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Digital Communications

Grades in English

Chapter 4

Multipulse modulations

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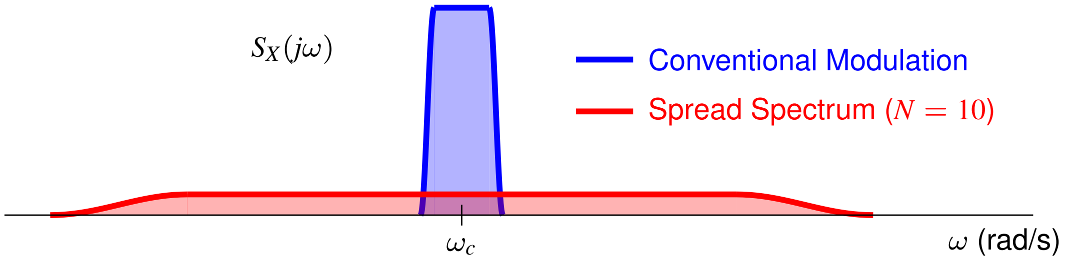
Index of contents

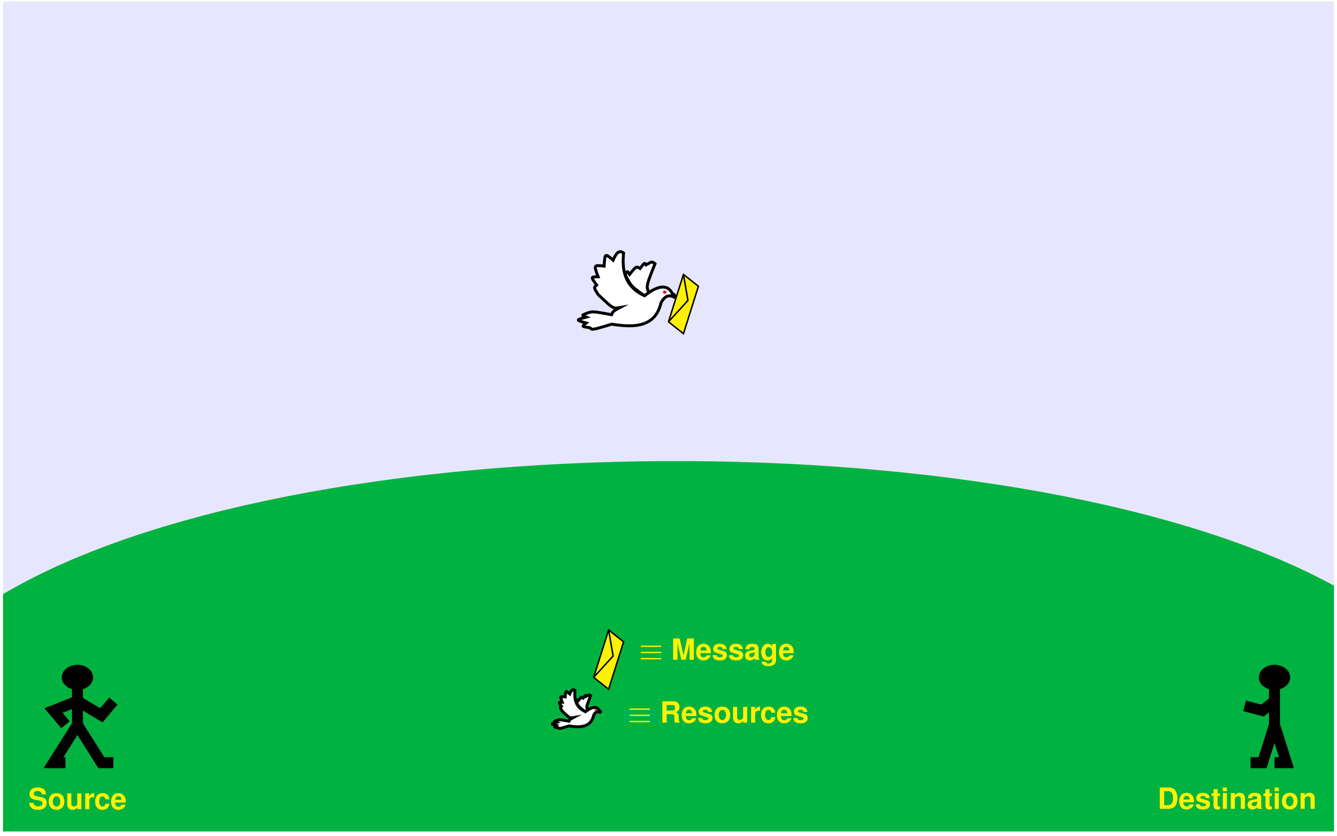
Spread spectrum modulations

- Bandwidth is deliberately much higher than in conventional modulations
    - Bandwidth is increased (*spread*) by a factor N
        - This allows some degree of inmunity to narrow band interferences / fading

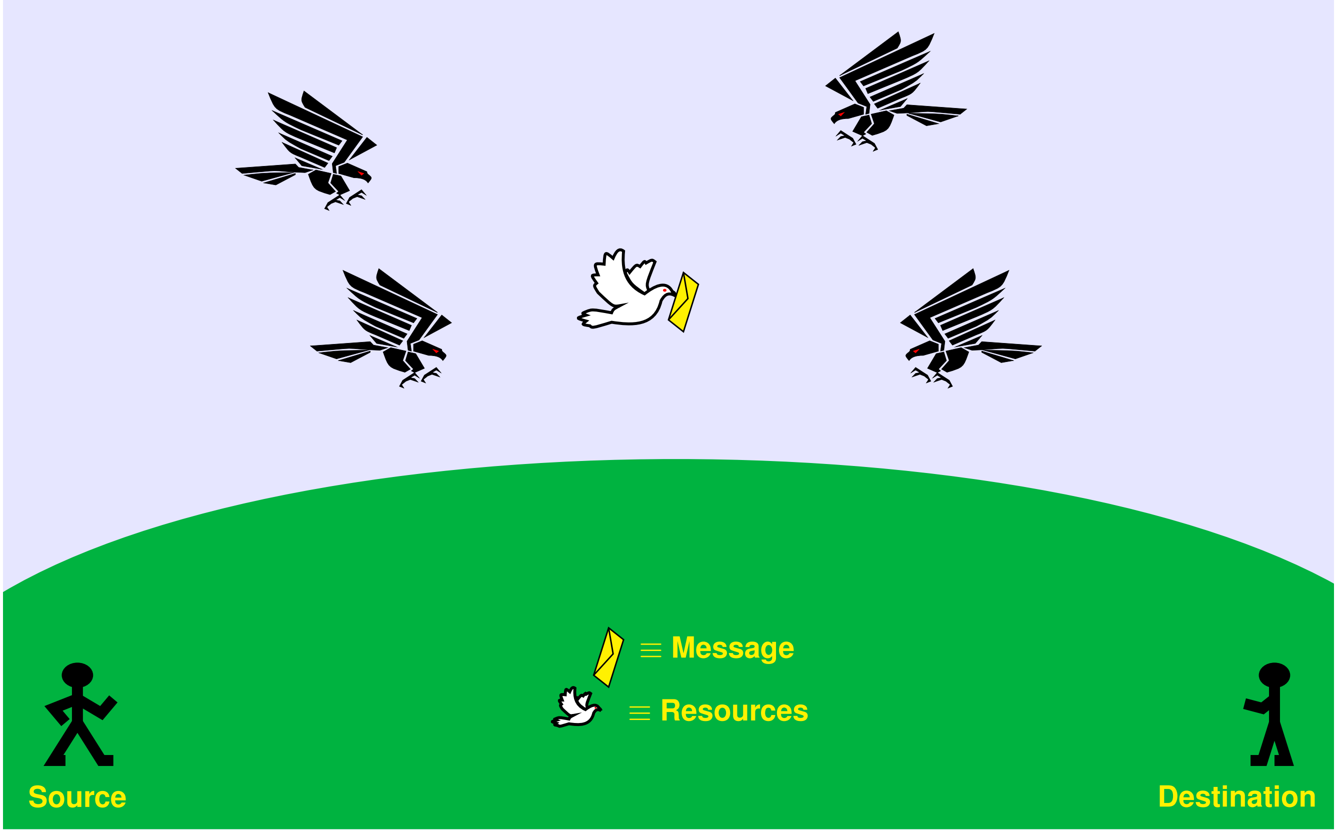


- Myth: spread spectrum modulation increases capacity
    - Reality:
        - Provides low sensibility to channel distortion (including jammers)
        - Allows secure communications
- Origin: to combat intentional interference (jamming) in military systems
    - Current days applications
        - Applications requiring robustness against local (in frequency) fadings
        - To limit power flux density in satellite downlinks
        - Multiple access
          - CDMA: Code Division Multiple Access

Ideal channel



“Aggressive” channel



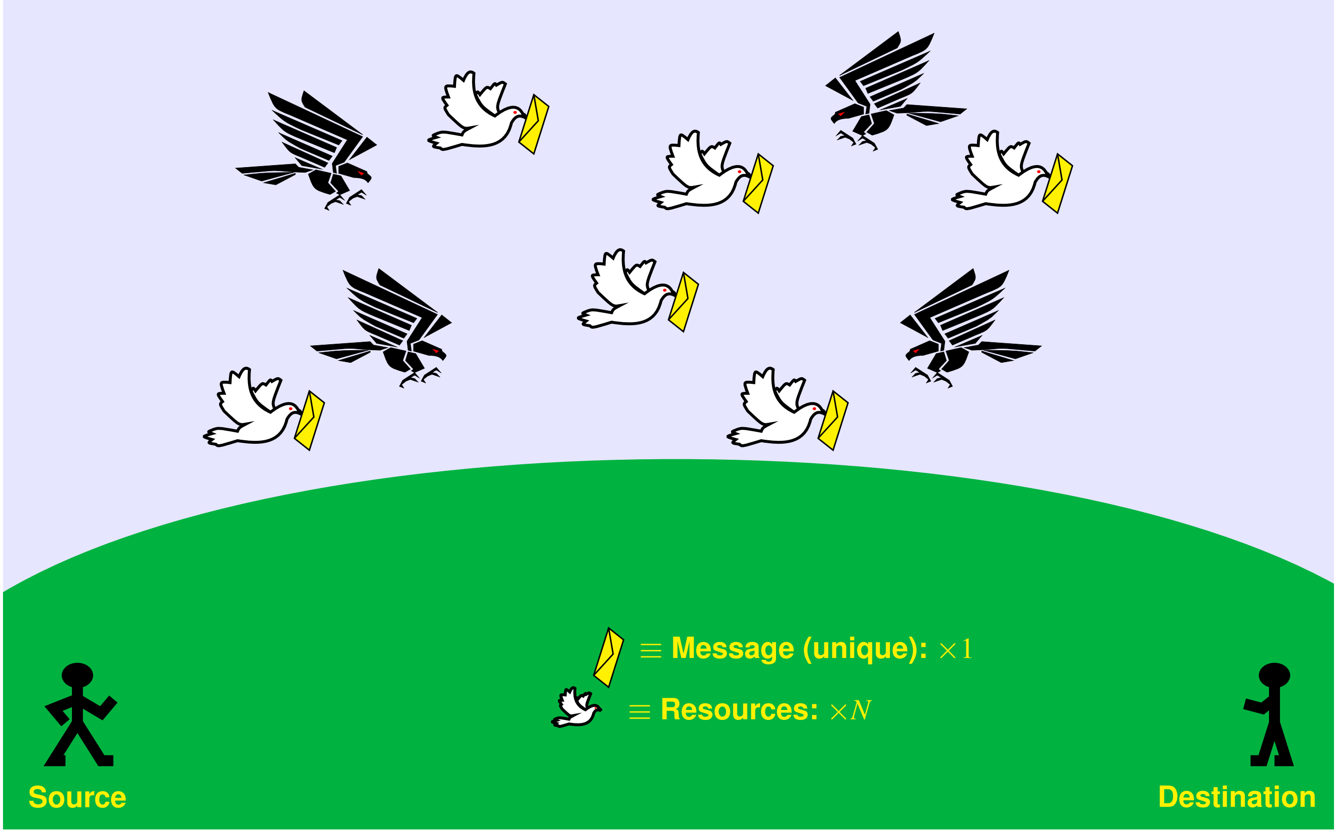
✉ $\equiv$ **Message**

🕊 $\equiv$ **Resources**

Source

Destination

“Aggressive” channel: spread spectrum



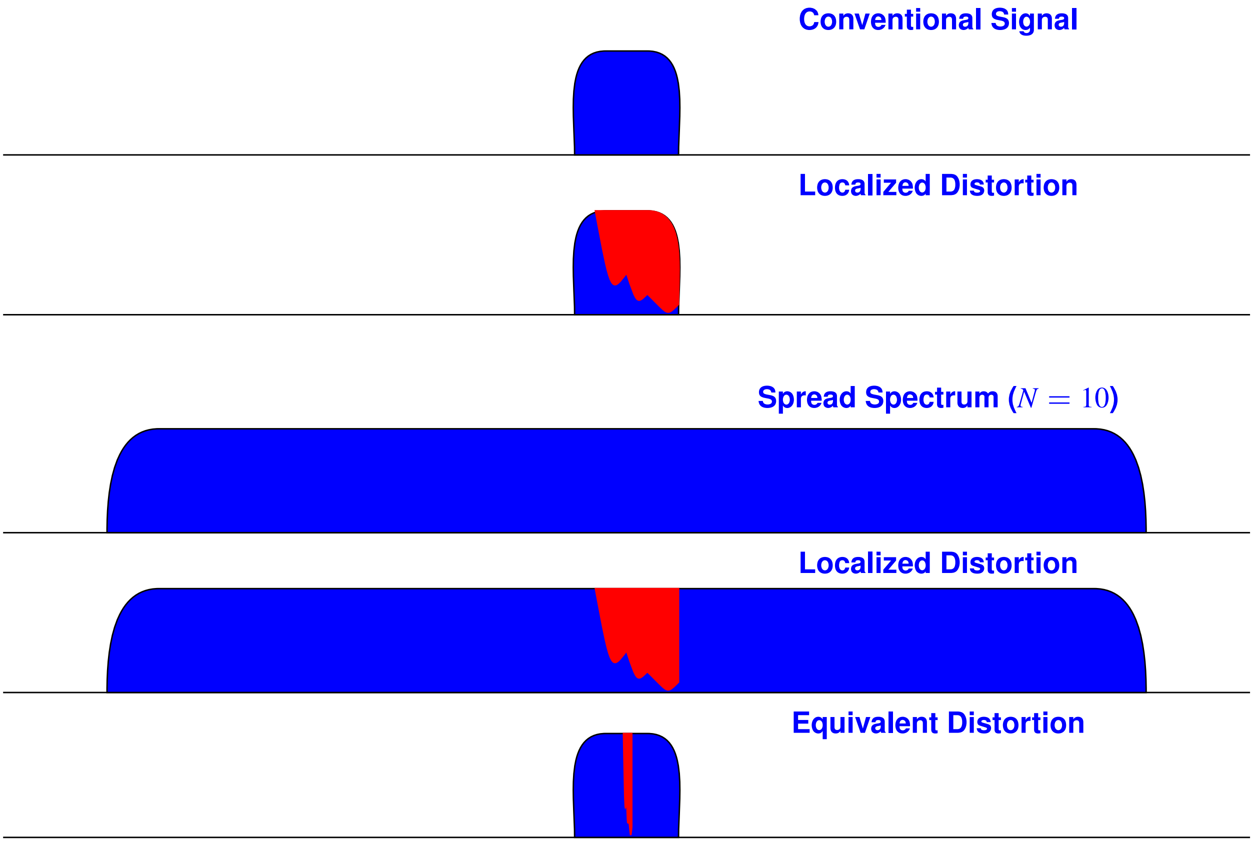
✉ $\equiv$ **Message (unique):** $\times 1$

🕊 $\equiv$ **Resources:** $\times N$

Source

Destination

Spread spectrum: intuitive idea



Increasing the bandwidth of a digital communication signal

- PAM signal - Time and frequency expressions ($R_s = \frac{1}{T}$ bauds)

$$s(t) = \sum_n A[n]\; g(t-nT),\; S_s(j\omega) = \frac{1}{T}\, S_A(e^{j\omega T})\; |G(j\omega)|^2$$

BB: $W = \frac{\pi}{T}(1+\alpha)$ rad/s
$B = \frac{R_s}{2}(1+\alpha)$ Hz

$$x(t) = \sqrt{2}\mathcal{R}e\{s(t)\; e^{j\omega_c}\},\; S_x(j\omega) = \frac{1}{2}\left[S_s(j\omega - j\omega_c) + S_s^*(-j\omega - j\omega_c)\right]$$

PB: $W = \frac{2\pi}{T}(1+\alpha)$ rad/s
$B = R_s(1+\alpha)$ Hz

  - Bandwidth using $h_{RRC}^{\alpha,T}(t)$
    - Transmission at $R_s = \frac{1}{T}$ bauds with roll-off factor α
- Goal: to increase bandwidth by an expansion factor N
  - Root-raised cosine trasnsmision filters

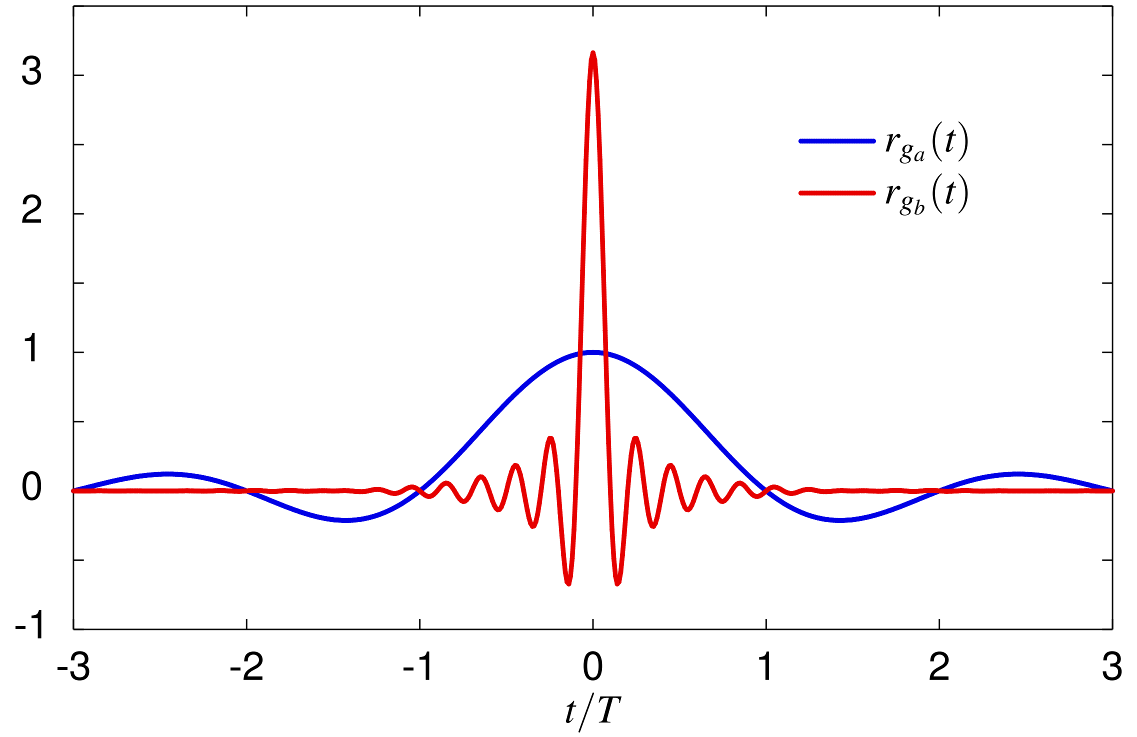
$$\text{BB.: } W = N \times \frac{\pi}{T}(1+\alpha) \text{ rad/s},\; \text{BP: } W = N \times \frac{2\pi}{T}(1+\alpha) \text{ rad/s}$$

$$\text{BB: } B = N \times \frac{R_s}{2}(1+\alpha) \text{ Hz},\; \text{BP: } B = N \times R_s(1+\alpha) \text{ Hz}$$

- A possible option: pulses fulfilling Nyquist ISI criterion at T/N
  - If Nyquist criterion is fulfilled at T/N it is also fulfilled at T
  - Bandwidth increases by a factor N
  - Problem: ambiguity function of pulses is localized in time
    ⇒ Power of the signal is localized in time

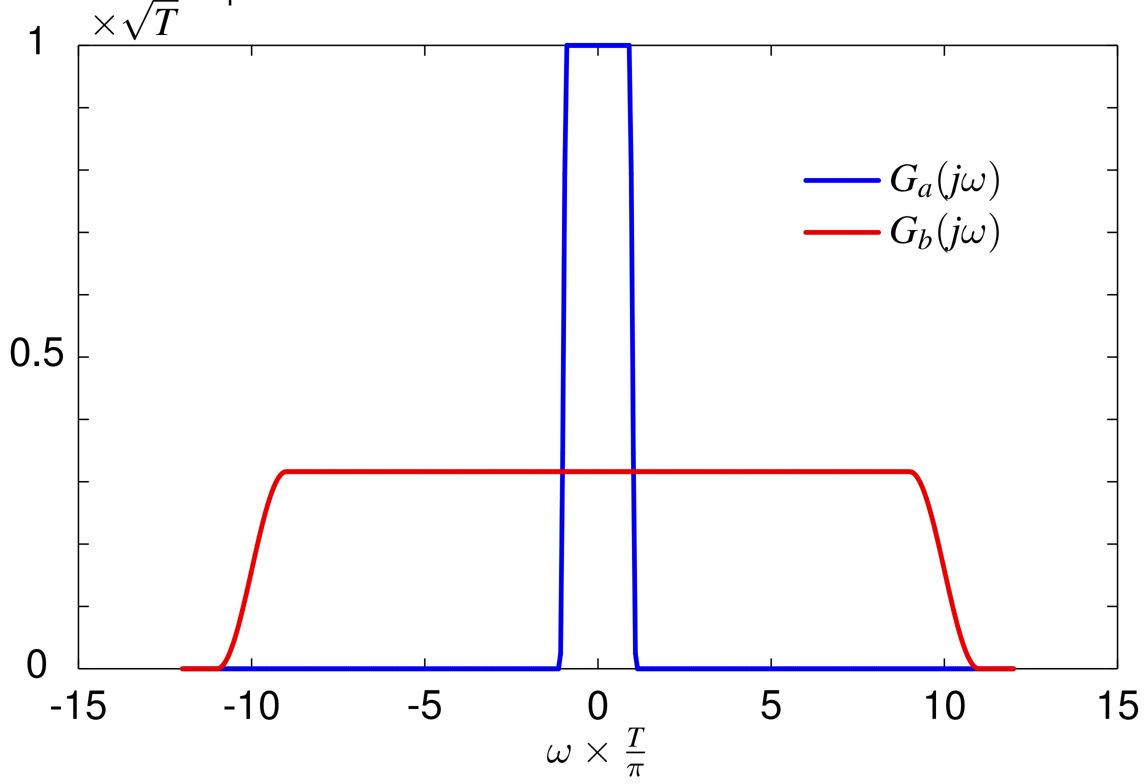
Raised cosine pulses: $g_a(t)$ at T and $g_b(t)$ at $\frac{T}{N}$ ($N = 10, \alpha = 0.1$)

