 2\ \pi\ h\ I[n]\ q_g(t-nT) \text{ rad}$$

Phase tree - Example - Squared pulse

- Example: squared pulse

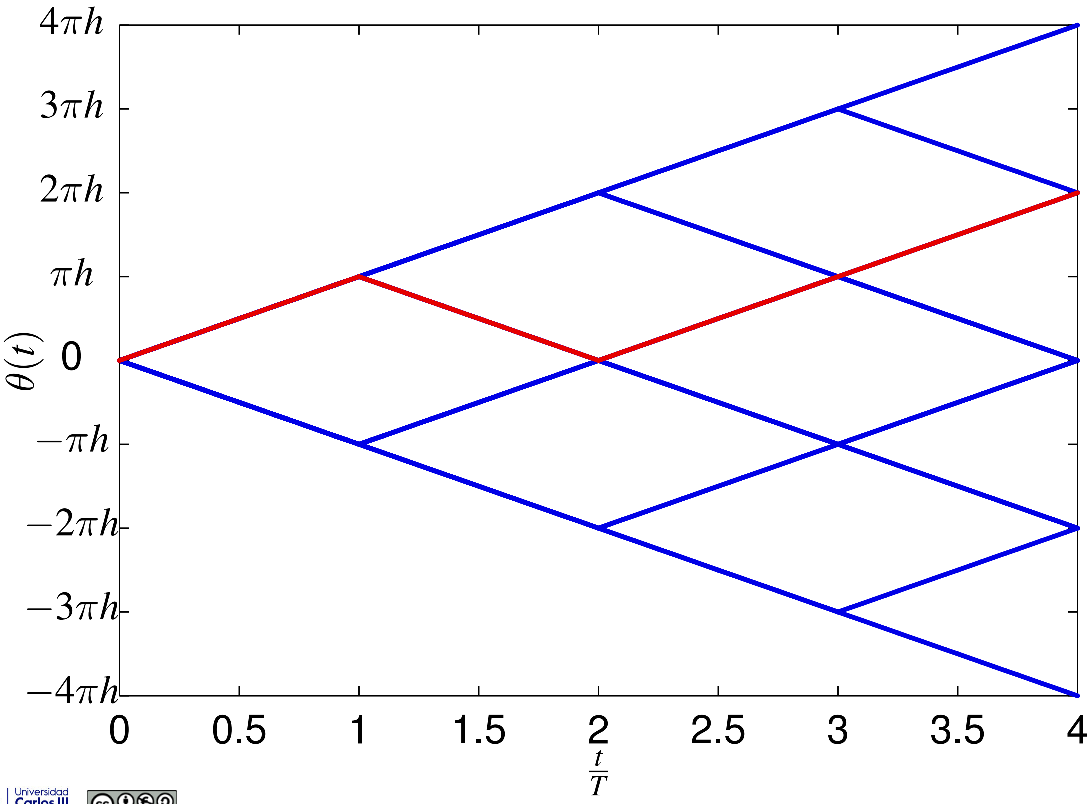
$$g(t) = \begin{cases} \frac{1}{2T}, & 0 \le t < T \\ 0, & \text{en otro caso} \end{cases}, \quad q_g(t) = \int_{-\infty}^{t} g(t)\ dt = \begin{cases} 0, & t < 0 \\ \frac{t}{2T}, & 0 \le t < T \\ 1/2, & t \ge T \end{cases}$$