Ambiguity function for root-raised cosine pulses with $\alpha = 0.1$

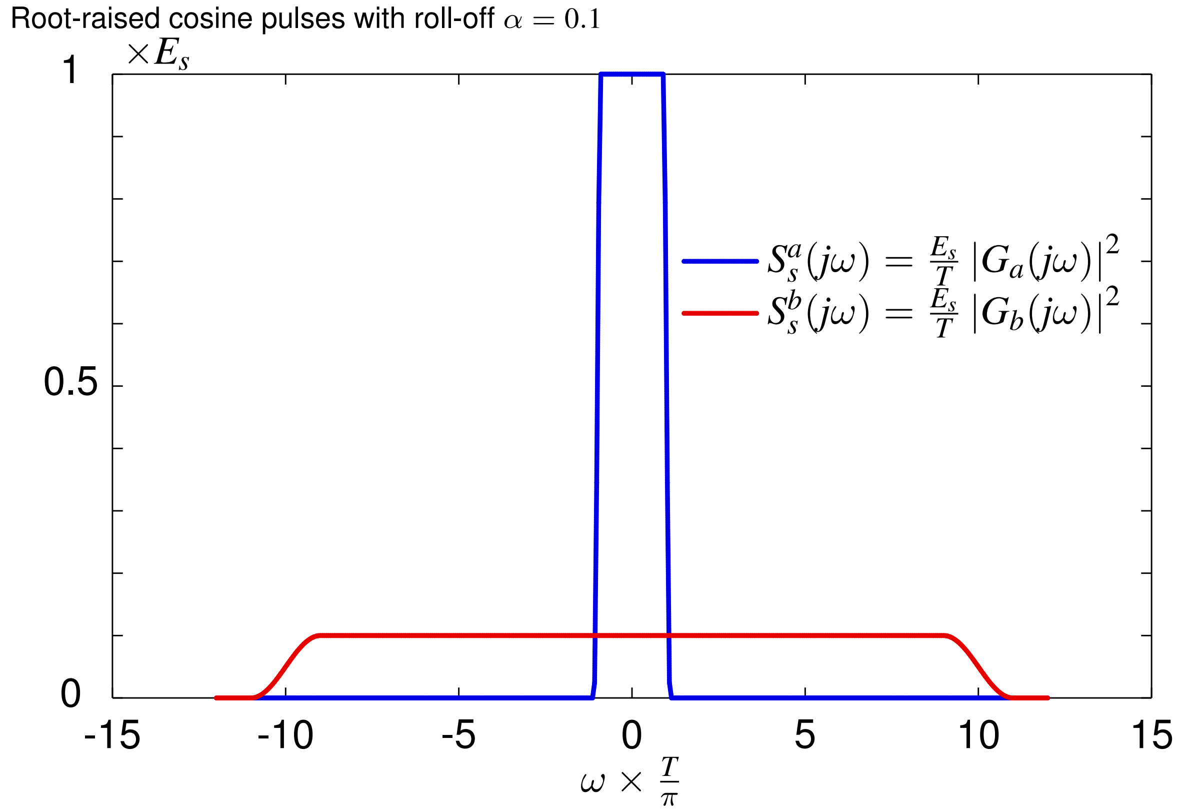


Frequency response of pulses at T and $\frac{T}{N}$ ($N = 10, \alpha = 0.1$)

Root-raised cosine pulses with roll-off $\alpha = 0.1$

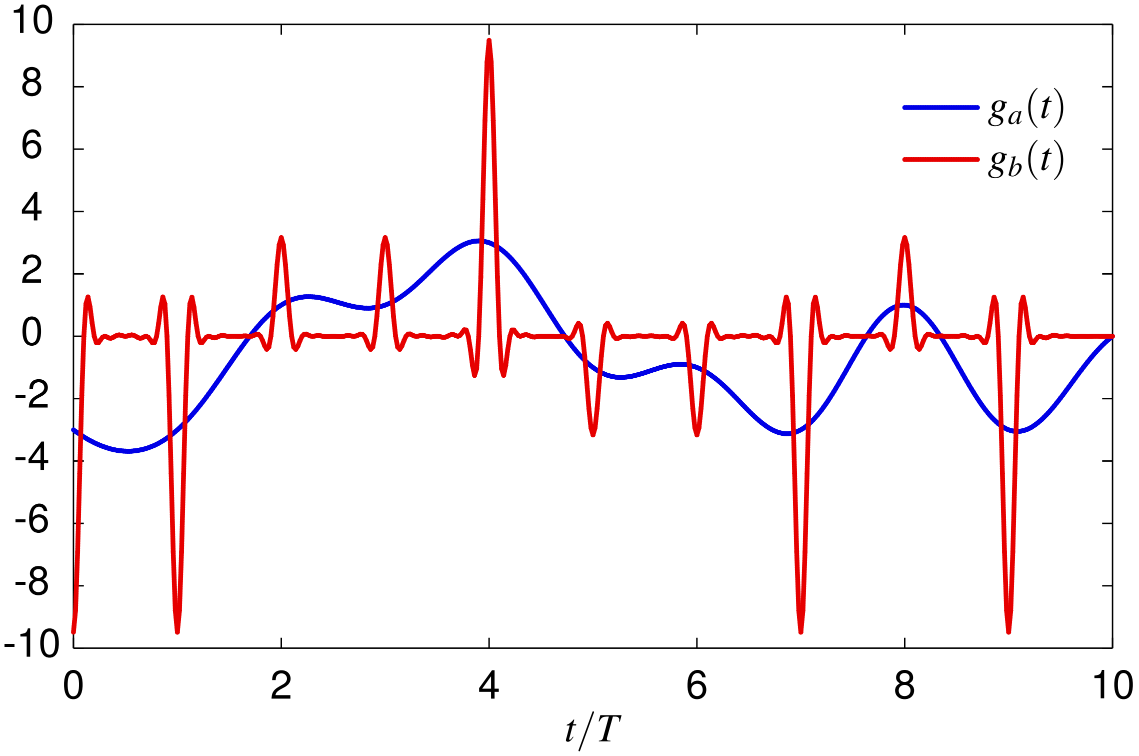


$S_s(j\omega)$ using pulses at T and $\frac{T}{N}$ ($N = 10, \alpha = 0.1$)

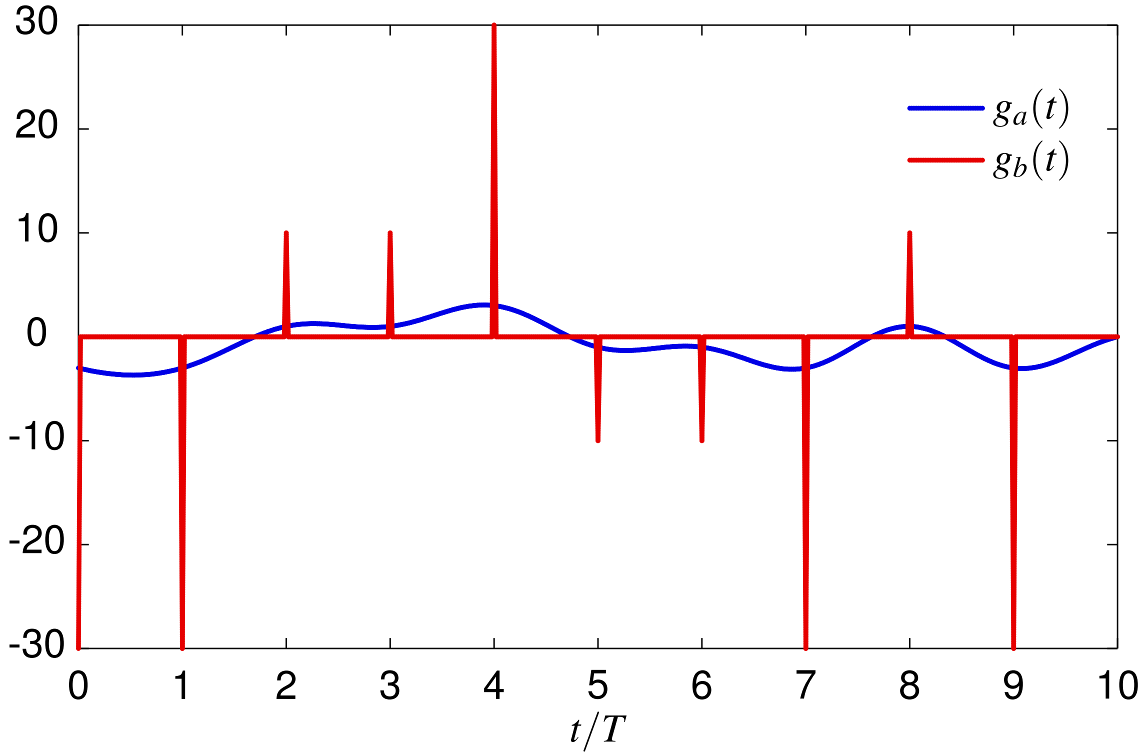


Root-raised cosine pulses with roll-off $\alpha = 0.1$

Example of waveforms: 4-PAM, $N = 10$, $\alpha = 0.5$



Example of waveforms: 4-PAM, $N = 100$, $\alpha = 0.5$



Direct sequence spread spectrum (DS-SS)

- This method is an alternative that avoids to localize power in time
- Family of pulses

$$\boxed{g(t) = \sum_{m=0}^{N-1} x[m]\; g_c(t - mT_c)}$$

Linear combination of N replicas of a pulse, $g_c(t)$, shifted multiples of $T_c = \frac{T}{N}$, with coefficients $x[m]$

  - $x[m]$: spreading sequence (*chip* sequence)
    - ★ N values: $\{x[0], x[1], \cdots, x[N-1]\}$
  - T_c: *chip* period $T_c = \frac{T}{N}$
  - $g_c(t)$: pulse such that $r_{g_c}(t)$ satisfies Nyquist at T_c

- The analytic expression for the modulated signal is

$$\boxed{s(t) = \sum_n A[n]\; g(t - nT) = \sum_n A[n] \underbrace{\sum_{m=0}^{N-1} x[m]\; g_c(t - mT_c - nT)}_{g(t-nT)}}$$

Example of pulse: squared $N = 4$

- Chip length: $T_c = \frac{T}{N} = \frac{T}{4}$
- Filter at chip rate

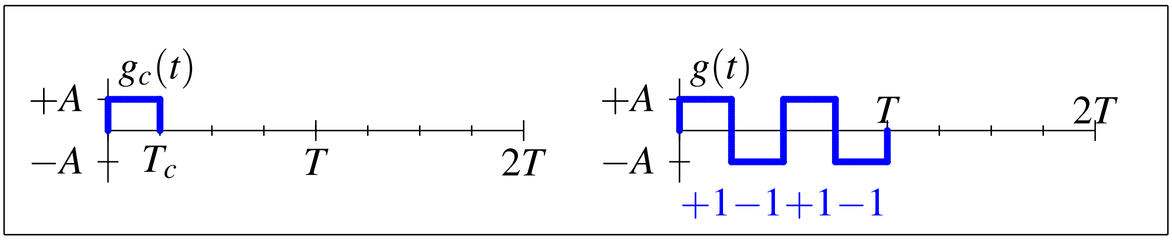
$$g_c(t) = \begin{cases} A & \text{if } 0 \le t < T_c \\ 0 & \text{other case} \end{cases}$$

- Transmitter filter (at symbol rate)

$$g(t) = \sum_{m=0}^{N-1} x[m]\ g_c(t - mT_c)$$

- Spreading sequence

n	0	1	2	3
$x[n]$	$+1$	-1	$+1$	-1

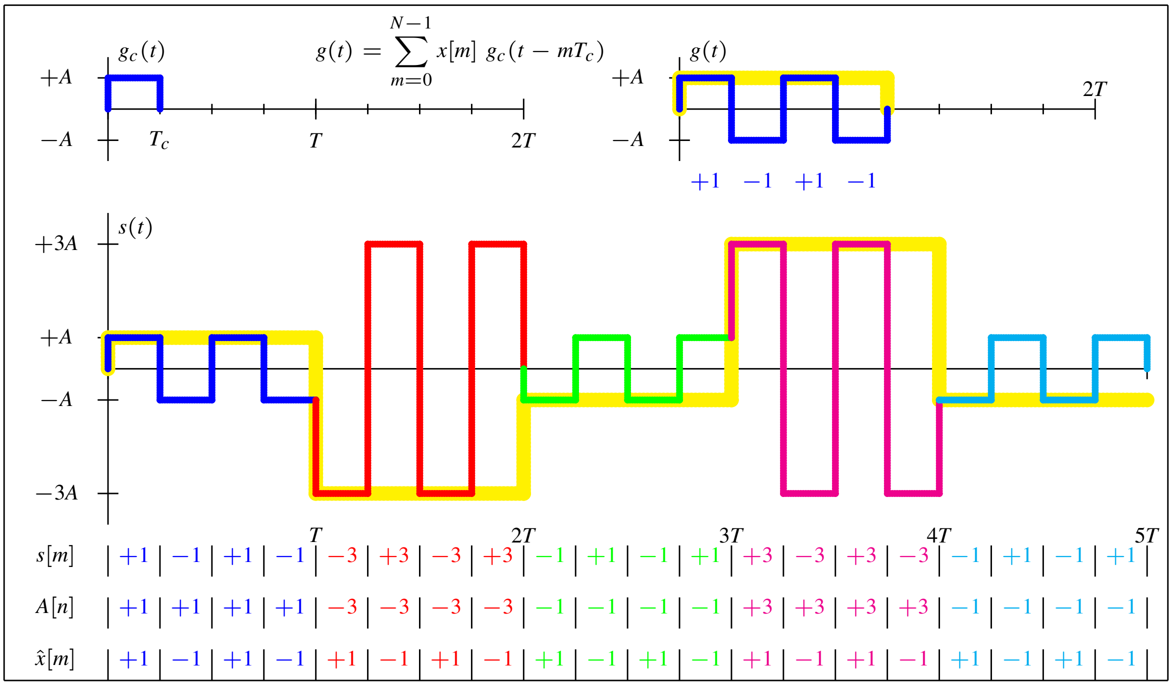


Generation of modulated signal $s(t)$ (Example $N = 4$)

- Data sequence to be transmitted:

n	0	1	2	3	4
$A[n]$	$+1$	-3	-1	$+3$	-1

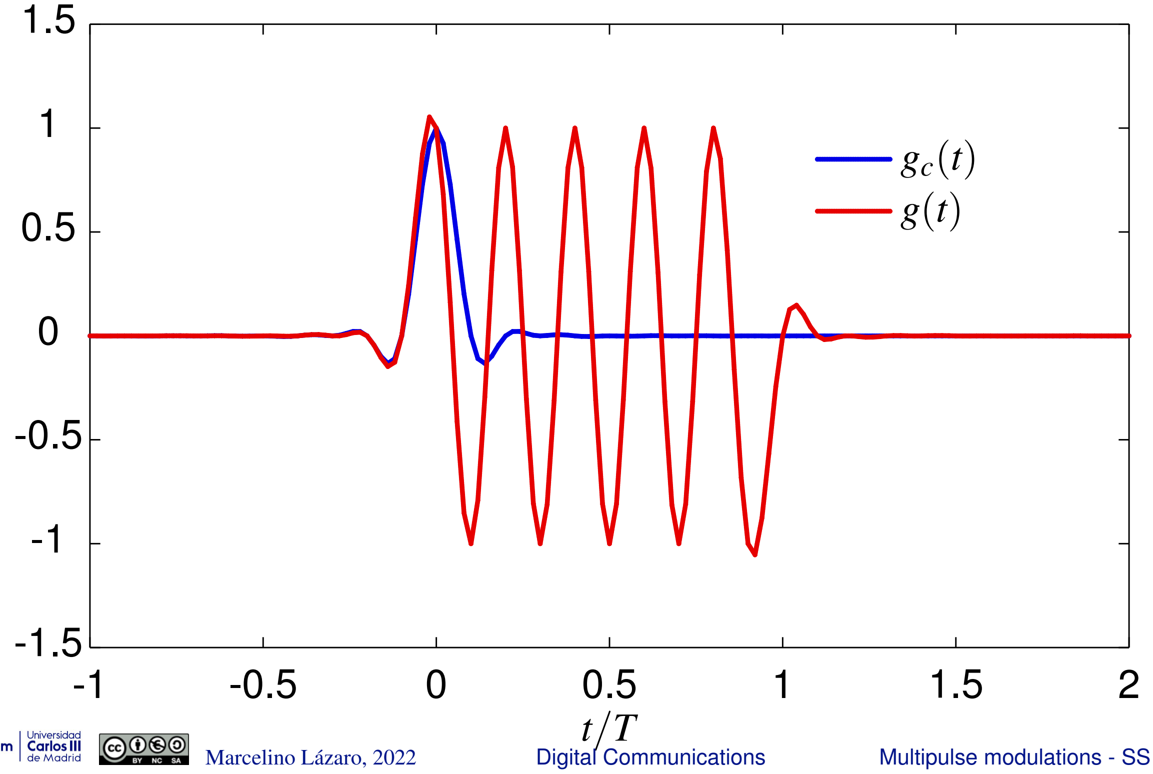
- Spreading sequence ($N = 4$): $x[0] = +1,\ x[1] = -1,\ x[2] = +1,\ x[3] = -1$



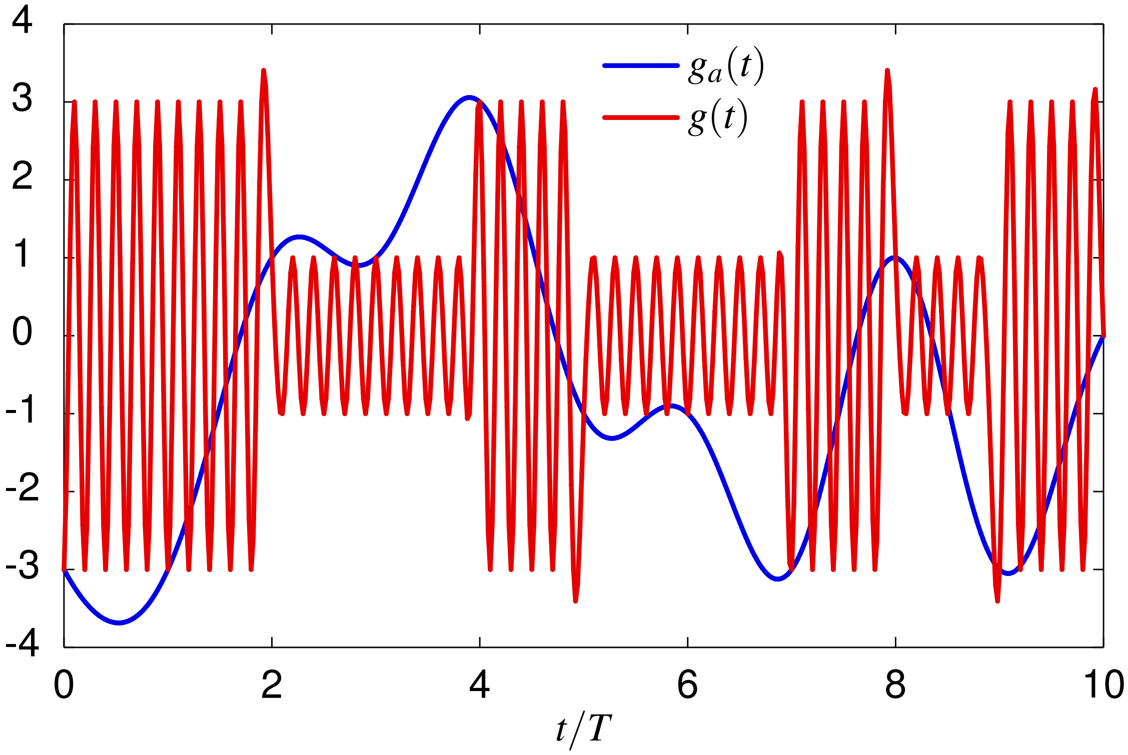
$s[m]$	$+1$	-1	$+1$	-1	-3	$+3$	-3	$+3$	-1	$+1$	-1	$+1$	$+3$	-3	$+3$	-3	-1	$+1$	-1	$+1$
$A[n]$	$+1$	$+1$	$+1$	$+1$	-3	-3	-3	-3	-1	-1	-1	-1	$+3$	$+3$	$+3$	$+3$	-1	-1	-1	-1
$\hat{x}[m]$	$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1

Example using raised cosine pulses $N = 10$, $\alpha = 0.5$

n	0	1	2	3	4	5	6	7	8	9
$x[n]$	$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1



Example of waveform using raised cosine pulses $N = 10$, $\alpha = 0.5$

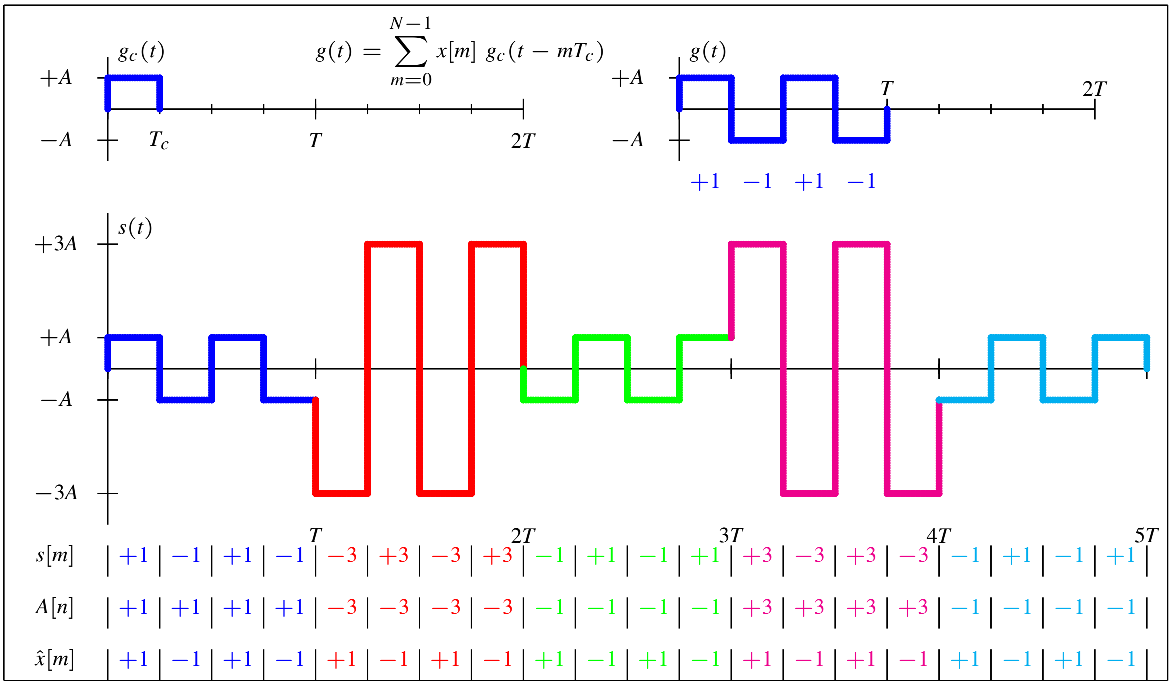


Generation of modulated signal $s(t)$ (Example $N = 4$)

- Data sequence to be transmitted:

n	0	1	2	3	4
$A[n]$	$+1$	-3	-1	$+3$	-1

- Spreading sequence ($N = 4$): $x[0] = +1,\ x[1] = -1,\ x[2] = +1,\ x[3] = -1$



Direct sequence spread spectrum - Alternative notation

- Some definitions
  - Periodical sequence $\tilde{x}[m]$ extending spreading sequence $x[m]$

$$\tilde{x}[m] = \sum_k x[m - kN]$$

  - Discrete-time causal window of N samples

$$w_N[m] = \begin{cases} 1 & \text{if } 0 \leq m \leq N-1 \\ 0 & \text{in other case} \end{cases}$$

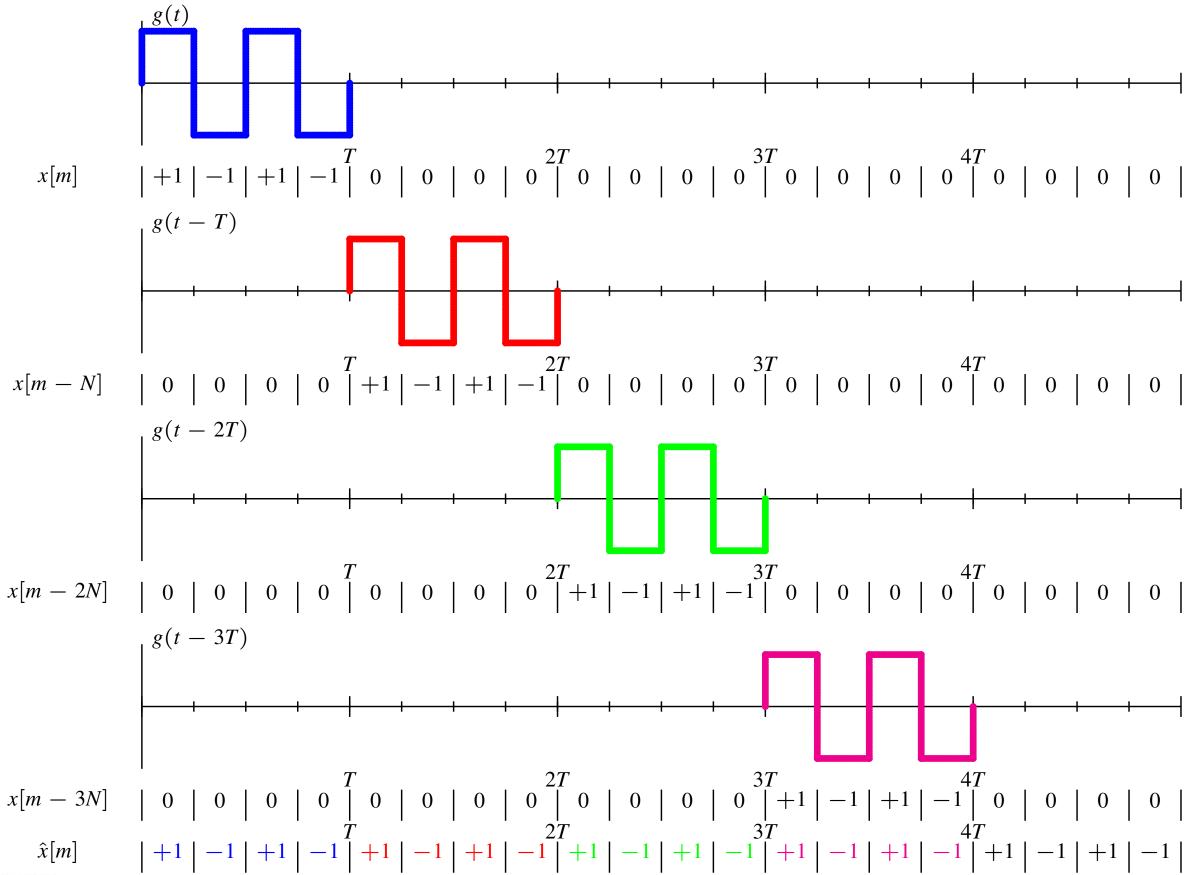
- Equation for $s(t)$ is re-written by re-writting $g(t - nT)$

$$s(t) = \sum_n A[n] \times \underbrace{\sum_{m=0}^{N-1} x[m]\ g_c(t - mT_c - nT)}_{g(t-nT)}$$

$$= \sum_n A[n] \times \sum_{m=nN}^{nN+N-1} x[m - nN]\ g_c(t - mT_c)$$

$$= \sum_n A[n] \times \sum_m \tilde{x}[m]\ w_N[m - nN]\ g_c(t - mT_c)$$

Expressions for shifted pulses (Example $N = 4$)



Direct sequence spread spectrum - Alternative notation (II)

- Previous expression is re-ordered

$$\boxed{\begin{aligned} s(t) &= \sum_n A[n] \sum_m \tilde{x}[m]\; w_N[m-nN]\; g_c(t-mT_c) \\ &= \sum_m \underbrace{\tilde{x}[m]\; \sum_n A[n]\; w_N[m-nN]}_{s[m]}\; g_c(t-mT_c) \\ &= \sum_m s[m]\; g_c(t-mT_c) \end{aligned}}$$

- Analysis of this expression
  - Generation of signal $s(t)$ transmitting sequence of samples $s[m]$ at chip rate, $R_c = \frac{1}{T_c}$, with transmitter filter at R_c
  - Generation of sequence of samples of the signal at chip rate, $s[m]$

$$\boxed{s[m] = \tilde{x}[m] \sum_n A[n]\; w_N[m-nN]}$$

DSSS Transmitter - Generation of samples and signal

- Samples of $s(t)$ at T_c (samples $s[m]$)

$$\boxed{s[m] = \tilde{x}[m] \sum_n A[n]\, w_N[m-nN], \qquad s[m] = A[n] \otimes x[m]}$$

- Generation of $s[m]$ in blocks of N samples

$$s[m] = \{\underbrace{s[0], s[1], \cdots, s[N-1]}_{s^{(0)}[m]}, \underbrace{s[N], s[N+1], \cdots, s[2N-1]}_{s^{(1)}[m]}, \cdots, \underbrace{s[nN], s[nN+1], \cdots, s[(n+1)N-1]}_{s^{(n)}[m]}, \cdots\}$$

Block of index n: $s^{(n)}[m]$, N samples $\{s^{(n)}[0], s^{(n)}[1], \cdots, s^{(n)}[N-1]\}$

Samples of block of index n: $s^{(n)}[m] = s[nN+m], \; m \in \{0, 1, \cdots, N-1\}$

Sequence of samples in the transmitter: $s[m] = \sum_n s^{(n)}[m-nN]$

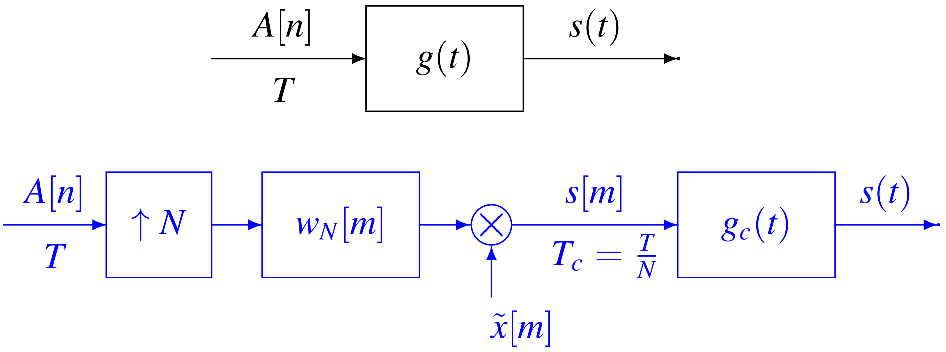
- Each symbol $A[n]$ generates the N samples of a block (index n)
  - Value of $A[n]$ is multiplied with the spreading sequence

$$\boxed{s^{(n)}[m] = A[n] \times x[m]}$$

$$\boxed{\left\{s^{(n)}[m]\right\}_{m=0}^{N-1} \equiv \{\underbrace{A[n]\times x[0]}_{s[nN]}, \underbrace{A[n]\times x[1]}_{s[nN+1]}, \underbrace{A[n]\times x[2]}_{s[nN+2]}, \cdots, \underbrace{A[n]\times x[N-1]}_{s[nN+N-1]}\}}$$

DSSS transmitter - Block diagram

- Block diagram for the transmitter



- Both structures are equivalent
  - Modulation of symbol sequence (at symbol rate $R_s = \frac{1}{T}$) with filter $g(t)$ defined at symbol rate
  - Modulation of sequence of samples (at chip rate $R_c = \frac{1}{T_c}$) with filter $g_c(t)$ defined at chip rate
- Implementation of second option is simpler
  - Implementation of transmitter filter $g(t)$ can be complex
    - Linear combination of N filters with $r_{g_c}(t)$ satisfying Nyquist at T_c
  - Implementation of transmitter filter $g_c(t)$ is simpler
    - Efficient generation of samples $s[m]$ from $A[n]$ by software

Spectrum of a DS-SS signal

- Power spectral density of baseband signal $s(t)$

$$\boxed{S_s(j\omega) = \frac{1}{T}\, S_A\left(e^{j\omega T}\right)\, |G(j\omega)|^2}$$

- Frequency response of pulse $g(t)$

$$\boxed{g(t) = \sum_{m=0}^{N-1} x[m]\; g_c(t - mT_c)}$$

$$\boxed{G(j\omega) = G_c(j\omega)\, \sum_{m=0}^{N-1} x[m]\; e^{-j\omega m T_c} = G_c(j\omega)\, X\left(e^{j\omega T_c}\right)}$$

- Power spectral density of the DS-SS signal

$$\boxed{S_s(j\omega) = \frac{1}{T}\, S_A\left(e^{j\omega T}\right)\, \left|X\left(e^{j\omega T_c}\right)\right|^2\, |G_c(j\omega)|^2}$$

Bandwidth of DS-SS signals

- Baseband bandwidth

$$\boxed{S_s(j\omega) = \frac{1}{T}\, S_A\left(e^{j\omega T}\right)\, \left|X\left(e^{j\omega T_c}\right)\right|^2\, |G_c(j\omega)|^2}$$

  - Bandwidth given by filter $g_c(t)$ at chip rate
    - ★ Filters of the raised-cosine family (root)

$$\boxed{B = \frac{R_s}{2}\,(1+\alpha) \times N \text{ Hz}}$$

- Bandpass bandwidth

$$\boxed{S_X(j\omega) = \frac{1}{2}\left[S_s(j\omega - j\omega_c) + S_s^*(-(j\omega + j\omega_c))\right]}$$

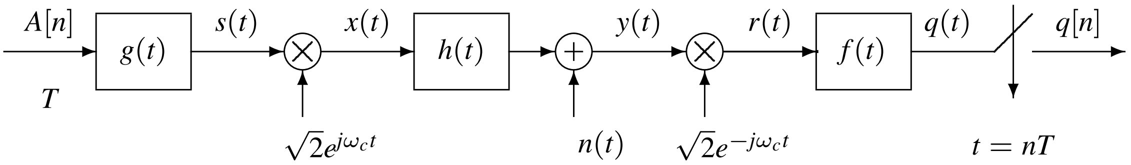
  - For a given symbol rate, bandwidth is doubled

$$\boxed{B = R_s\,(1+\alpha) \times N \text{ Hz}}$$

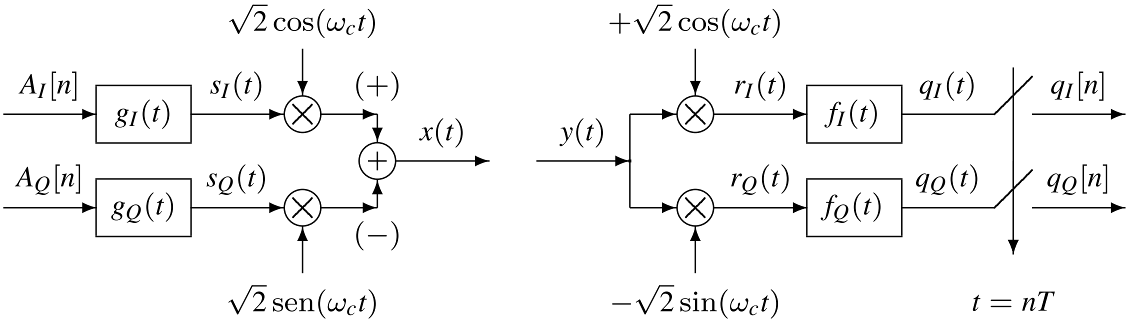
    - ★ Now each T sec. two symbols ($A_I[n]$ and $A_Q[n]$) are transmitted
      - Spectral efficiency is the same one as in a baseband transmission

Bandpass transmission

- Complex representation



- Representation using in-phase and quadrature components



- Spread spectrum sequence and filter $g(t)$ are complex

$$\boxed{x[m] = x_I[m] + jx_Q[m], \;\; g(t) = g_I(t) + jg_Q(t)}$$

  - Usually, $x_I[m] = x_Q[m]$ (althought it is not necessary)

Baseband (complex) receiver

- Receiver filter is $f(t) = g^*(-t)$, and $r(t)$ is the baseband received signal

$$g(t) = \sum_{m=0}^{N-1} x[m]\; g_c(t - mT_c) \Rightarrow g^*(-t) = \sum_{m=0}^{N-1} x^*[m]\; g_c^*(-t - mT_c) = \sum_{m=0}^{N-1} x^*[m]\; g_c(-t - mT_c)$$

REMARK: $g_c^*(-t) = g_c(-t)$, because $g_c(t)$ is a real filter

$$\boxed{\begin{aligned} q[n] &= (r(t) * g^*(-t))\,|_{t=nT} = \sum_{m=0}^{N-1} x^*[m]\; (r(t) * g_c(\underbrace{-t - mT_c}_{-(t+mT_c)}))\Big|_{t=nT} \\ &= \sum_{m=0}^{N-1} x^*[m] \left(\underbrace{r(t) * g_c(-t)}_{v(t)} \right)\Bigg|_{t=\underbrace{nT + mT_c}_{(nN+m)T_c}} \end{aligned}}$$

  - Signal at the output of the receiver filter (at chip rate): $v(t) = r(t) * g_c(-t)$
  - Discrete-time signal sampled at T_c: $v[m] = v(t)\big|_{t=mT_c=m\frac{T}{N}} = v(mT_c)$

- The output of the demodulator can be written as follows

$$\boxed{q[n] = \sum_{m=0}^{N-1} x^*[m]\; v[nN + m]}$$

Block processing of N samples

- Samples of sequence $v[m]$ are processed in blocks of N samples

$$v[m]=\{\underbrace{v[0],v[1],\cdots,v[N-1]}_{v^{(0)}[m]},\underbrace{v[N],v[N+1],\cdots,v[2N-1]}_{v^{(1)}[m]},\cdots,\underbrace{v[nN],v[nN+1],\cdots,v[(n+1)N-1]}_{v^{(n)}[m]},\cdots\}$$

Block of index n: $v^{(n)}[m],\ N$ samples $\{v^{(n)}[0],v^{(n)}[1],\cdots,v^{(n)}[N-1]\}$

Samples of block of index n: $v^{(n)}[m]=v[nN+m],\ m\in\{0,1,\cdots,N-1\}$

Sequence with samples at the receiver: $v[m]=\sum_n v^{(n)}[m-nN]$

- Computation of $q[n]$: processing of block of index n

$$q[n]=\sum_{m=0}^{N-1}x^*[m]\ v[nN+m]=\sum_{m=0}^{N-1}x^*[m]\ v^{(n)}[m]$$

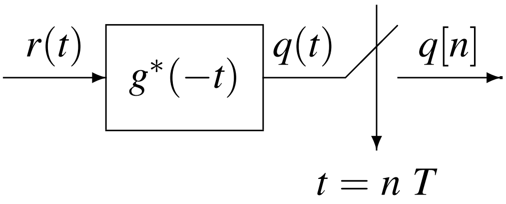
$$q[n]=\underbrace{v^{(n)}[0]}_{v[nN]}\times x^*[0]+\underbrace{v^{(n)}[1]}_{v[nN+1]}\times x^*[1]+\underbrace{v^{(n)}[2]}_{v[nN+2]}\times x^*[2]+\cdots+\underbrace{v^{(n)}[N-1]}_{v[nN+N-1]}\times x^*[N-1]$$

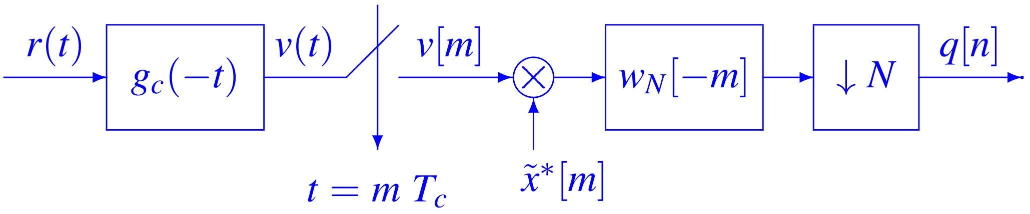
Baseband receiver - Block diagram

- Output of the demodulator at symbol rate

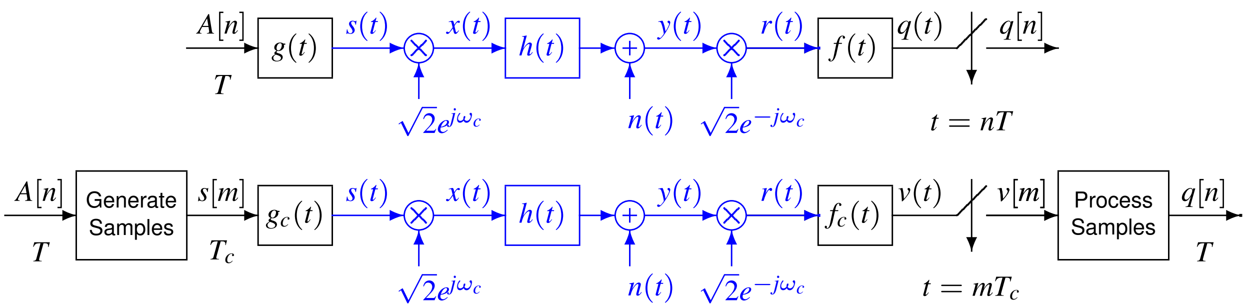
$$\begin{aligned}q[n]&=\sum_{m=0}^{N-1}x^*[m]\ v[nN+m]\\&=(v[m]\ \tilde{x}^*[m])*w_N[-m-nN]\end{aligned}$$

- Block diagram for the receiver





Equivalent discrete channels



- Receiver filters - matched filters: $f(t) = g^*(-t), \; f_c(t) = g_c(-t)$
- Joint transmitter/receiver/channel responses
  - Transmission of $A[n]$ at symbol rate: $p(t) = g(t) * h_{eq}(t) * f(t) = r_g(t) * h_{eq}(t)$
  - Transmission of $s[m]$ at chip rate: $d(t) = g_c(t) * h_{eq}(t) * f_c(t) = r_{g_c}(t) * h_{eq}(t)$
- Equivalent discrete channels
  - At symbol rate

$$\boxed{p[n] = p(t)\big|_{t=nT} = p(nT) \text{ relates } q[n] \text{ with } A[n]\text{: } q[n] = A[n] * p[n] + z[n]}$$

  - At chip rate

$$\boxed{d[m] = d(t)\big|_{t=mT_c} = d(mT_c) \text{ relates } v[m] \text{ with } s[m]\text{: } v[m] = s[m] * d[m] + z_c[m]}$$

Characteristic of noise at the receiver (bandpass)

- Channel introduces additive Gaussian noise $n(t)$ with PSD $S_n(j\omega)$
- If the following conditions happen
  - $f(t) = g^*(-t)$
  - $n(t)$ is white and $S_n(j\omega) = N_0/2$ W/Hz

$z[n]$ is white and Gaussian, with variance

$$\boxed{\sigma_z^2 = N_0 \; \mathcal{E}\left\{g(t)\right\}}$$

Real and imaginary part are independent with variance $\frac{N_0}{2} \; \mathcal{E}\left\{g(t)\right\}$

$z_c[n]$ is white and Gaussian, with variance

$$\boxed{\sigma_{z_c}^2 = N_0 \; \mathcal{E}\left\{g_c(t)\right\}}$$

Real and imaginary part are independent with variance $\frac{N_0}{2} \; \mathcal{E}\left\{g_c(t)\right\}$

Energy of pulses $g_c(t)$ and $g(t)$

- Definition of the pulse at symbol rate $$\boxed{g(t) = \sum_{m=0}^{N-1} x[m]\; g_c(t - mT_c)}$$
- Energy of pulse $g(t)$

$$\begin{aligned}
\mathcal{E}\left\{g(t)\right\} &= \int_{-\infty}^{\infty} |g(t)|^2\, dt = \int_{-\infty}^{\infty} g(t)\; g^*(t)\; dt \\
&= \int_{-\infty}^{\infty} \left(\sum_{m=0}^{N-1} x[m]\; g_c(t - mT_c)\right) \left(\sum_{\ell=0}^{N-1} x^*[\ell]\; g_c(t - \ell T_c)\right)\; dt \\
&= \sum_{m=0}^{N-1}\sum_{\ell=0}^{N-1} x[m]\; x^*[\ell] \int_{-\infty}^{\infty} g_c(t - mT_c)\; g_c(t - \ell T_c)\; dt \\
&= \sum_{m=0}^{N-1}\sum_{\ell=0}^{N-1} x[m]\; x^*[\ell]\; \underbrace{\int_{-\infty}^{\infty} g_c(\tau)\; g_c(\underbrace{\tau - \ell T_c + mT_c}_{-([\ell-m]T_c-\tau)})\; d\tau}_{r_{g_c}([\ell-m]T_c)} \\
&= \left[\sum_{m=0}^{N-1} |x[m]|^2\right] \times \mathcal{E}\left\{g_c(t)\right\}
\end{aligned}$$

<u>REMARK</u>: If $r_{g_c}(t)$ satisfies Nyquist at T_c, $r_{g_c}([\ell - m]T_c)$ is null for $\ell \neq m$, and for $\ell = m$ (property of an ambiguity function) takes the value of the energy of $g_c(t)$, i.e. $r_{g_c}([\ell - m]T_c = \mathcal{E}\left\{g_c(t)\right\}\, \delta[\ell - m]$

Equivalent discrete channel at symbol rate, $p[n]$

- Receiver filter at symbol rate is $f(t) = g^*(-t)$

$$g(t) = \sum_{m=0}^{N-1} x[m]\; g_c(t - mT_c), \;\; f(t) = g^*(-t) = \sum_{\ell=0}^{N-1} x^*[\ell]\; g_c(-t - \ell T_c)$$