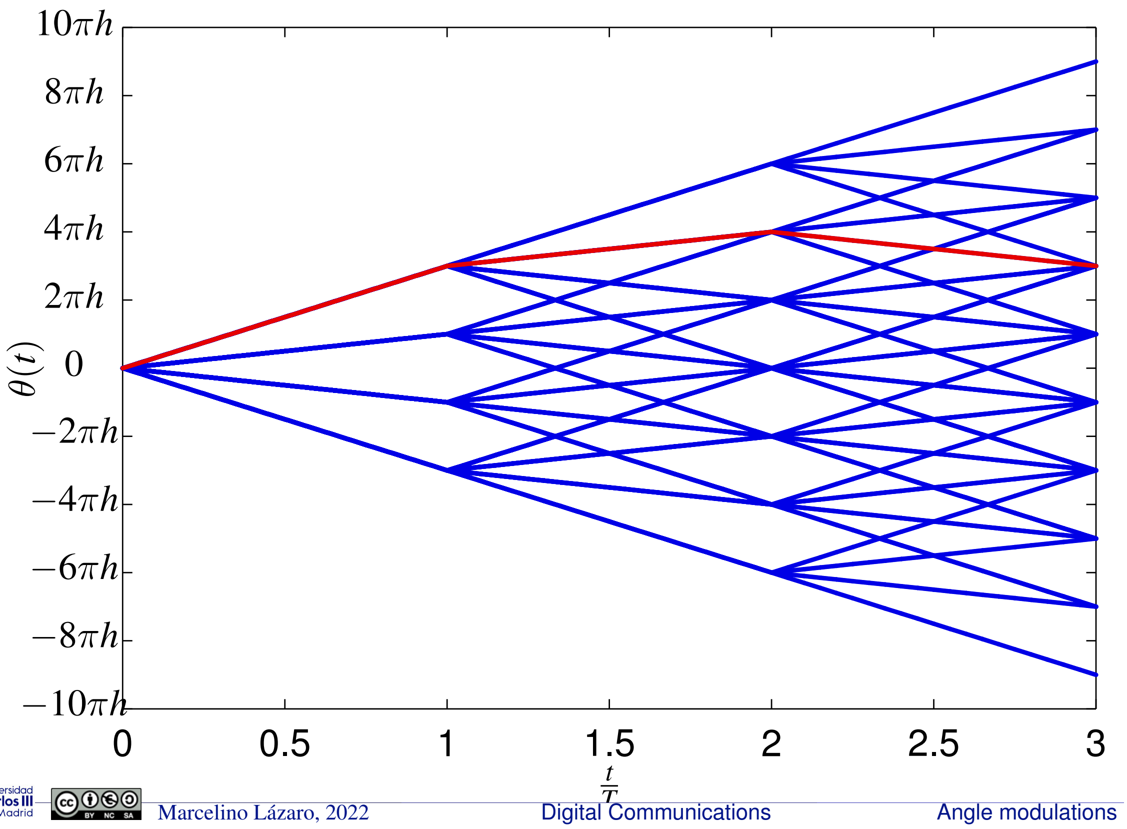
Example of phase tree - squared pulse - binary

- Highlighted sequence: $I[0] = +1,\ I[1] = -1,\ I[2] = +1,\ I[3] = +1$



Example of phase tree - squared pulse - 4-ary

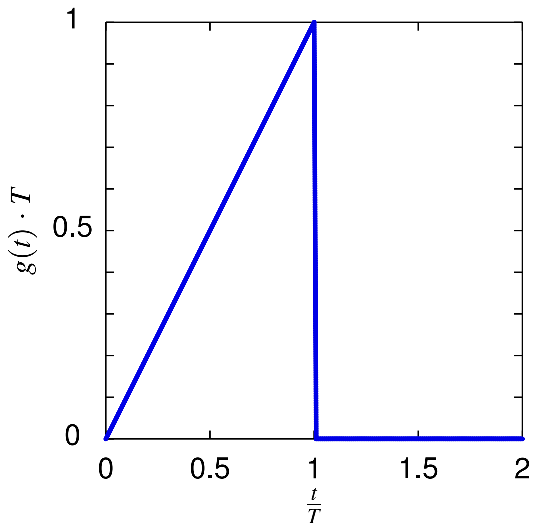
- Highlighted sequence: $I[0] = +3,\ I[1] = +1,\ I[2] = -1$

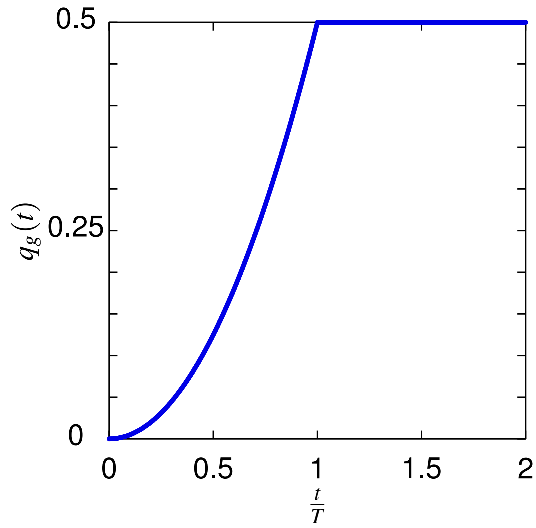


Phase tree - Example - Triangle pulse

- Example: triangle pulse

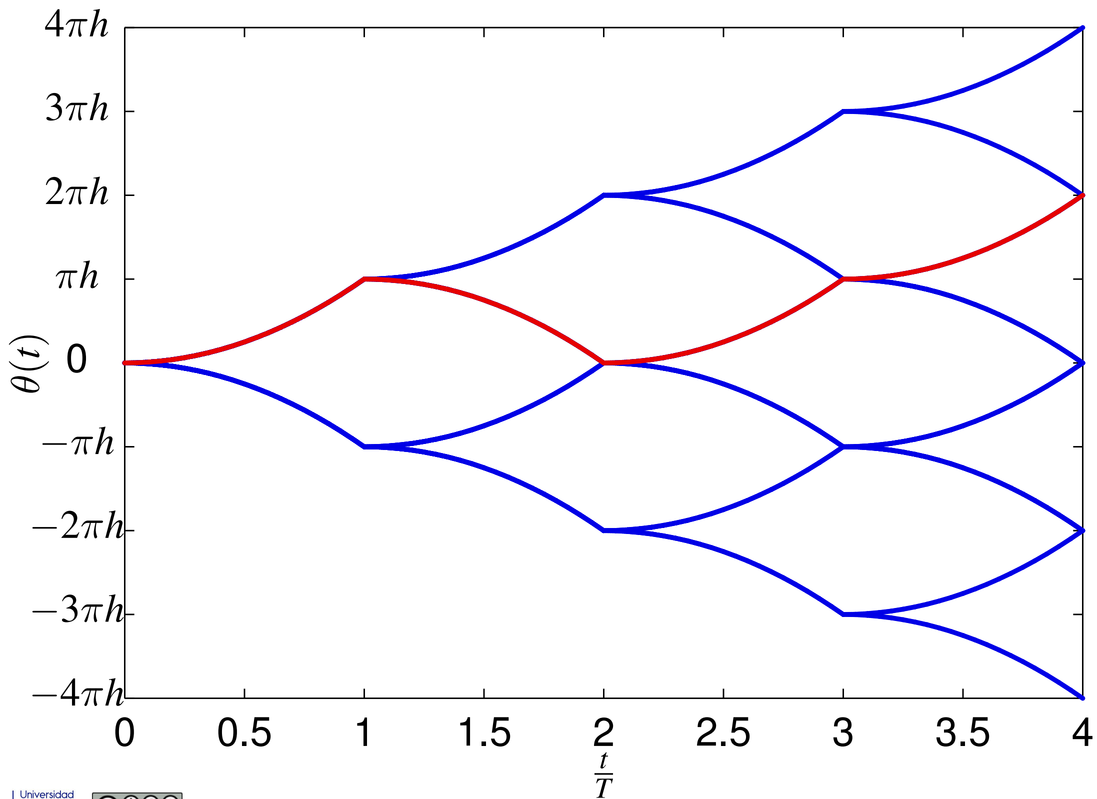
$$g(t) = \begin{cases} \frac{t}{T^2}, & 0 \le t < T \\ 0, & \text{en otro caso} \end{cases}, \quad q_g(t) = \int_{-\infty}^{t} g(t)\, dt = \begin{cases} 0, & t < 0 \\ \frac{t^2}{2T^2}, & 0 \le t < T \\ 1/2, & t \ge T \end{cases}$$





Phase tree - Example - Triangle pulse - binary

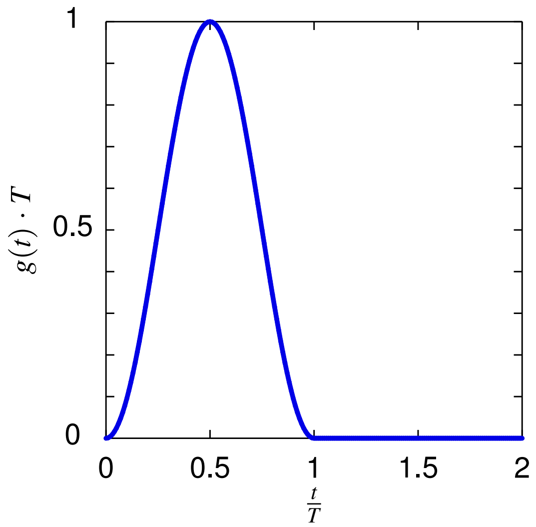
- Highlighted sequence: $I[0] = +1,\ I[1] = -1,\ I[2] = +1,\ I[3] = +1$