- Equivalent discrete channel

$$p[n] = \left(g(t) * h_{eq}(t) * f(t)\right)\Big|_{t=nT}$$

$$\begin{aligned}
p[n] &= \sum_{m=0}^{N-1}\sum_{\ell=0}^{N-1} x[m]\; x^*[\ell]\; \left(g_c(t - mT_c) * h_{eq}(t) * g_c(\underbrace{-t - \ell T_c}_{-(t+\ell T_c)})\right)\Bigg|_{t=nT} \\
&= \sum_{m=0}^{N-1}\sum_{\ell=0}^{N-1} x[m]\; x^*[\ell]\; \left(\underbrace{g_c(t) * h_{eq}(t) * g_c(-t)}_{d(t)}\right)\Bigg|_{t=\underbrace{nT + \ell T_c - mT_c}_{(nN+\ell-m)T_c}}
\end{aligned}$$

$$\boxed{p[n] = \sum_{m=0}^{N-1}\sum_{\ell=0}^{N-1} x[m]\; x^*[\ell]\; d[nN + \ell - m]}$$

Equivalent discrete channel $p[n]$ - Example

- Equivalent discrete channel at symbol rate

$$\boxed{p[n]=\sum_{m=0}^{N-1}\sum_{\ell=0}^{N-1} x[m]\; x^*[\ell]\; d[nN+\ell-m]}$$

- Spreading factor $N=10$
  - Spreading sequence

$$x[0],x[1],x[2],x[3],x[4],x[5],x[6],x[7],x[8],x[9]$$

- Equivalent discrete channel at chip rate

$$d[m]=a\;\delta[m]+b\;\delta[m-2]+c\;\delta[m-14]$$

  - Non-null values of $d[nN+\ell-m]$
    - ★ $nN+\ell-m=0$
    - ★ $nN+\ell-m=2$
    - ★ $nN+\ell-m=14$

Equivalent discrete channel $p[n]$ - Example (II)

- Case $nN+\ell-m=0 \Rightarrow d[nN+\ell-m]=a$
  - $n=0\rightarrow \ell-m=0\rightarrow \ell=m$

$$\sum_{m=0}^{N-1} x[m]\; x^*[m]=\sum_{m=0}^{9}|x[m]|^2=a_1$$

- Case $nN+\ell-m=2 \Rightarrow d[nN+\ell-m]=b$
  - $n=0\rightarrow \ell-m=2\rightarrow \ell=m+2$

$$\sum_{m=0}^{N-1} x[m]\; x^*[m+2]=\sum_{m=0}^{7} x[m]\; x^*[m+2]=b_1$$

  - $n=1\rightarrow N+\ell-m=2\rightarrow \ell=m-8$

$$\sum_{m=0}^{N-1} x[m]\; x^*[m-8]=\sum_{m=8}^{9} x[m]\; x^*[m-8]=b_2$$

Equivalent discrete channel $p[n]$ - Example (III)

- Case $nN + \ell - m = 14 \Rightarrow d[nN + \ell - m] = c$
  - $n = 1 \rightarrow N + \ell - m = 14 \rightarrow \ell = m + 4$

$$\sum_{m=0}^{N-1} x[m]\, x^*[m+4] = \sum_{m=0}^{5} x[m]\, x^*[m+4] = c_1$$

  - $n = 2 \rightarrow 2N + \ell - m = 14 \rightarrow \ell = m - 6$

$$\sum_{m=0}^{N-1} x[m]\, x^*[m-6] = \sum_{m=6}^{9} x[m]\, x^*[m-6] = c_2$$

- Equivalent discrete channel

$$\begin{aligned} p[n] = &\,(a \times a_1 + b \times b_1)\ \delta[n] \\ &+ (b \times b_2 + c \times c_1)\ \delta[n-1] \\ &+ (c \times c_2)\ \delta[n-2] \end{aligned}$$

Equivalent discrete channel $p[n]$ - Example (IV)

- Spreading sequence

m	0	1	2	3	4	5	6	7	8	9
$x[m]$	$+1$	-1	$+1$	-1	-1	-1	$+1$	$+1$	-1	-1

- Values related with $p[n]$, for $a = 1$, $b = -\frac{1}{2}$, $c = \frac{1}{4}$

$$a_1 = 10,\ b_1 = -2,\ b_2 = 0,\ c_1 = +2,\ c_2 = 0$$

$$p[n] = 11\ \delta[n] + \frac{1}{2}\delta[n-1]$$

  - Terms related with ISI

$$\boxed{\sum_{m=0}^{N-1} x[m]\, x^*[m-k_a],\ \sum_{m=0}^{N-1} x[m]\, x^*[m+k_b]}$$

Conditions to eliminate ISI

- There is not intersymbol interference if $d[m] = C\ \delta[m]$
  - There is not interference over $A[n]$ whenever there is not interference over $s[m]$
- When $d[m] \neq C\ \delta[m]$, ISI is avoided if

$$\boxed{\sum_{m=0}^{N-1} x[m]\ x^*[m \pm k] = C\ \delta[k]}$$

which is fulfilled if
  - the ambiguity function of $x[m]$ is a delta function
  - Spectrum of $x[m]$ is constant $\boxed{\left|X(e^{j\omega})\right|^2 = C}$
- Examples of sequence with (almost) flat spectrum
  - $x[m] = e^{j\theta}\ \delta[m-k]$ (problem: time localization)
  - Pseudo-noise sequences: $\left|X(e^{j\omega})\right|^2 \approx C$

Energy of pulses $g_c(t)$ and $g(t)$ - Practical implication

- Transmission through an ideal channel ($h_{eq}(t) = \delta(t)$)
- Equivalent discrete channel at chip rate

$$d[m] = d(t)\big|_{t=mT_c} = \left(g_c(t) * h_{eq}(t) * g_c(-t)\right)\big|_{t=mT_c} = r_{g_c}(t)\big|_{t=mT_c} = \mathcal{E}\left\{g_c(t)\right\}\ \delta[m]$$

- Equivalent discrete channel at symbol rate

$$\boxed{p[n] = \sum_{m=0}^{N-1} \sum_{\ell=0}^{N-1} x[m]\ x^*[\ell]\ d[nN + \ell - m]}$$

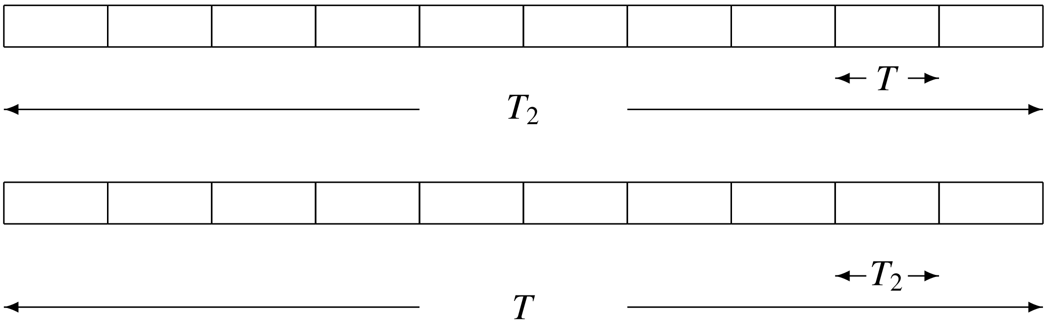
  - Given $d[m]$, non-null terms only for $nN + \ell - m = 0$
    - ★ Only possible when $n = 0$ and $\ell = m$

$$\boxed{p[n] = \left[\sum_{m=0}^{N-1} |x[m]|^2 \times \mathcal{E}\left\{g_c(t)\right\}\right]\ \delta[n] = \mathcal{E}\left\{g(t)\right\}\ \delta[n]}$$

<u>REMARK</u>: $p[n] = p(t)\big|_{t=nT} = \left(g(t) * h_{eq}(t) * g(-t)\right)\big|_{t=nT} = r_g(t)\big|_{t=nT} = \mathcal{E}\left\{g(t)\right\}\ \delta[n]$

Frequency hopping spread spectrum (FH-SS)

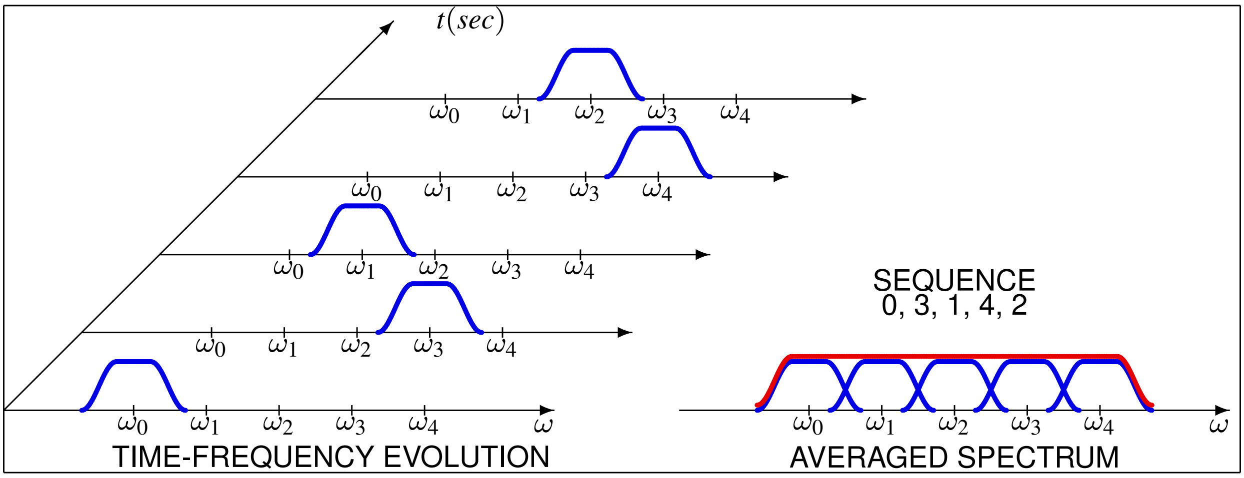
- Designed to work in channels with attenuation “valleys”
  - Idea: to alternate good and bad portions of the spectrum
    - ★ Carrier frequency changes periodically
    - ★ Period for “hopping” in carrier frequencies: T_2
- Clasification
  - Slow frequency hopping: $T_2/T = N \in \mathbb{Z} > 1$
  - Fast frequency hopping: $T/T_2 = N \in \mathbb{Z} > 1$



- It can be implemented using a variety of basic modulations
  - An example: Continuous phase FSK (CPFSK)

FH-SS spectrum - cyclic time evolution

- Frequency hops are guided by the spreading sequence
  - Pseudorandom sequence defining the order of carriers in the hopping
  - Has to be known by both transmitter and receiver
- Example using 5 carriers



Expressions using CPFSK modulations

- M-ary CPFSK signal: $I[n] \in \{\pm 1, \pm 3, \cdots, \pm(M-1)\}$

$$x(t) = \sqrt{\frac{2E_s}{T}} \sum_n \sin\left(\omega_c t + I[n]\frac{\pi}{T}t\right)\ w_T(t-nT)$$

- Slow frequency hopping signal

$$x(t) = \sqrt{\frac{2E_s}{T}} \sum_m \sum_{n=0}^{T_2/T-1} \sin\left(\omega_c t + x[m]\frac{\pi}{T}t + I[n+mN]\frac{\pi}{T}t\right)\ w_T(t-nT-mT_2)$$

- Fast frequency hopping signal

$$x(t) = \sqrt{\frac{2E_s}{T}} \sum_n \sum_{m=0}^{T/T_2-1} \sin\left(\omega_c t + x[m+nN]\frac{\pi}{T_2}t + I[n]\frac{\pi}{T_2}t\right)\ w_{T_2}(t-nT-mT_2)$$

- $x[m]$: deterministic sequence organizing the changes in frequency ($2kM$)

Multiple medium access - CDMA

- One of the applications of spread spectrum is multiple medium access
    - Several users access simultaneously to the system using the same frequency band
        - Code Division Medium Access (CDMA)
- Each user uses a different spreading sequence
    - User code
- Conditions to select sequences for different uses are particular for each kind of spread spectrum modulation

CDMA - DS-SS

- Basic modulation parameter are identical for all users
  - $g_c(t)$, T, T_c
- Multiuser signals in CDMA: L users
  - Each user has a different spreading sequence $\boxed{x_i[m]}$
  - Transmission filters for users at symbol time are given by

$$\boxed{g_i(t) = \sum_{m=0}^{N-1} x_i[m]\ g_c(t - m\ T_c)}$$

  - Complex baseband signal

$$s(t) = \sum_{i=0}^{L-1} s_i(t)$$

$$\boxed{s_i(t) = \sum_n A_i[n]\ g_i(t - nT) = \sum_n A_i[n] \sum_{m=0}^{N-1} x_i[m]\ g_c(t - mT_c - nT)}$$

- Separation of the signals of different users
  - Orthogonal transmission filters: $\boxed{\langle g_i(t), g_j(t)\rangle = 0, \quad \forall i \neq j}$

Condition for orthogonality of the pulses

- Inner product of two different filters at symbol period is

$$\langle g_i(t), g_j(t)\rangle = \int_{-\infty}^{\infty} g_i(t)\ g_j^*(t)\ dt$$

$$\boxed{\begin{array}{l} g_i(t) = \sum_{m=0}^{N-1} x_i[m]\ g_c(t - m\ T_c) \\ g_j(t) = \sum_{\ell=0}^{N-1} x_j[\ell]\ g_c(t - \ell\ T_c) \end{array}}$$

$$= \sum_{m=0}^{N-1}\sum_{\ell=0}^{N-1} x_i[m]\ x_j^*[\ell]\ \int_{-\infty}^{\infty} g_c(t - mT_c)\ g_c^*(t - \ell T_c)\ dt$$

$$= \sum_{m=0}^{N-1}\sum_{\ell=0}^{N-1} x_i[m]\ x_j^*[\ell]\ \left(g_c(t - mT_c) * g_c^*(-t - \ell T_c)\right)_{t=0}$$

$$= \sum_{m=0}^{N-1} x_i[m]\ x_j^*[m]$$

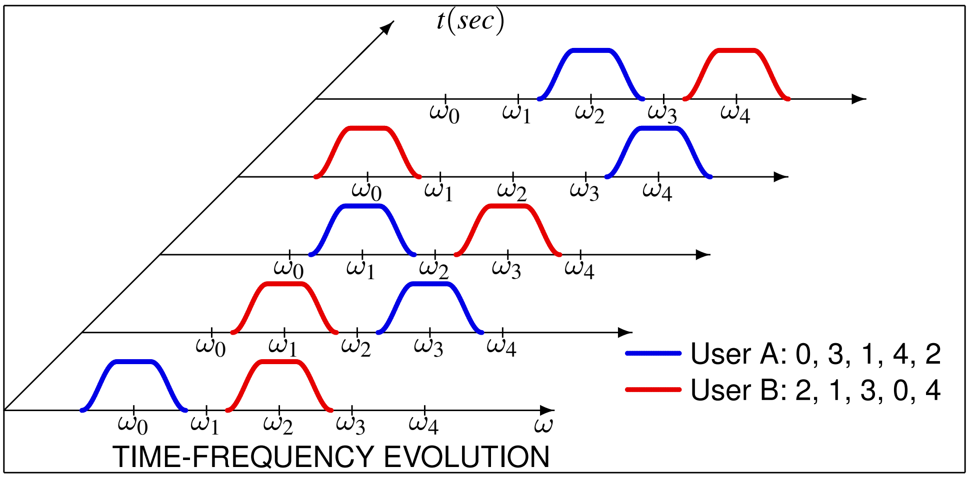
- Filters are orthogonal if spreading sequences are orthogonal

$$\boxed{\sum_{m=0}^{N-1} x_i[m]\ x_j^*[m] = \begin{cases} C, & i = j \\ 0, & i \neq j \end{cases} = C\ \delta[i - j], \quad i,j \in \{0, 1, \cdots, L-1\}}$$

- Several kind of sequences are used in practical systems
  - Gold sequences (1967), Kasami code, Welch sequences,...

CDMA - FH-SS

- Different users employ different spreading sequences
  - Sequences can not produce spectral overlapping at any moment
- A simple example with 5 carriers and 2 users



Modulation using multiple carriers - FDM

FDM - Frequency division multiplex

- Bandwidth for the FDM system: $\boxed{W^{FDM} \text{ rad/s}}$
- Available bandwidth is divided in N subchannels
  - Bandwidth for each subchannel: $\boxed{W = \frac{W^{FDM}}{N} \text{ rad/s}}$
  - Data sequence $A[n]$ is divided in N sequences
    - ★ Transmission of a different modulated signal in each subchannel
  - Subchannel symbol rate: $\boxed{R_s = \frac{1}{T} \text{ bauds}}$
  - Bandwidth / transmission rate relationship
    - ★ Using filter of the raised cosine family: $\boxed{W = \frac{2\pi}{T}(1+\alpha) \text{ rad/s}}$
- Total FDM system rate:

$$\boxed{R_s^{FDM} = \frac{1}{T^{FDM}} = N \times R_s \text{ bauds}} \quad \boxed{T^{FDM} = \frac{T}{N} \text{ seconds}}$$

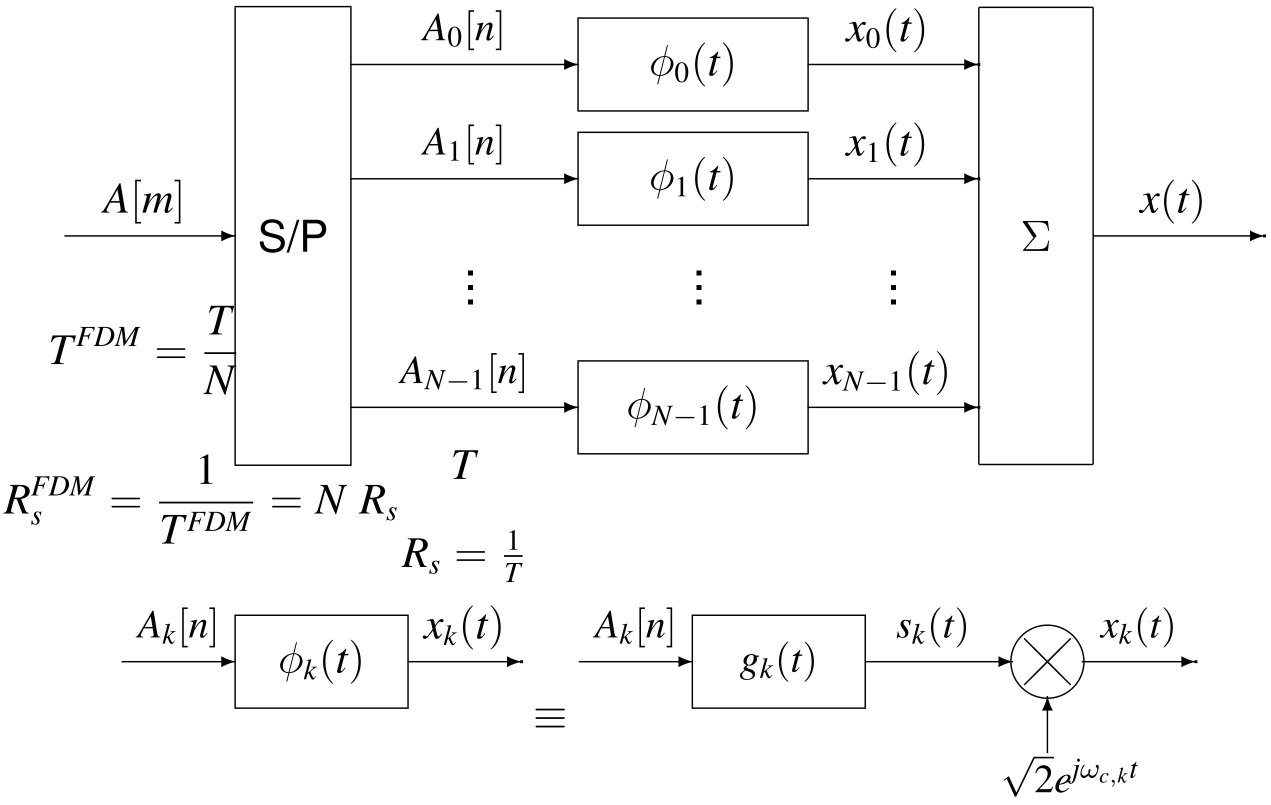
Modulation using multiple carriers - FDM (II)

- Transmitter
  - Serial to parallel conversion: $\boxed{A[m] \rightarrow \{A_0[n], \cdots, A_{N-1}\}}$
    - Conversion of rates:

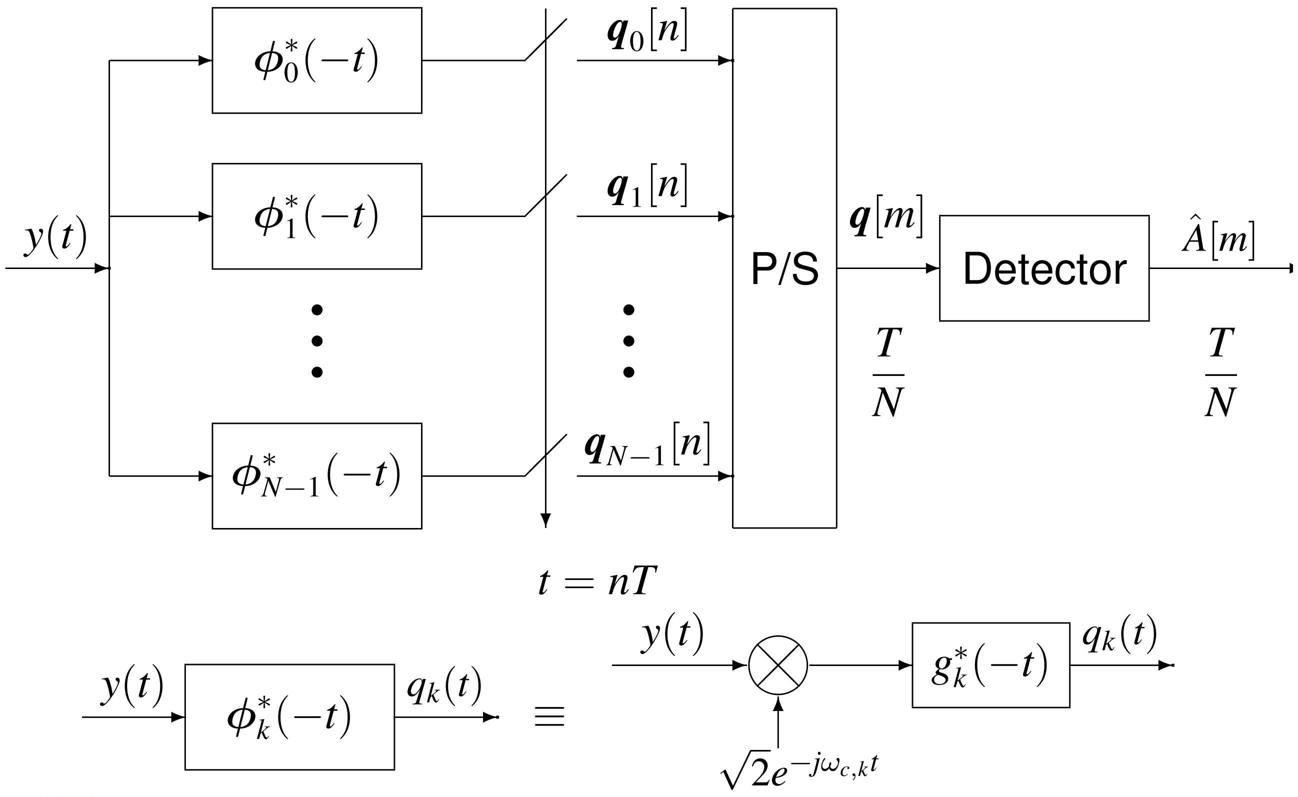
      $\boxed{R_s^{FDM} \text{ bauds (FDM system)} \rightarrow R_s \text{ bauds (per channel)}}$
  - N branches with bandpass PAM signals
    - Transmission filter in k-th branch is $\phi_k(t)$, $k = 0, \cdots, N-1$
      - Parameters: shaping filter $g_k(t)$, central frequency $\omega_{c,k}$
    - The modulated signal in k-th branch is $s_k(t)$
- Receiver
  - Set of N matched filters
  - Parallel to serial conversion: $\boxed{\{\hat{A}_0[n], \cdots, \hat{A}_{N-1}\} \rightarrow \hat{A}[m]}$
    - Conversion of rates:

      $\boxed{R_s \text{ bauds (per channel)} \rightarrow R_s^{FDM} \text{ bauds (FDM system)}}$

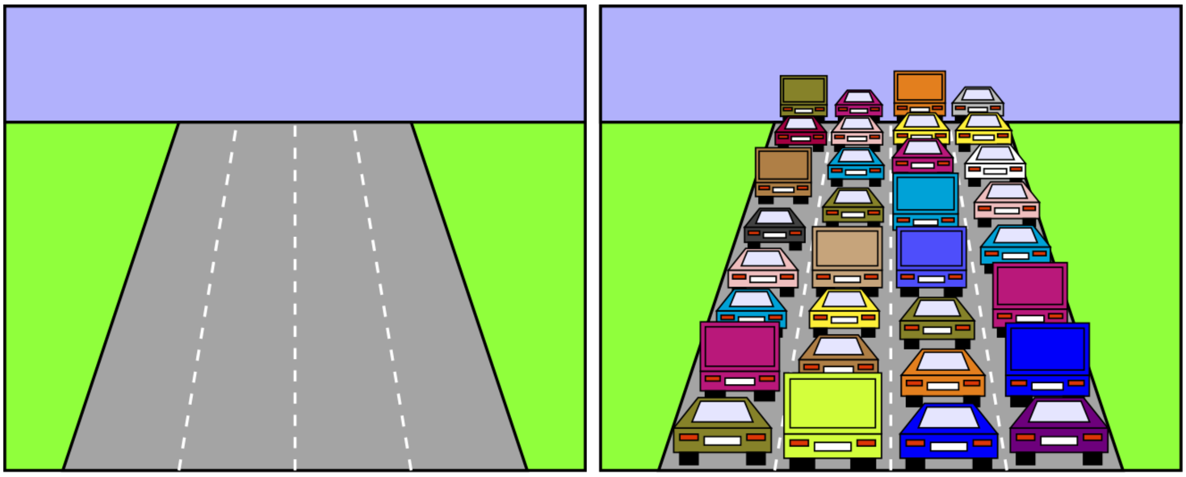
FDM modulator



FDM demodulator



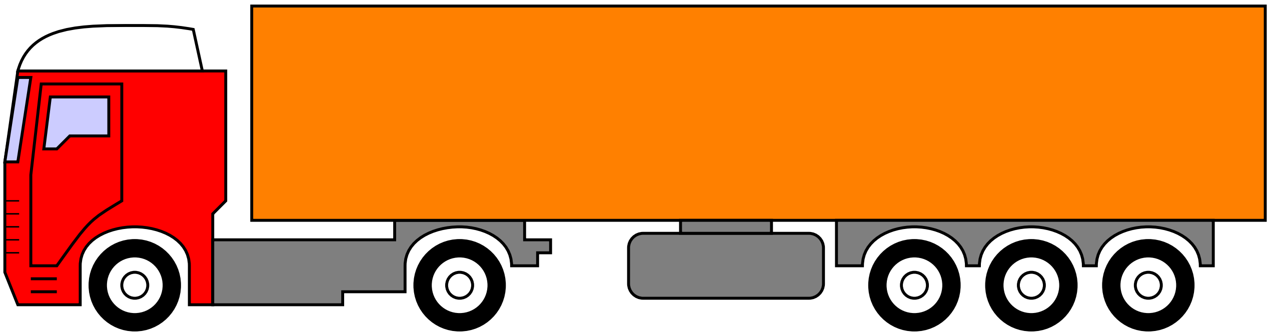
Some kinds of “channels”



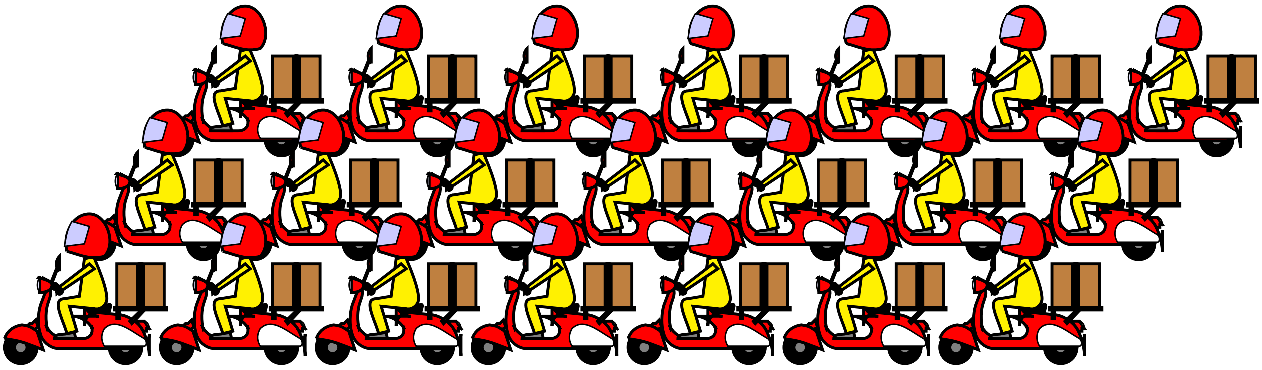
Channel A

Channel B

Alternatives

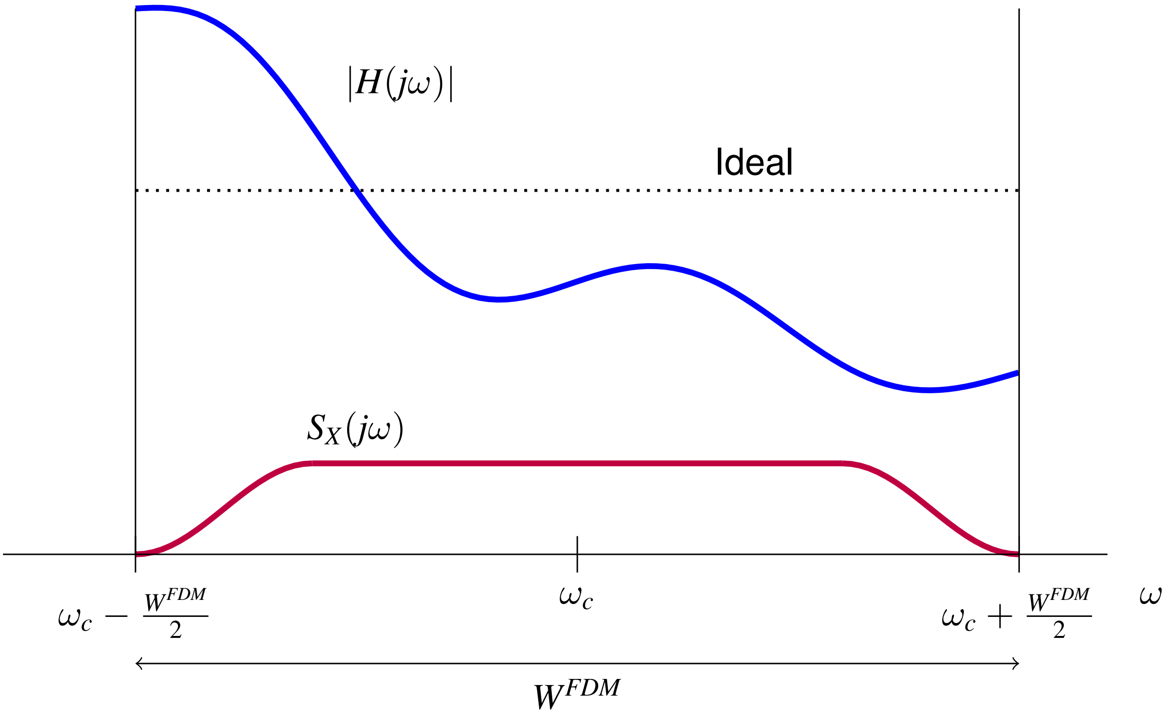


Single Carrier

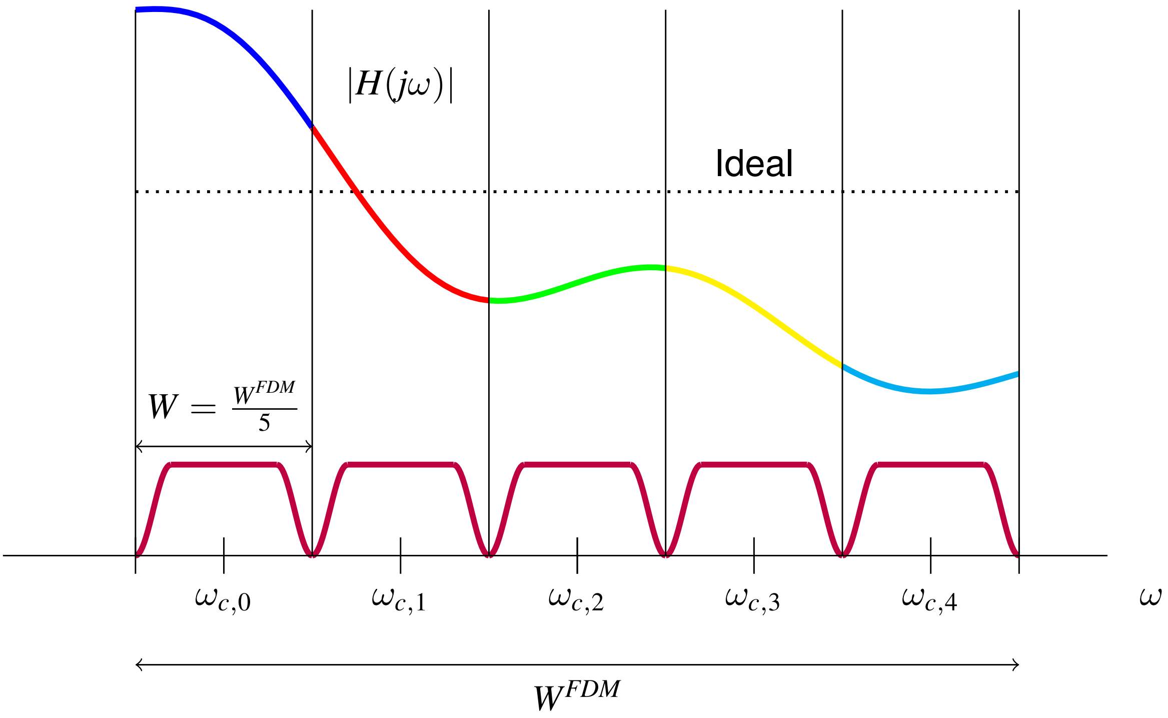


Multi Carrier

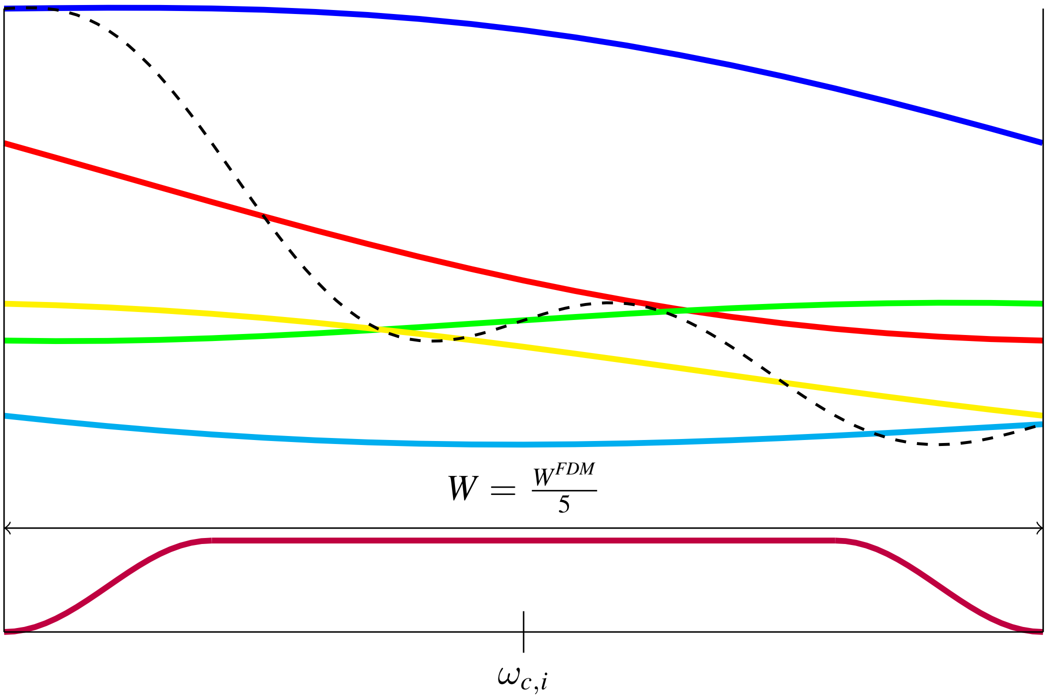
FDM - Channel distortion



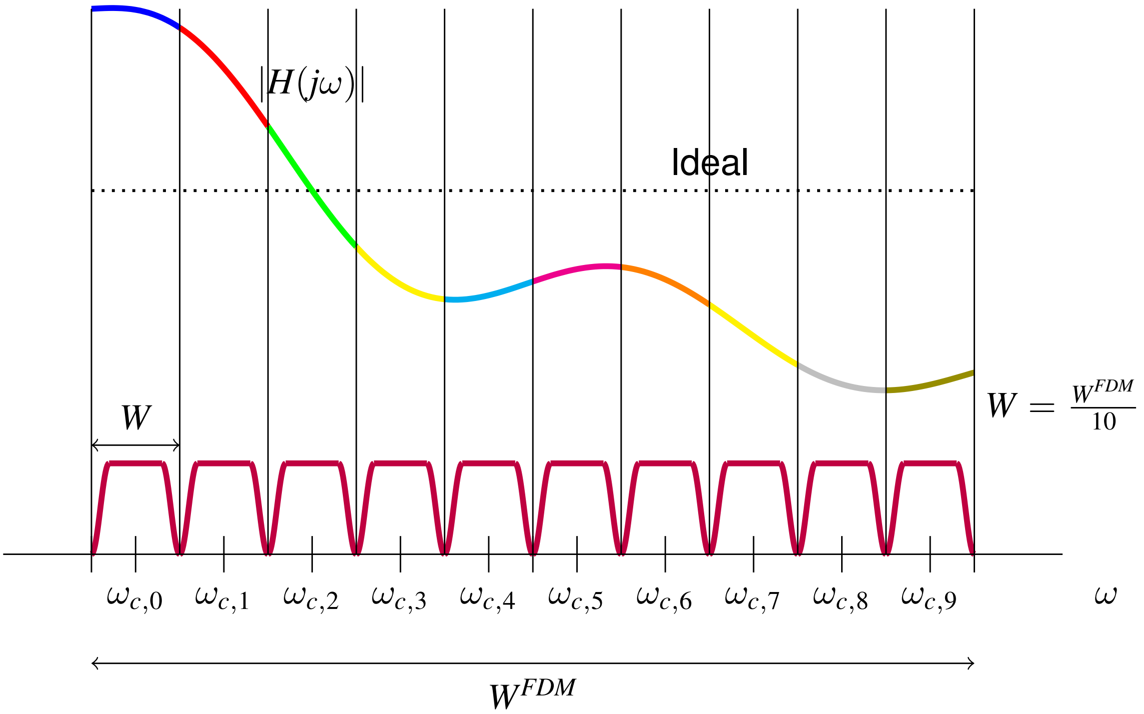
FDM - Channel distortion (II)



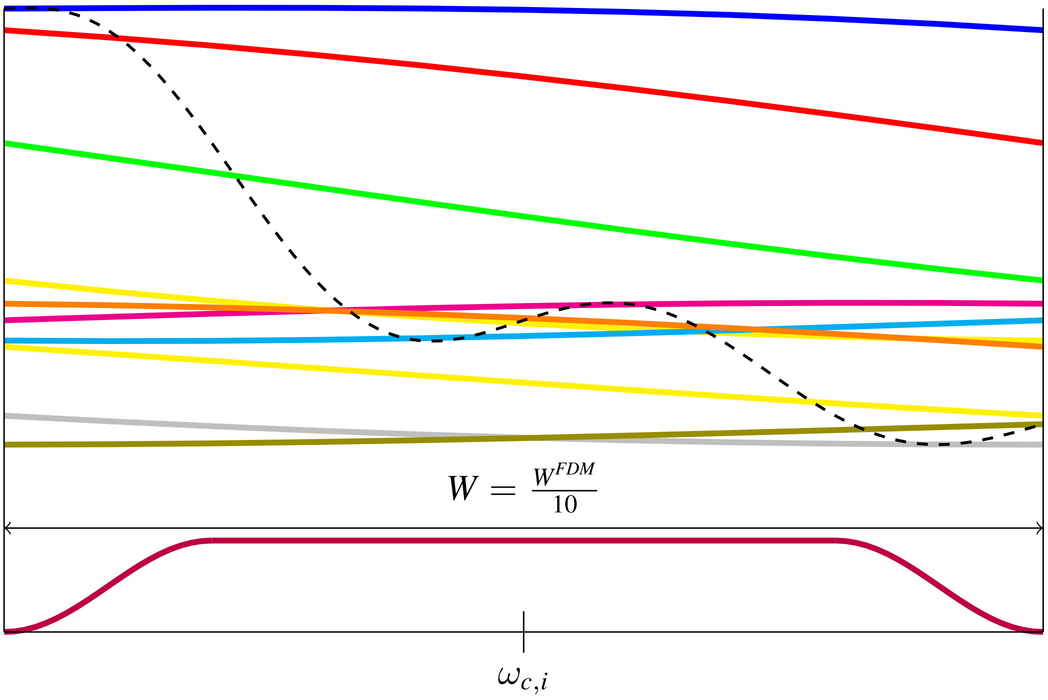
FDM - Channel distortion (III)



FDM - Channel distortion (IV)



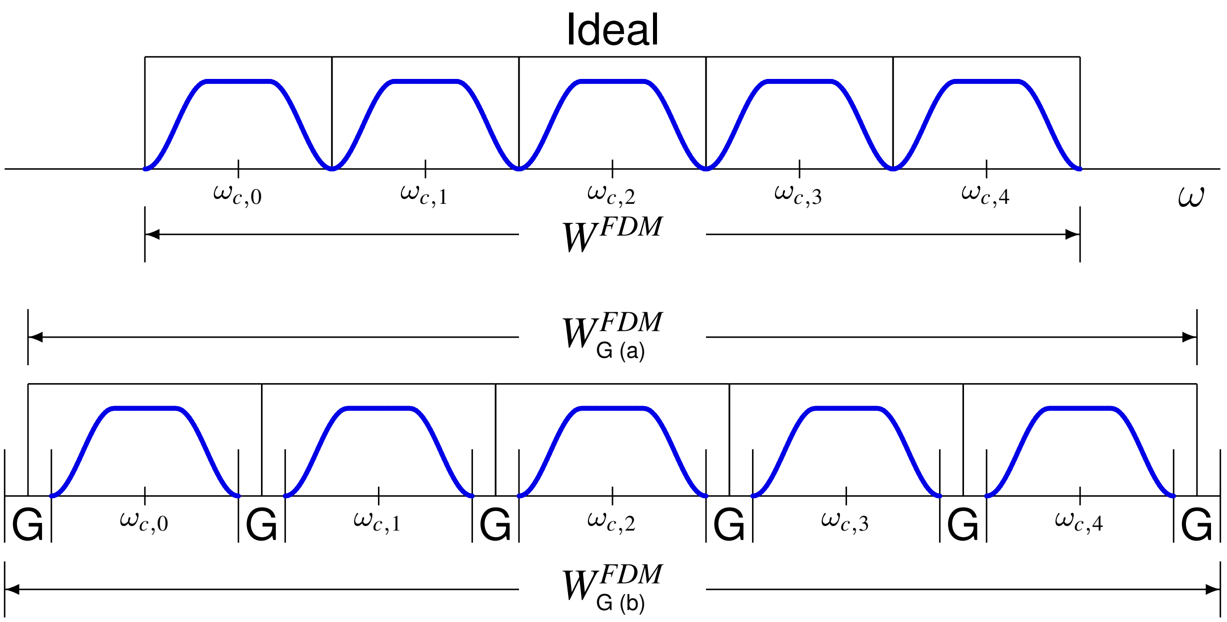
FDM - Channel distortion (V)



Drawbacks of FDM solution

- Hardware complexity
  - N transmission filters (bandpass: in phase and quadrature components)
  - N modulators / demodulators (bandpass)
  - N receiver filters (bandpass)
  - N synchronous samplers (bandpass)
- Ideal filters are required to optimize bandwidth use
  - Without ideal filters, guard intervals must be introduced to separate channels
    - ★ Loss of spectral efficiency
- Alternative solution:
  - Orthogonal FDM modulation (OFDM)
    - ★ N orthogonal pulses (allowing spectral overlapping)
    - ★ Efficient use of available bandwidth
    - ★ Efficient implementation : low hardware complexity

FDM - Guard bands



$$W^{FDM}_{\text{G (a)}} = W^{FDM} + N \times \text{G}$$

$$W^{FDM}_{\text{G (b)}} = W^{FDM} + (N+1) \times \text{G}$$

NOTE: in some systems, guards at both extremes of the band are half size (a)

Continuous time OFDM

- Modulated signal in terms of complex baseband signal

$$\boxed{x(t) = \sqrt{2}\,\mathcal{R}e\{s(t)\,e^{j\omega_c t}\}}$$

Usual notation for bandpass modulated signals

- Complex baseband signal
  - Addition of N signals, one for each data sequence $A_k[n]$

$$\boxed{s(t) = \sum_{k=0}^{N-1} \underbrace{\sum_n A_k[n]\;\phi_k(t-nT)}_{s_k(t)}}$$

Each signal $s_k(t)$ is a PAM signal with transmission filter $\phi_k(t)$

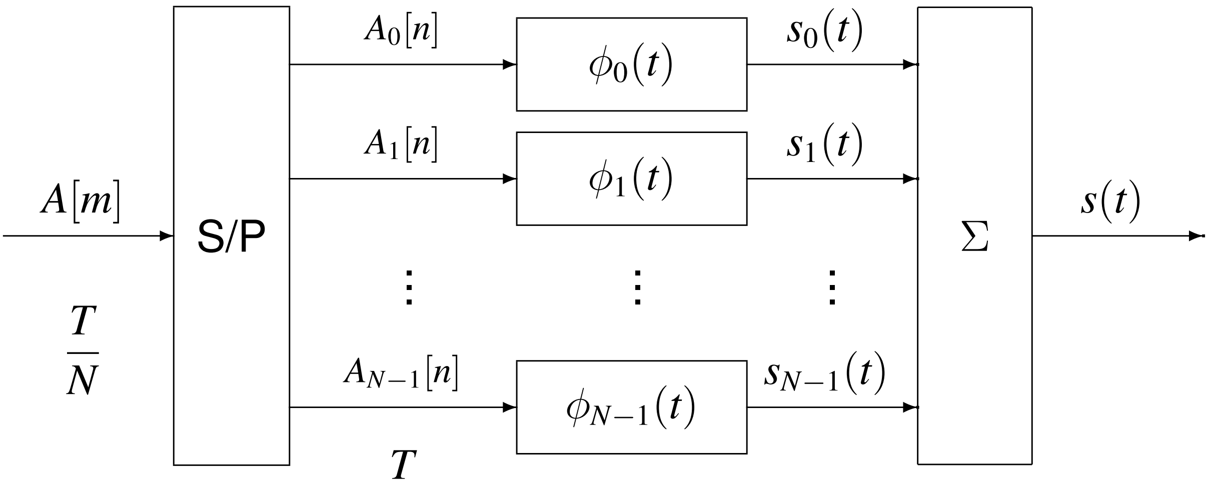
- N transmission filters: prototype filter $\times$ N different carriers
  - Pulse $\phi_k(t)$: k cycles in T sec. of a normalized complex exponential

$$\boxed{\phi_k(t) = \frac{1}{\sqrt{T}}\,w_T(t)\;e^{j\frac{2\pi k}{T}\,t}}$$

$w_T(t)$: continuous time causal window of T seconds $w_T(t) = \begin{cases} 1 & 0 \le t < T \\ 0 & \text{other case} \end{cases}$

OFDM modulator (continuous time)

- Serial to parallel conversion of data sequence $A[m]$
  - Sequences $A_k[n]$, $k \in \{0, 1, \cdots, N-1\}$
  - Symbol lenght for OFDM symbol
    - ★ Symbol time for each sequence $A_k[n]$: T sec.
    - ★ Rate per OFDM carrier: $R_s = \frac{1}{T}$ bauds
  - Total transmission rate
    - ★ Total rate: $R_s^{TOTAL} = N \times R_s$ bauds
    - ★ Total symbol time (for $A[m]$): $T^{TOTAL} = \frac{T}{N}$ sec



Orthonormality of pulses

- OFDM pulses can be seen as an orthonormal basis
  - Inner product

$$\begin{aligned}\langle \phi_k, \phi_\ell \rangle =& \frac{1}{T}\int_0^T e^{j\frac{2\pi k}{T}\,t}\, e^{-j\frac{2\pi \ell}{T}\,t}\, dt = \frac{1}{T}\int_0^T e^{j\frac{2\pi(k-\ell)}{T}\,t}\, dt \\ =& \frac{1}{T}\int_0^T \cos\left(\frac{2\pi(k-\ell)}{T}\,t\right) dt + j\frac{1}{T}\int_0^T \sin\left(\frac{2\pi(k-\ell)}{T}\,t\right) dt \\ =& \begin{cases} 0 & k \neq \ell \\ 1 & k = \ell \end{cases} = \delta[k-\ell]\end{aligned}$$

- Relationship of pulses with prototype filter $\phi_0(t)$

$$\phi_k(t) = \phi_0(t) \times e^{\frac{2\pi k}{T}\,t}$$

$$\phi_k(t-nT) = \phi_0(t-nT)\, e^{\frac{2\pi k}{T}\,(t-nT)} = \phi_0(t-nT)\, e^{\frac{2\pi k}{T}\,t}$$

Spectrum of continuous time OFDM

- Frequency response for the pulses

$$|\Phi_k(j\omega)|^2 = T\ \text{sinc}^2\left(\frac{\left(\omega - \frac{2\pi k}{T}\right)T}{2\pi}\right),\ k = 0,\cdots,N-1$$

- $A_k[n]$ and $A_\ell[n]$ are not correlated and $A_k[n]$ is assumed to be white $\forall k$

$$S_s(j\omega) = \frac{1}{T}\sum_{k=0}^{N-1} E_{s,k}\, |\Phi_k(j\omega)|^2$$

$E_{s,k}$: mean energy per symbol of constellation for $A_k[n]$

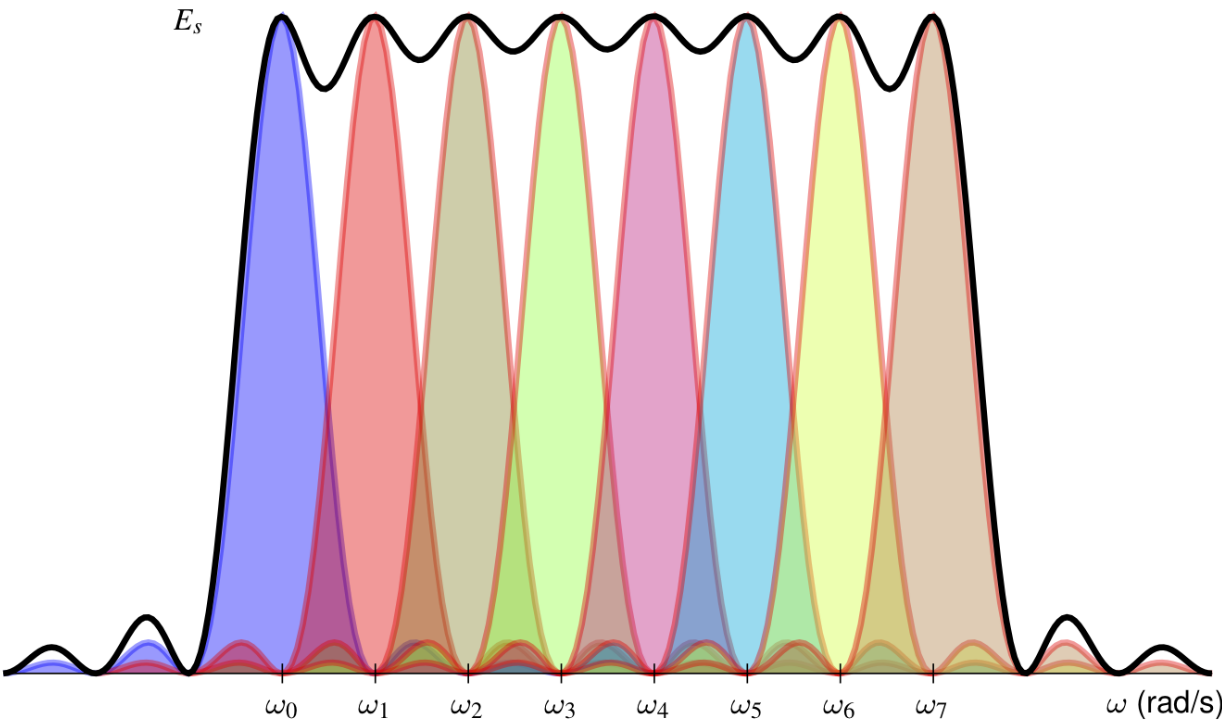
- Power of the transmitted signal

$$P_S = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_s(j\omega)\, d\omega = \frac{1}{T}\sum_{k=0}^{N-1} E_{s,k}\, \frac{1}{2\pi}\int_{-\infty}^{\infty} |\Phi_k(j\omega)|^2 d\omega = \frac{1}{T}\sum_{k=0}^{N-1} E_{s,k}$$

- When constellatios of all N sequences are identical

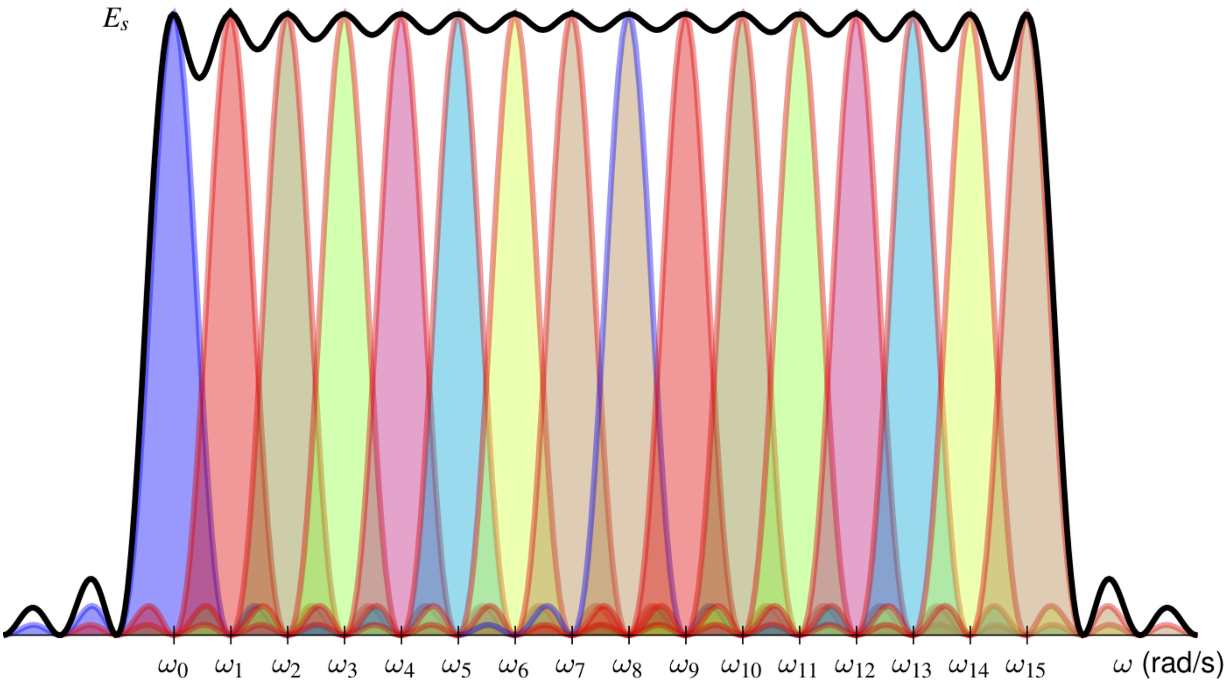
$$P_S = \frac{E_s}{T} \times N = E_s \times R_s \times N \text{ Watts}$$

Spectrum of continuous time OFDM - $N = 8$

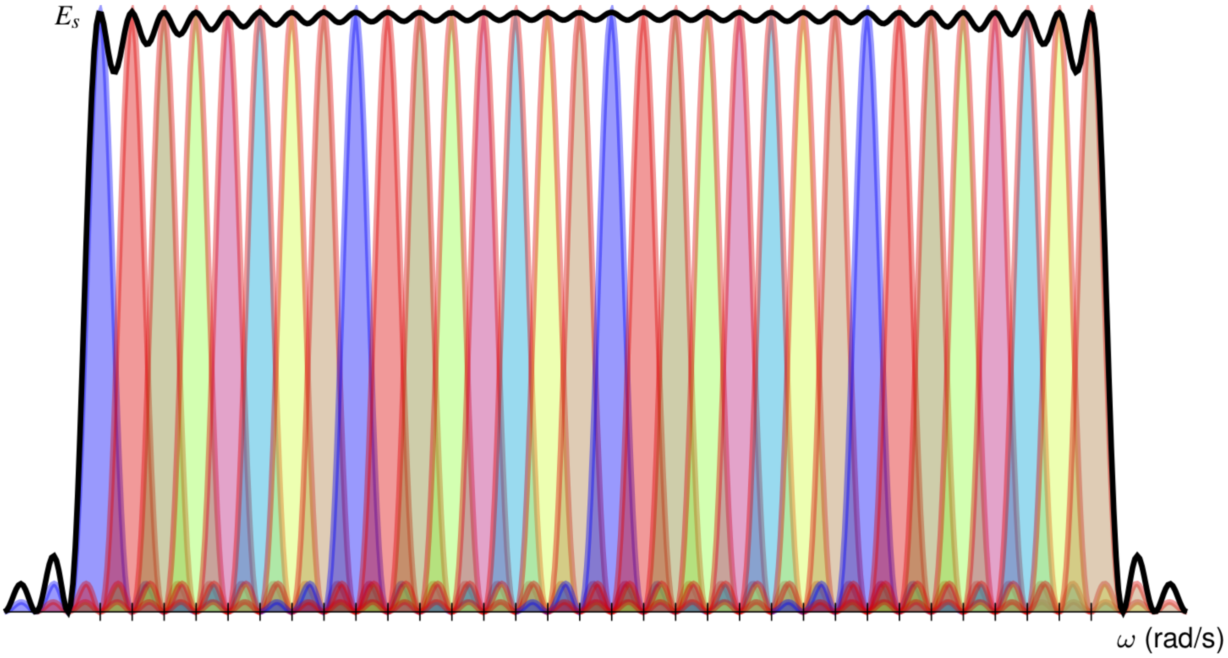


$$\omega_k = \frac{2\pi}{T} k$$

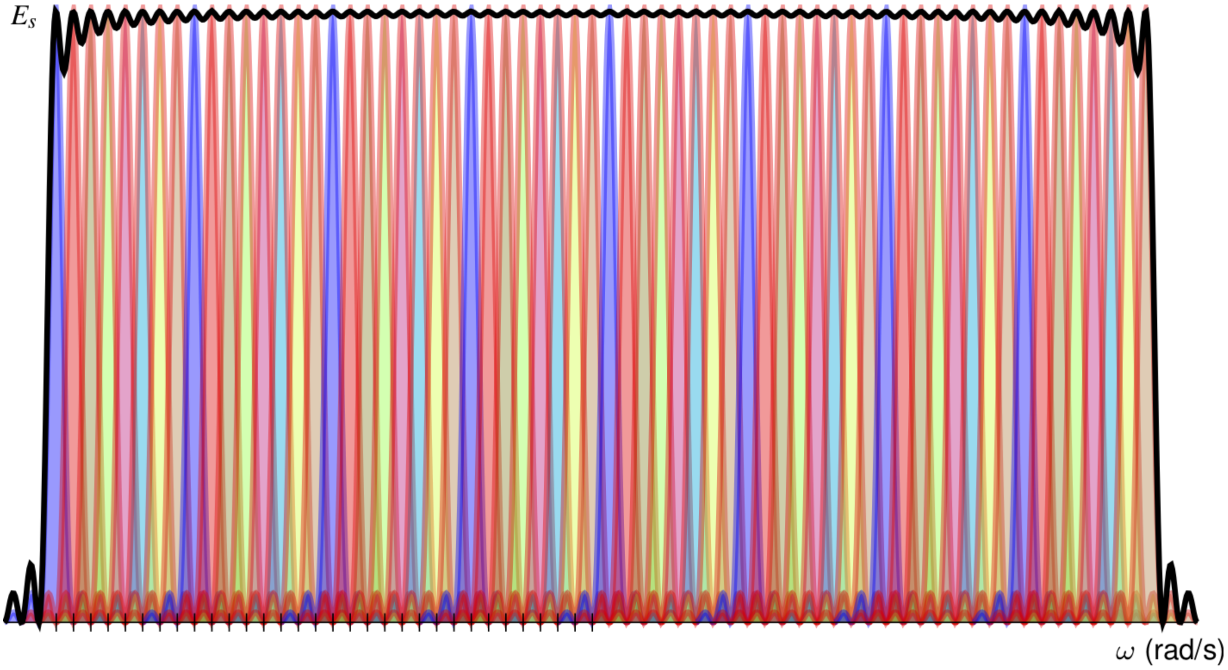
Spectrum of continuous time OFDM - $N = 16$



Spectrum of continuous time OFDM - $N = 32$



Spectrum of continuous time OFDM - $N = 64$



Spectrum is asymptotically flat

- We consider infinite carriers and identical constellations

$$\begin{aligned} S_s(j\omega) =& E_s \sum_{k=-\infty}^{\infty} \operatorname{sinc}^2\left(\frac{\left(\omega-\frac{2\pi k}{T}\right)T}{2\pi}\right) \\ =& E_s \operatorname{sinc}^2\left(\frac{\omega T}{2\pi}\right) * \sum_{k=-\infty}^{\infty} \delta\left(\omega-\frac{2\pi k}{T}\right) \end{aligned}$$

- This PSD is flat if the following condition is fulfilled

$$\frac{E_s}{T}\left(\phi_0(t)*\phi_0(-t)\right) \sum_{k=-\infty}^{\infty} \delta(t-kT) = C\,\delta(t)$$

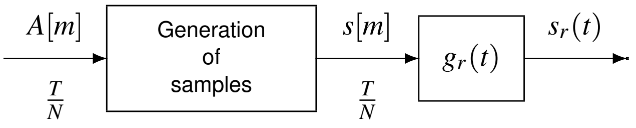
Discrete time OFDM modulation

- Approximation: bandwidth can be considered limited
  - Approximated bandwidth $\boxed{W^{OFDM} \approx \frac{2\pi}{T} \times N \text{ rad/s}}$
- Alternative for signal generation
  - Synthesis of samples of the signal, $s[m]$, at sampling rate given by Nyquist With the assumed approximation this means to sample at T/N s
  - Digital / analog conversion (reconstruction filter at T/N)
- Procedure to generate the signal
  - Computation of samples of the signal (generation by *software*)
    - ★ Dependent on the transmitted data sequences $A_k[n]$

$$\boxed{s[m] = s(t)\big|_{t=m\frac{T}{N}} = s\left(m\frac{T}{N}\right)}$$

  - Recontruction of the signal (D/A conversion)
    - ★ Ideal reconstruction filter (interpolation with sincs at $\frac{T}{N}$)

$$\boxed{g_r(t) = \operatorname{sinc}\left(\frac{N}{T}\,t\right) \quad \overset{\mathcal{FT}}{\leftrightarrow} \quad G_r(j\omega) = \frac{T}{N}\,\Pi\left(\frac{\omega T}{2\pi N}\right)}$$



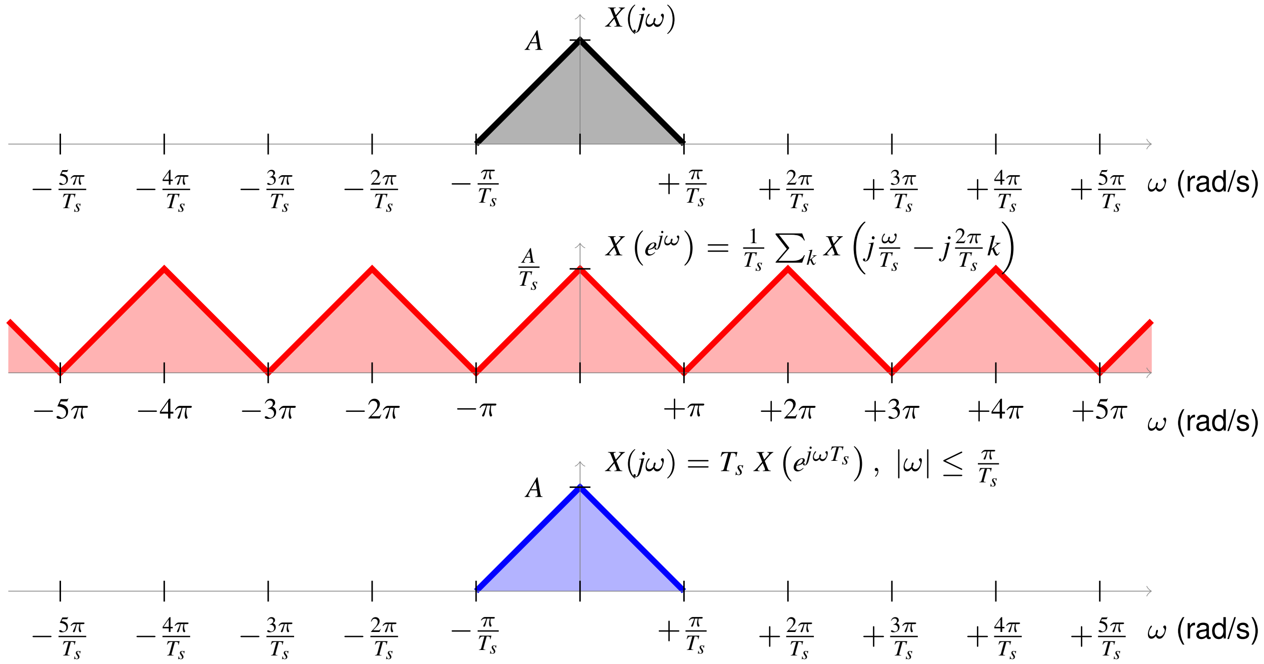
$$s_r(t) = \sum_m s[m]\, g_r\left(t - m\frac{T}{N}\right)$$

REMARK: Reconstructed signal $\boxed{s_r(t) \neq s(t)}$ ($s_r(t)$ is a signal with bandwidth $\frac{2\pi}{T} \times N$ rad/s)

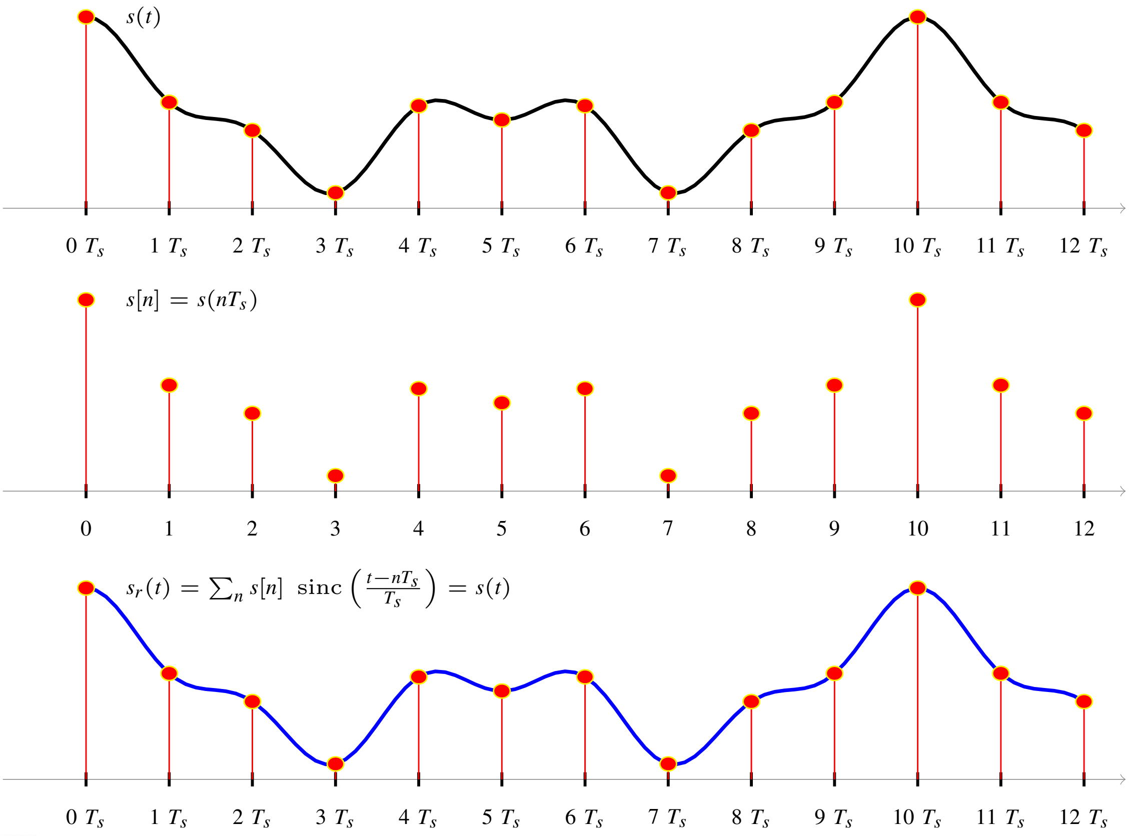
Sampling in the frequency domain satisfying Nyquist