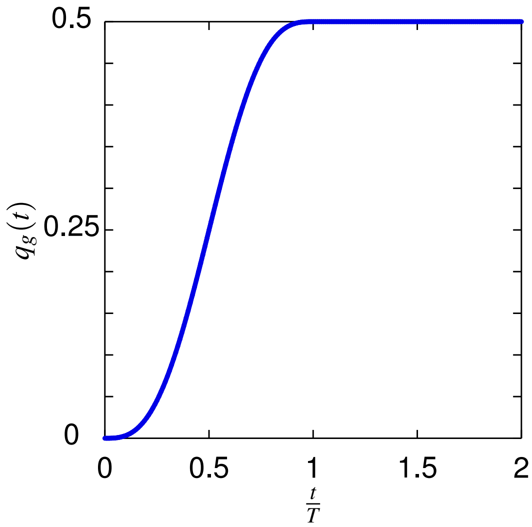


Phase tree - Example - smoother pulses

- Example: raised cosine pulse ($L = 1$)

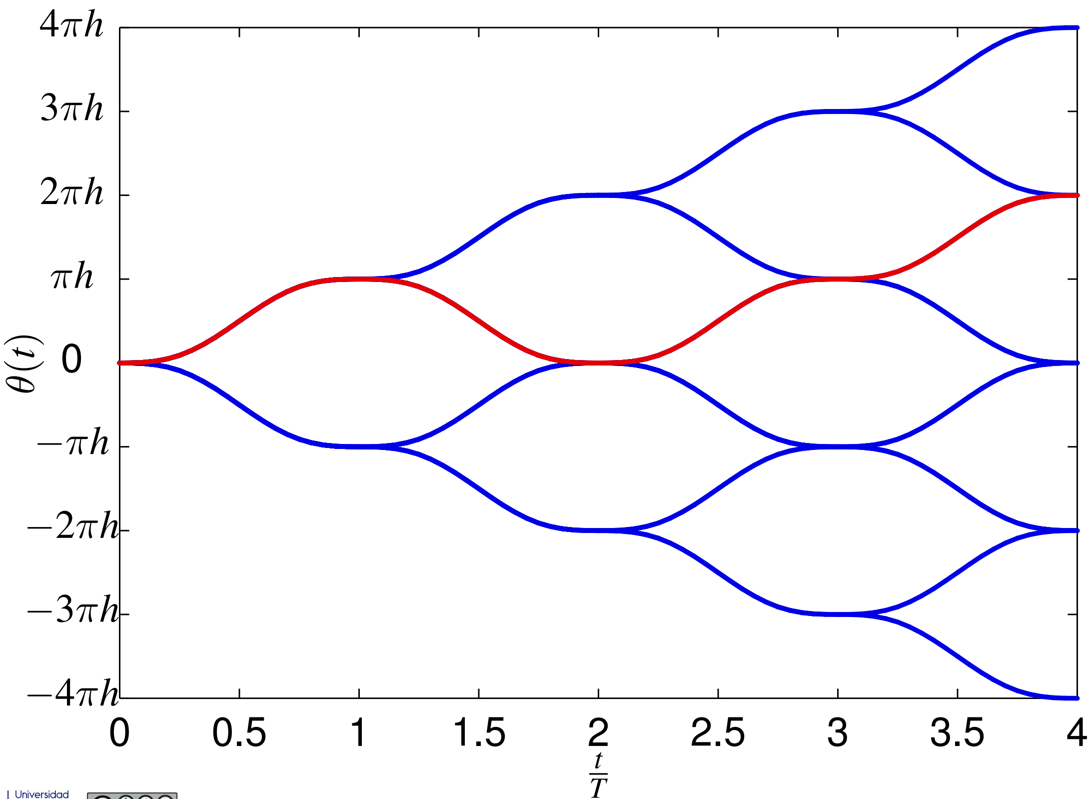
$$g(t) = \frac{1}{2T}\left[1 - \cos\left(\frac{2\pi t}{T}\right)\right] w_T(t),\ q_g(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{2T}\left[t - \frac{T}{2\pi}\sin\left(\frac{2\pi t}{T}\right)\right] & 0 \le t < T \\ 1/2, & t \ge T \end{cases}$$