- Sampling of a real baseband signal with bandwidth B Hz ($W = 2\pi B$ rad/s)

$$f_s = \frac{1}{T_s} \geq 2B = \frac{W}{\pi} \text{ samples/s (Nyquist)}$$



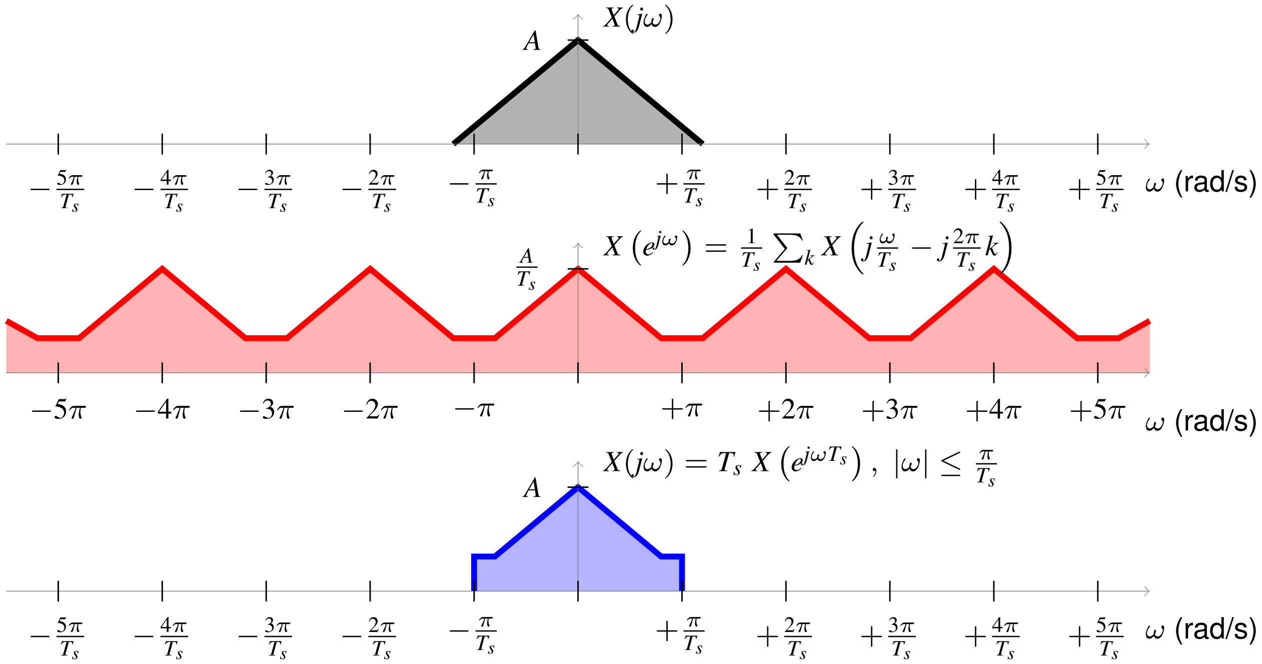
Reconstruction from sampling satisfying Nyquist



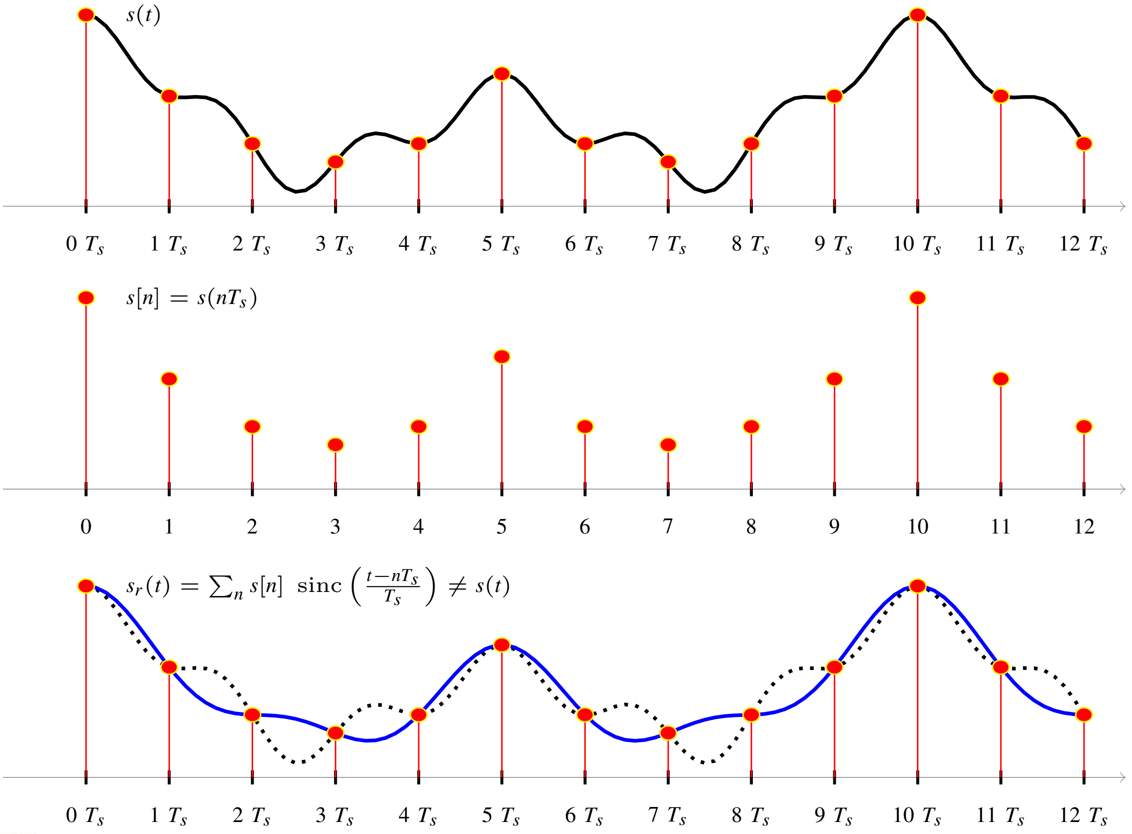
Sampling in the frequency domain with spectral overlapping

- Sampling of a real baseband signal with bandwidth B Hz ($W = 2\pi B$ rad/s)

$$f_s = \frac{1}{T_s} < 2B = \frac{W}{\pi} \text{ samples/s (aliasing)}$$



Reconstruction from sampling with spectral overlapping



Nyquist sampling rates for complex signals

- Signal with bandwidth B Hz ($W = 2\pi B$ rad/s)
- Sampling rate allowing a perfect reconstruction from samples (Nyquist sampling rate)
  - Real baseband signals

$$\boxed{f_s = \frac{1}{T_s} \geq 2B = \frac{W}{\pi} \text{ samples/s}}$$

  - Complex bandpass signals

$$\boxed{f_s = \frac{1}{T_s} \geq B = \frac{W}{2\pi} \text{ samples/s}}$$

Samples of the signal in the first symbol interval

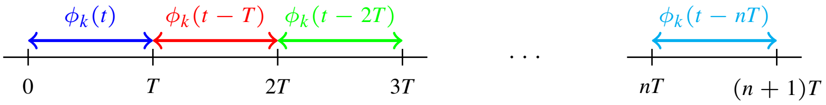
- Analytic expression for samples in interval $0 \leq t < T$
  - First N samples: $m \in \{0, 1, 2, \cdots, N-1\}$: Associated time intants

$$\boxed{t = m\frac{T}{N} \rightarrow \left\{0, \frac{T}{N}, 2\frac{T}{N}, \cdots, (N-1)\frac{T}{N}\right\}}$$

- Continuous time OFDM signal

$$s(t) = \sum_{k=0}^{N-1} \sum_n A_k[n]\, \phi_k(t - nT), \quad \phi_k(t) = \frac{1}{\sqrt{T}}\, w_T(t)\, e^{\frac{2\pi k}{T} t}$$

NOTE: Support for basis functions: in $0 \leq t < T$, $\phi_k(t - nT)$ is only non-null for $n = 0$



- Signal and samples in the first symbol interval

$$s(t) = \sum_{k=0}^{N-1} A_k[0]\, \phi_k(t), \quad s[m] = s\left(m\frac{T}{N}\right) = \sum_{k=0}^{N-1} A_k[0]\, \phi_k\left(m\frac{T}{N}\right)$$

  - Equivalent expression for these samples

$$\boxed{s[m] = \frac{1}{\sqrt{T}} \sum_{k=0}^{N-1} A_k[0]\, e^{j\frac{2\pi k}{T}\, m\frac{T}{N}} = \frac{1}{\sqrt{T}} \sum_{k=0}^{N-1} A_k[0]\, e^{j\frac{2\pi k}{N}\, m}}$$

Samples through the inverse DFT

- DFT and inverse DFT (IDFT) for sequences of N samples
  For $m \in \{0, 1, \cdots, N-1\}$ y $k \in \{0, 1, \cdots, N-1\}$

$$\boxed{X[k] = \text{DFT}_N\{x[m]\}[k] = \sum_{m=0}^{N-1} x[m]\ e^{-j\frac{2\pi k}{N}\ m}}$$

$$\boxed{x[m] = \text{IDFT}_N\{X[k]\}[m] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\ e^{j\frac{2\pi k}{N}\ m}}$$

- Samples of the OFDM signal (in the first symbol interval)

$$\boxed{s[m] = \frac{1}{\sqrt{T}}\sum_{k=0}^{N-1} A_k[0]\ e^{j\frac{2\pi k}{N}\ m}}$$

- Identification of terms in the IDFT

$$X[k] \equiv A_k[0],\ \text{different scaling factors } \frac{1}{N} \text{ vs } \frac{1}{\sqrt{T}}$$

Therefore

$$\boxed{\{s[m]\}_{m=0}^{N-1} = \frac{N}{\sqrt{T}} \times \text{IDFT}_N\ \{A_k[0]\}_{k=0}^{N-1}}$$

General expression for samples of the OFDM signal

- Samples of the OFDM signal

$$\begin{aligned} s[m] &= \sum_{n}\sum_{k=0}^{N-1} A_k[n]\ \phi_k(mT/N - nT) \\ &= \frac{1}{\sqrt{T}}\sum_{n}\sum_{k=0}^{N-1} A_k[n]\ e^{j\frac{2\pi k}{N}\ (m-nN)}\ w_N[m-nN] \end{aligned}$$

$w_N[m]$: discrete-time causal window of N samples $w_N[m] = \begin{cases} 1 & 0 \leq m \leq N-1 \\ 0 & \text{other case} \end{cases}$

  - Generation of sequence $s[m]$ as a process for blocks of N samples

$$\boxed{\frac{N}{\sqrt{T}} \times \text{IDFT}_N\ (\{A_0[n], A_1[n], \cdots, A_{N-1}[n]\}) \rightarrow \{s[nN], s[nN+1], \cdots, s[(n+1)N-1]\}}$$

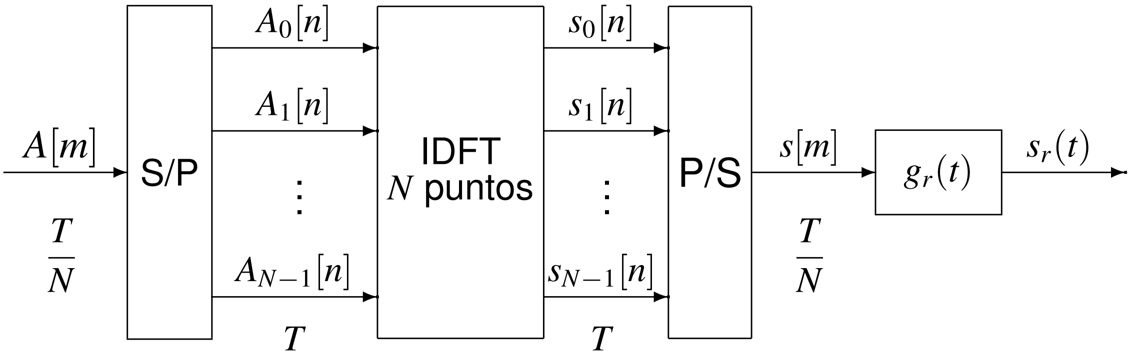
Notation: block of index n: $\boxed{s^{(n)}[m] = s[nN+m]}$

$$\boxed{\frac{N}{\sqrt{T}} \times \text{IDFT}_N\left(\{A_k[n]\}_{k=0}^{N-1}\right) \rightarrow \left\{s^{(n)}[m]\right\}_{m=0}^{N-1}}$$

Modulator for discrete-time OFDM

Generation of samples

$$\left\{ \begin{array}{ccccc} \underbrace{A_0[0], A_1[0], \cdots, A_{N-1}[0]} & \underbrace{A_0[1], A_1[1], \cdots, A_{N-1}[1]} & \cdots & \underbrace{A_0[n], A_1[n], \cdots, A_{N-1}[n]} & \cdots \\ A_k[0] & A_k[1] & & A_k[n] & \\ \downarrow & \downarrow & \cdots & \downarrow & \cdots \\ \frac{N}{\sqrt{T}} \times \text{IDFT}_N & \frac{N}{\sqrt{T}} \times \text{IDFT}_N & \cdots & \frac{N}{\sqrt{T}} \times \text{IDFT}_N & \cdots \\ \downarrow & \downarrow & \cdots & \downarrow & \cdots \\ s^{(0)}[m] & s^{(1)}[m] & & s^{(n)}[m] & \\ \overbrace{s[0], s[1], \cdots, s[N-1]} & \overbrace{s[N], s[N+1], \cdots, s[2N-1]} & \cdots & \overbrace{s[nN], s[nN+1], \cdots, s[nN+N-1]} & \cdots \end{array} \right\}$$



$$s_r(t) = \sum_m s[m]\, g_r(t - mT/N)$$

Example discrete OFDM synthesis: $N = 4$

- Data sequence

m	0	1	2	3	4	5	6	7	8	9	10	11
$A[m]$	A	B	C	D	E	F	G	H	I	J	K	L

- Serial to Parallel conversion

n	0	1	2
$A_0[n]$	A	E	I
$A_1[n]$	B	F	J
$A_2[n]$	C	G	K
$A_3[n]$	D	H	L

- OFDM samples

$$n = 0 : s^{(0)}[m] = \{a, b, c, d\} = \frac{N}{\sqrt{T}}\text{IDFT}_4\{A_0[0], A_1[0], A_2[0], A_3[0]\} = \frac{N}{\sqrt{T}}\text{IDFT}_4\{A, B, C, D\}$$

$$n = 1 : s^{(1)}[m] = \{e, f, g, h\} = \frac{N}{\sqrt{T}}\text{IDFT}_4\{A_0[1], A_1[1], A_2[1], A_3[1]\} = \frac{N}{\sqrt{T}}\text{IDFT}_4\{E, F, G, H\}$$

$$n = 2 : s^{(1)}[m] = \{i, j, k, \ell\} = \frac{N}{\sqrt{T}}\text{IDFT}_4\{A_0[2], A_1[2], A_2[2], A_3[2]\} = \frac{N}{\sqrt{T}}\text{IDFT}_4\{I, J, K, L\}$$

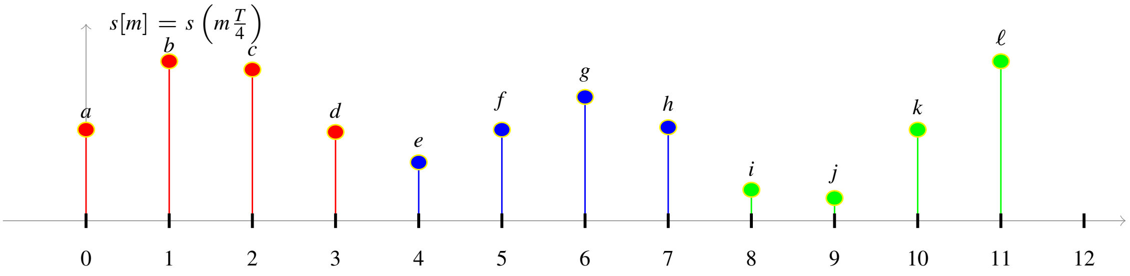
m	0	1	2	3	4	5	6	7	8	9	10	11
$s[m]$	a	b	c	d	e	f	g	h	i	j	k	l

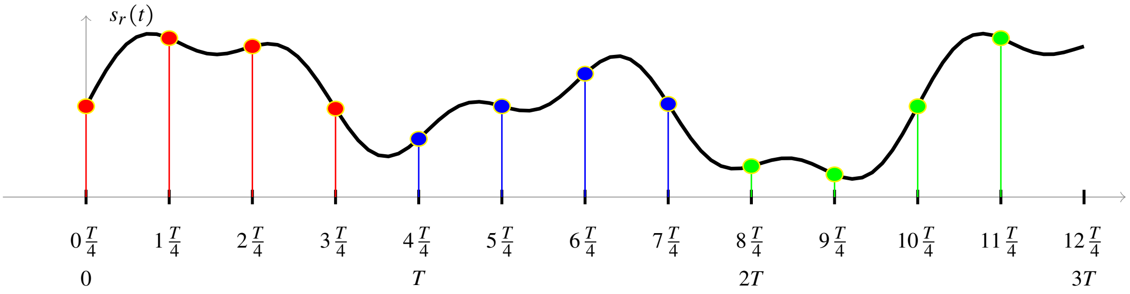
Example discrete time OFDM synthesis: $N=4$

$$n=0: s^{(0)}[m]=\{a,b,c,d\}=\frac{N}{\sqrt{T}}\text{IDFT}_4\{A_0[0],A_1[0],A_2[0],A_3[0]\}=\frac{N}{\sqrt{T}}\text{IDFT}_4\{A,B,C,D\}$$

$$n=1: s^{(1)}[m]=\{e,f,g,h\}=\frac{N}{\sqrt{T}}\text{IDFT}_4\{A_0[1],A_1[1],A_2[1],A_3[1]\}=\frac{N}{\sqrt{T}}\text{IDFT}_4\{E,F,G,H\}$$

$$n=2: s^{(2)}[m]=\{i,j,k,\ell\}=\frac{N}{\sqrt{T}}\text{IDFT}_4\{A_0[2],A_1[2],A_2[2],A_3[2]\}=\frac{N}{\sqrt{T}}\text{IDFT}_4\{I,J,K,L\}$$





Discrete time orthonormal basis

- Discrete time basis functions

$$\boxed{\xi_k[m]=\frac{1}{\sqrt{N}}\,w_N[m]\,e^{j\frac{2\pi k}{N}\,m},\qquad k=0,1,\cdots,N-1}$$

- Orthonormal basis

$$\boxed{\langle\xi_k,\xi_\ell\rangle=\sum_m \xi_k[m]\,\xi_\ell^*[m]=\frac{1}{N}\sum_{m=0}^{N-1}e^{j\frac{2\pi(k-\ell)}{N}\,m}=\delta[k-\ell]}$$

- Signal samples: expansion in the orthonormal basis

$$\boxed{s[m]=\sqrt{\frac{N}{T}}\sum_n\sum_{k=0}^{N-1}A_k[n]\,\xi_k[m-nN]}$$

Equivalent continuous time orthonormal basis

- Reconstructed signal is

$$s_r(t) = \sum_m s[m]\; g_r(t - mT/N)$$

$$\boxed{s_r(t) = \sqrt{\frac{N}{T}} \sum_n \sum_{k=0}^{N-1} A_k[n] \; \sum_m \xi_k[m - nN]\; g_r(t - mT/N)}$$

$$\boxed{s(t) = \sum_n \sum_{k=0}^{N-1} A_k[n]\; \phi_k(t - nT)} \qquad \boxed{s_r(t) = \sum_n \sum_{k=0}^{N-1} A_k[n]\; \hat{\phi}_k(t - nT)}$$

- Equivalent continuous time basis functions

$$\boxed{\hat{\phi}_k(t) = \sqrt{\frac{N}{T}} \sum_m \xi_k[m]\; g_r(t - mT/N) = \frac{1}{\sqrt{T}} \sum_{m=0}^{N-1} e^{j\frac{2\pi k}{N}\, m} \operatorname{sinc}\left(\frac{N}{T}\left(t - m\frac{T}{N}\right)\right)}$$

Orthonormality of equivalent basis functions

$$\begin{aligned}\langle \hat{\phi}_k, \hat{\phi}_\ell \rangle &= \int_{-\infty}^{\infty} \hat{\phi}_k(t)\; \hat{\phi}_\ell^*(t) dt \\ &= \frac{1}{T} \sum_{m=0}^{N-1} \sum_{i=0}^{N-1} e^{j\frac{2\pi k}{N}\, m} e^{-j\frac{2\pi \ell}{N}\, i} \int_{-\infty}^{\infty} g_r(t - mT/N)\; g_r(t - iT/N) dt\end{aligned}$$

$$\int_{-\infty}^{\infty} g_r(\tau - mT/N)\; g_r(\tau - iT/N) d\tau = (g_r(t) * g_r(-t))|_{t=(m-i)T/N}$$

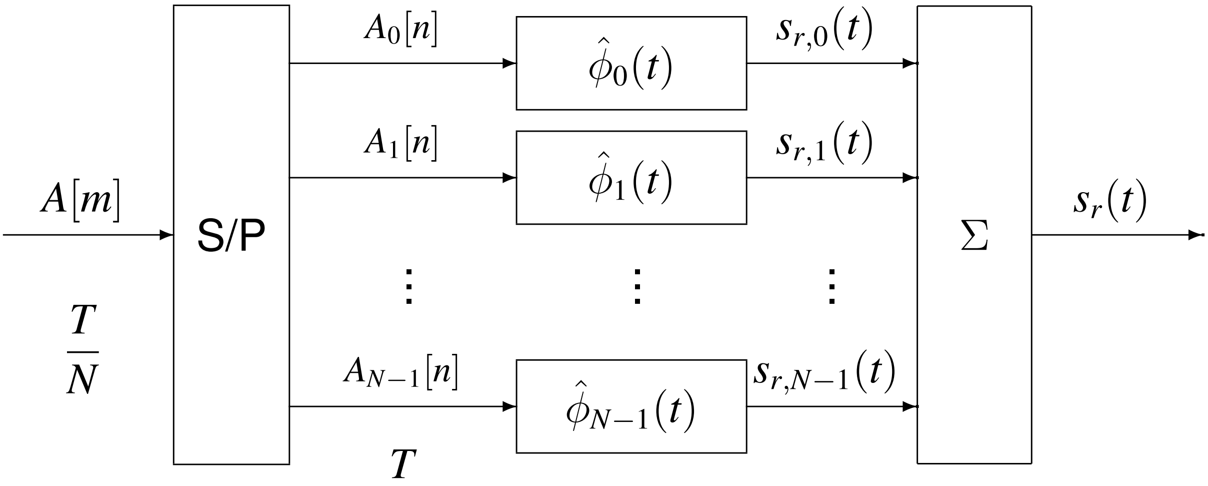
Since $g(t)$ fulfills Nyquist criterion for ISI

$$\int_{-\infty}^{\infty} g_r(\tau - mT/N)\; g_r(\tau - iT/N) d\tau = \frac{T}{N}\, \delta[m - i]$$