Phase tree - Example - smoother pulses - binary

- Highlighted sequence: $I[0] = +1,\ I[1] = -1,\ I[2] = +1,\ I[3] = +1$



Partial response CPM

- Duration of pulse $g(t)$ is extended up to L symbol periods ($L > 1$)
- Phase $\theta(t, \mathbf{I})$ in symbol interval $[nT, (n+1)T]$ is now

$$\begin{aligned}\theta(t,\mathbf{I}) &= 2\pi h \sum_{m=-\infty}^{n} I[m]\ q_g(t-mT) \\ &= \theta[n] + \theta(t,n) \text{ rad}\end{aligned}$$

  - $\theta[n]$: phase that is accumulated up to nT due to finished pulses

$$\theta[n] = \pi h \sum_{m=-\infty}^{n-L} I[m] \text{ rad}$$

  - $\theta(t, n)$: contribution of pulses that have not finished at the begining of the interval

$$\theta(t,n) = 2\pi h \sum_{m=n-L+1}^{n} I[m]\ q_g(t-mT) \text{ rad}$$

Pulses for partial response CPM

- Raised cosine pulses

$$g(t) = \frac{1}{2LT}\left[1-\cos\left(\frac{2\pi t}{LT}\right)\right]\ w_{LT}(t)$$

  - Smoothing phase transitions
- Gaussian MSK (GMSK)

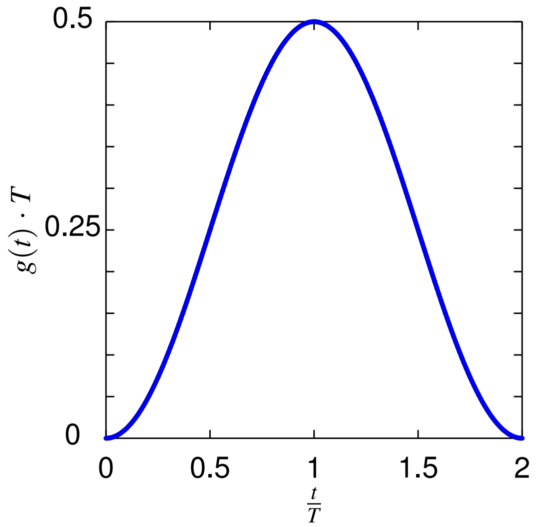
$$g(t) = \frac{1}{2T}\left[Q\left(\frac{2\pi\beta(t-T/2)}{\sqrt{\ln 2}}\right) - Q\left(\frac{2\pi\beta(t+T/2)}{\sqrt{\ln 2}}\right)\right]$$

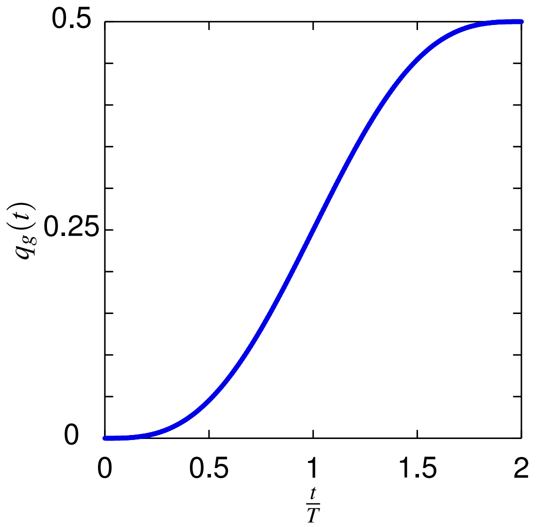
  - Employed in GSM ($\beta = 0.3$) and DECT ($\beta = 0.2$)
  - Squared pulse filtered with a Gaussian impulse response

Phase tree - Partial response CPM - Example

- Example: raised cosine pulse ($L = 2$)

$$g(t) = \frac{1}{4T}\left[1 - \cos\left(\frac{2\pi t}{2T}\right)\right] w_{2T}(t),\ \ q_g(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{4T}\left[t - \frac{2T}{2\pi}\sin\left(\frac{2\pi t}{2T}\right)\right] & 0 \le t < T \\ 1/2, & t \ge T \end{cases}$$





Phase tree - Partial response CPM - Example - Binary

- Highlighted sequence: $I[0] = +1,\ I[1] = -1,\ I[2] = +1,\ I[3] = +1$