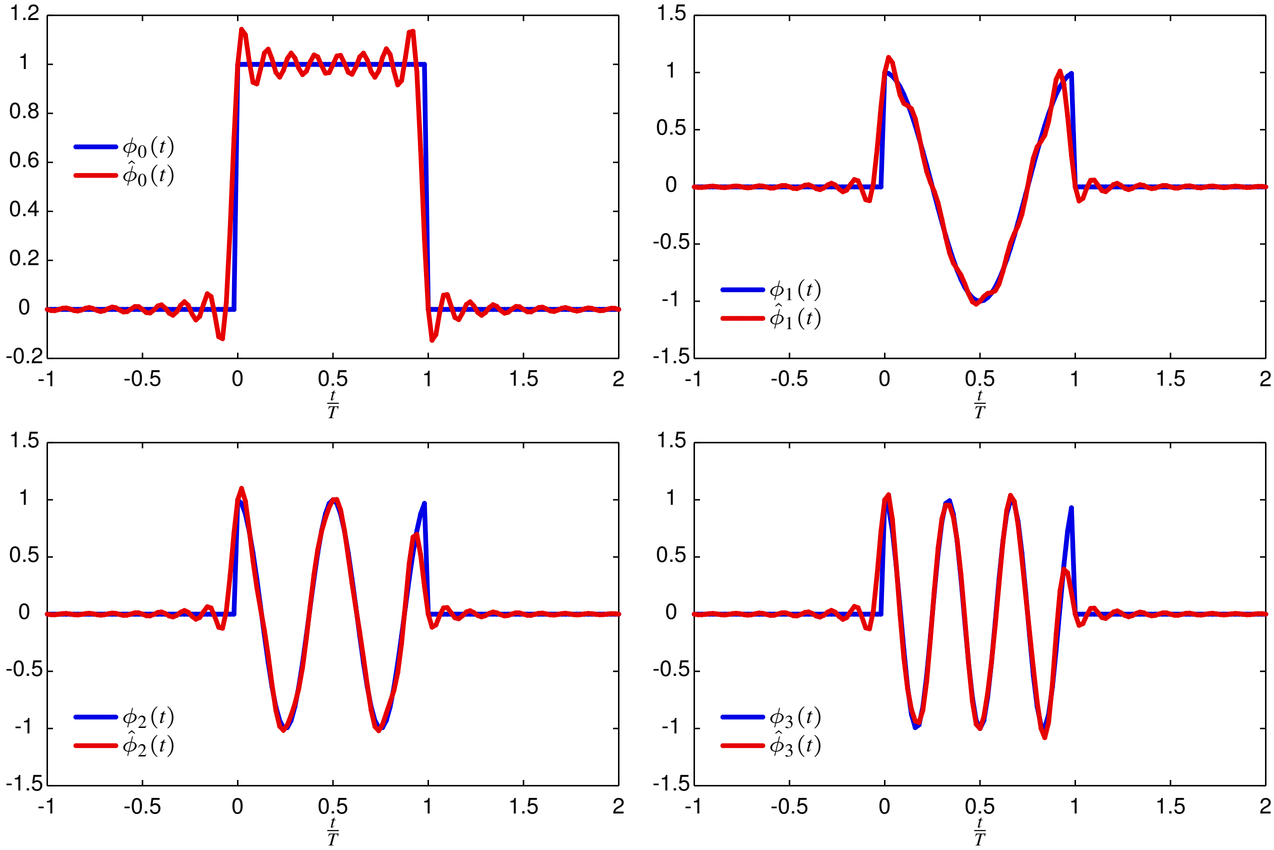
$$\boxed{\langle \hat{\phi}_k, \hat{\phi}_\ell \rangle = \frac{1}{N} \sum_{m=0}^{N-1} e^{j\frac{2\pi (k-\ell)}{N}\, m} = \delta[k - \ell]}$$

Discrete time OFDM modulator (equivalent)

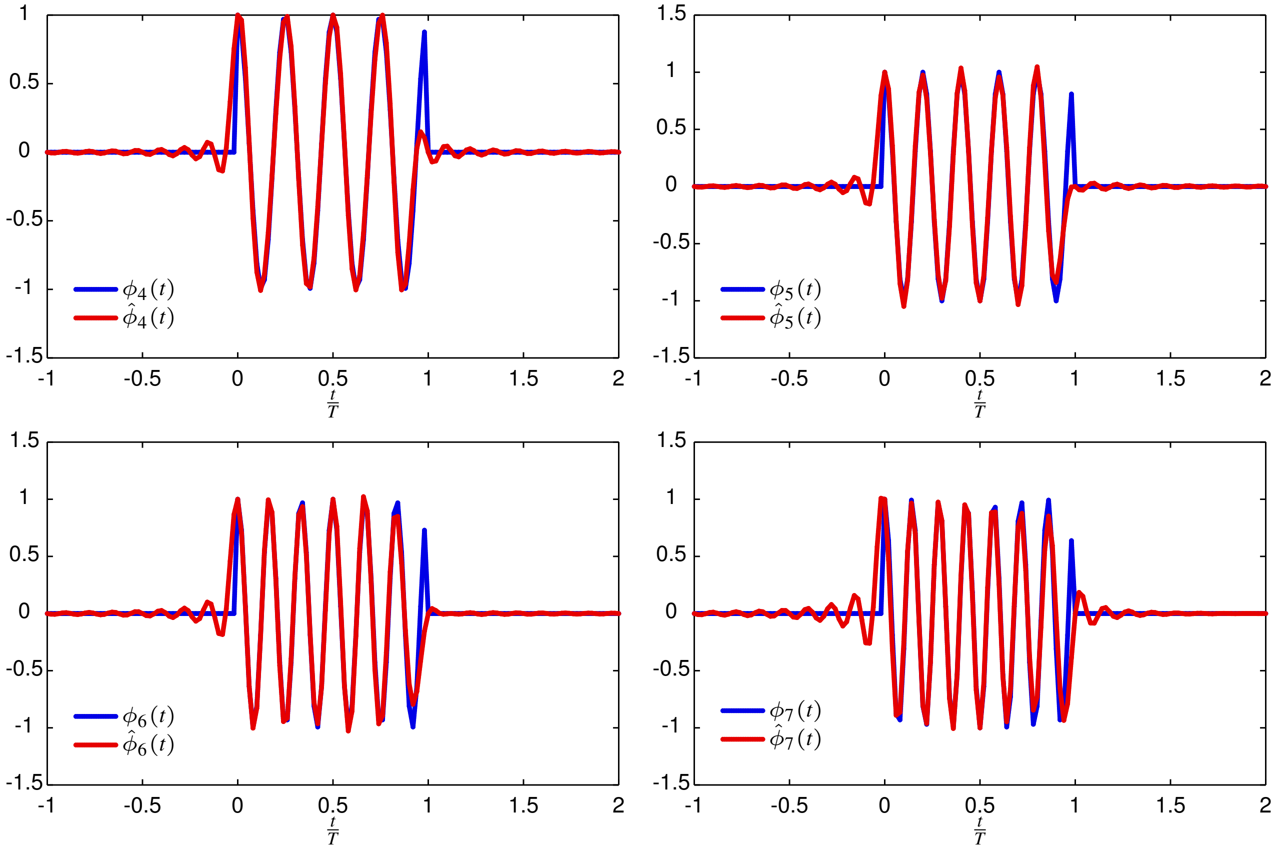
- Structure of modulator as a function of the equivalent basis
  - Conceptual scheme
  - It is not used for synthesis (implementation), but for analysis



Comparison with continuous time basis functions - Real part



Comparison with continuous time basis functions - Real part



Spectrum of discrete time OFDM

- Power spectral density

$$\boxed{S_{sr}(j\omega) = \frac{1}{T}\sum_{k=0}^{N-1} E_{s,k}\left|\hat{\Phi}_k(j\omega)\right|^2}$$

- Frequency response of discrete time basis functions

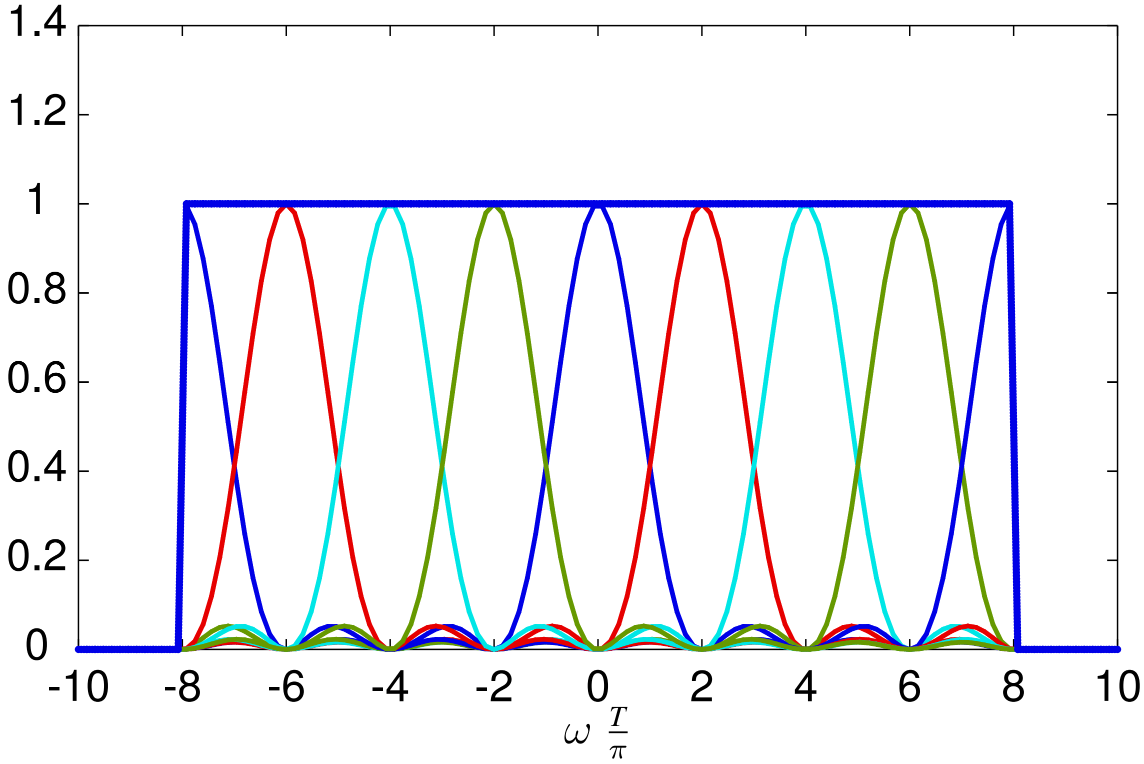
$$\boxed{\left|\Xi_k\left(e^{j\omega}\right)\right|^2 = \frac{1}{N}\frac{\sin^2[(\omega-2\pi k/N)N/2]}{\sin^2[(\omega-2\pi k/N)/2]}}$$

- Frequency response of equivalent continuous time basis functions

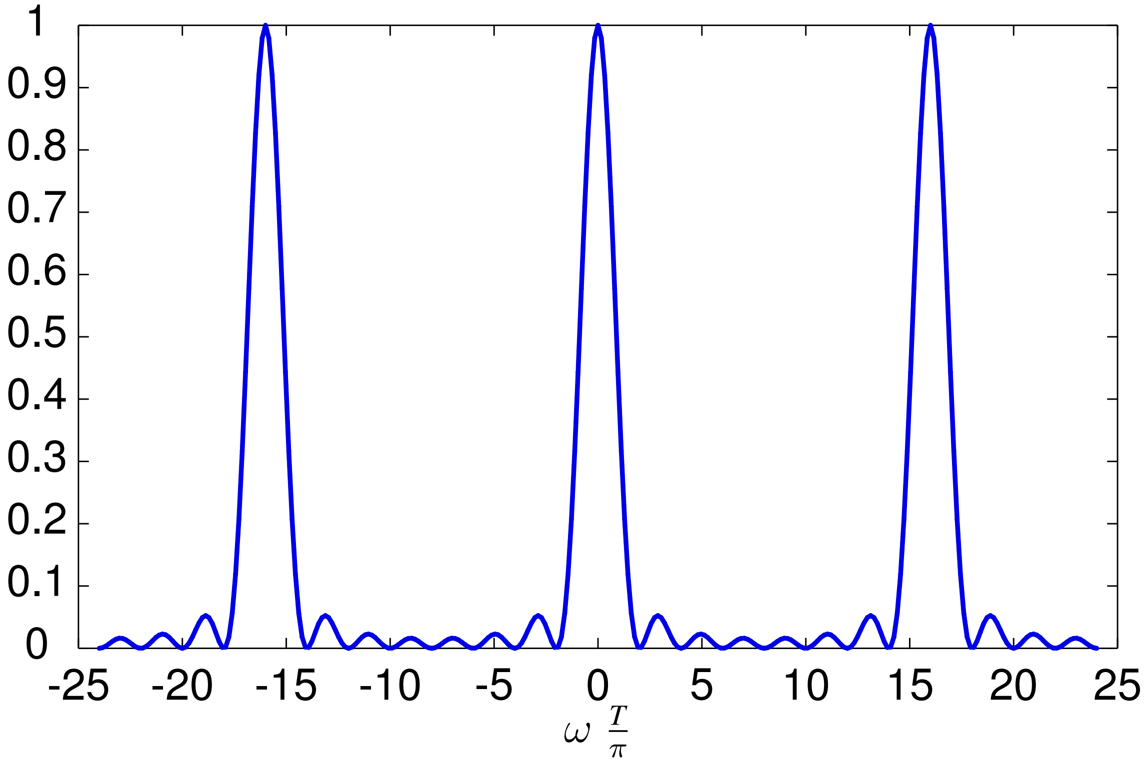
$$\boxed{\left|\hat{\Phi}_k(j\omega)\right|^2 = \frac{N}{T}\left|\Xi_k\left(e^{j\omega\frac{T}{N}}\right)\right|^2\left(\frac{T}{N}\right)^2\Pi\left(\frac{\omega T}{2\pi N}\right)}$$

$$\boxed{\left|\hat{\Phi}_k(j\omega)\right|^2 = \frac{T}{N^2}\frac{\sin^2[(\omega-2\pi k/T)T/2]}{\sin^2[(\omega-2\pi k/T)T/2N]}, \quad |\omega| < \frac{\pi}{T}N}$$

Spectrum of discrete time OFDM - $N = 8$

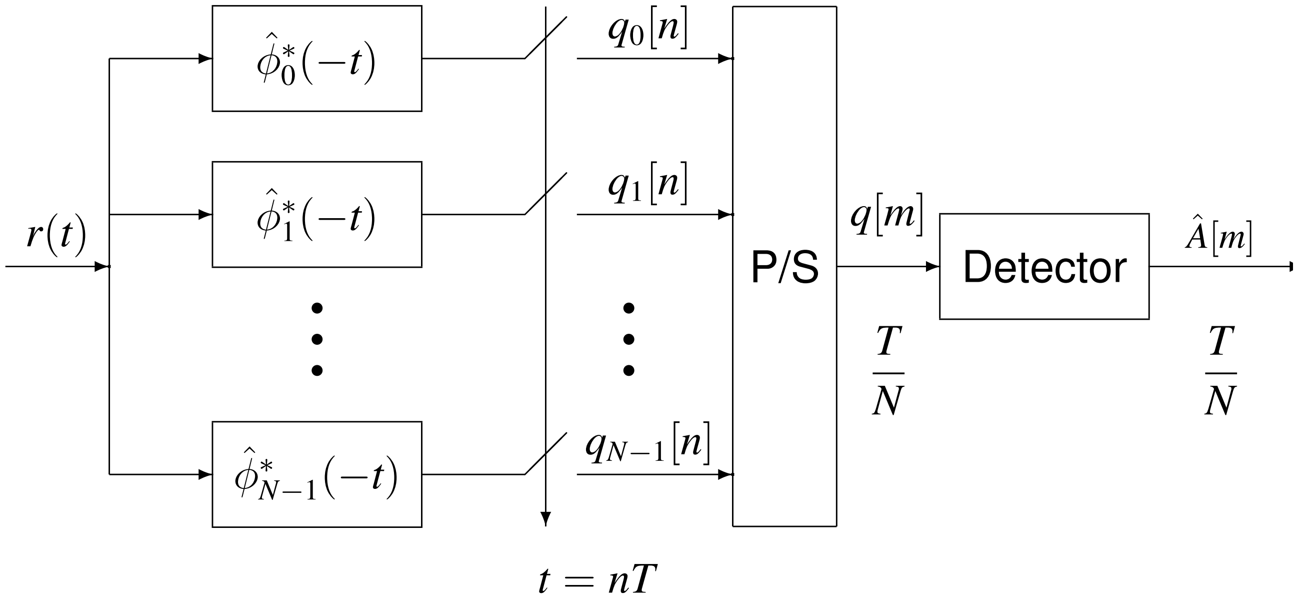


Spectrum of discrete time OFDM - Periodicity of $\left|\Xi_k\left(e^{j\omega T}\right)\right|^2$



Receiver for discrete-time OFDM

- Structure for the receiver considering the equivalent basis (bank of N matched filters)
  - Conceptual scheme
  - It is not used for synthesis (implementation), but for analysis



Noise in the receiver

- Receiver filter for carrier of index k: $\sqrt{2}\, f_k(t)$
- Power spectral density for noise sequence in this carrier, $z_k[n]$

$$S_{z_k}(e^{j\omega}) = \frac{2}{T} \sum_i S_n\left(j\frac{\omega}{T} - j\frac{\omega_c}{T} - j\frac{2\pi i}{T}\right) \left|F_k\left(j\frac{\omega}{T} - j\frac{2\pi i}{T}\right)\right|^2$$

- Matched filter at receiver: $f_k(t) = \hat{\phi}_k^*(-t)$
  - Normalized, $r_{f_k}(t)$ satisfies Nyquist
- $n(t)$: white, Gaussian, stationnary $S_n(j\omega) = N_0/2$
  - $z_k[n]$ white, Gaussian, circularly symmetric

$$\sigma_{z_k}^2 = N_0,\ k = 0, \cdots, N-1$$

  - A consequence of the orthogonality of pulses for each carrier

$$E\{z_i[n]\ z_k^*[n]\} = 0,\ \text{if } i \neq k$$

Receiver for discrete-time OFDM

- Matched filters

$$\hat{\phi}_k^*(-t) = \sqrt{\frac{N}{T}} \sum_m \xi_k^*[m]\, g_r(-t - mT/N), \text{ with } \xi_k[m] = \frac{1}{\sqrt{N}}\, e^{j\frac{2\pi k}{N}\, m}\, w_N[m]$$

- Analytic expression for the output of the demodulators

$$\begin{aligned} q_k[n] &= \left(r(t) * \hat{\phi}_k^*(-t)\right)\Big|_{t=nT} \\ &= \sqrt{\frac{N}{T}} \sum_m \xi_k^*[m] \left(r(t) * g_r\left(-t - m\frac{T}{N}\right)\right)\Big|_{t=nT} \\ &= \sqrt{\frac{1}{T}} \sum_{m=0}^{N-1} e^{-j\frac{2\pi k}{N}\, m} \left(r(t) * g_r(-t)\right)\big|_{t=nT+m\frac{T}{N}} \end{aligned}$$

- Output of the matched filter $g_r(-t)$ is defined as $v(t) = r(t) * g(-t)$
- Definition of sequence $v[m] = v(t)\big|_{t=m\frac{T}{N}} = r(t) * g_r(-t)\big|_{t=m\frac{T}{N}}$, leads to

$$q_k[n] = \frac{1}{\sqrt{T}} \sum_{m=0}^{N-1} e^{-j\frac{2\pi k}{N}\, m}\, v[nN + m]$$

Observations $q_k[n]$ through a DFT

- DFT and inverse DFT (IDFT) for sequences of N samples
  For $m \in \{0, 1, \cdots, N-1\}$ and $k \in \{0, 1, \cdots, N-1\}$

$$X[k] = \text{DFT}_N\{x[m]\}[k] = \sum_{m=0}^{N-1} x[m]\, e^{-j\frac{2\pi k}{N}\, m}$$

$$x[m] = \text{IDFT}_N\{X[k]\}[m] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi k}{N}\, m}$$

- Observations $q_k[n]$

$$q_k[n] = \frac{1}{\sqrt{T}} \sum_{m=0}^{N-1} e^{-j\frac{2\pi k}{N}\, m}\, v[nN + m]$$

- Identification of terms in the DFT

$$x[m] \equiv v[nN + m] = v^{(n)}[m], \text{ scaling factor } \frac{1}{\sqrt{T}}$$

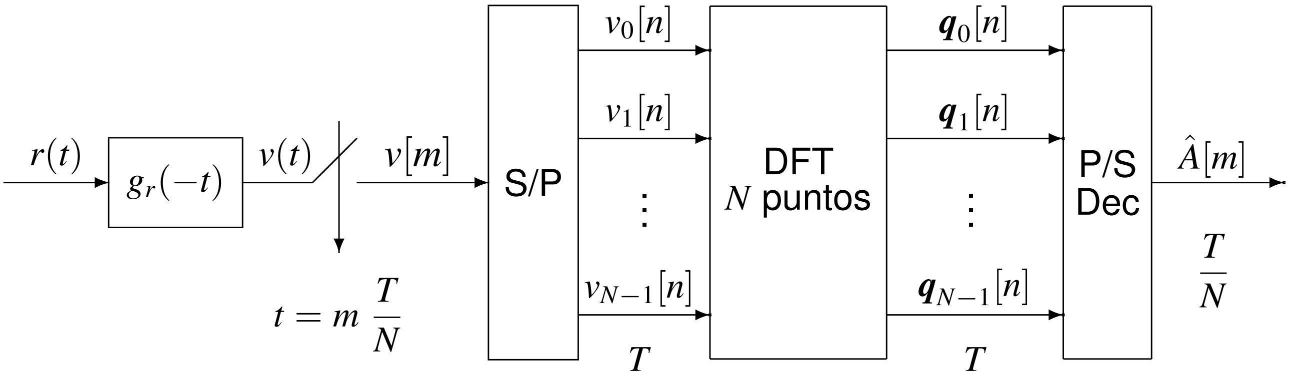
  Therefore

$$\{q_k[n]\}_{k=0}^{N-1} = \frac{1}{\sqrt{T}} \times \text{DFT}_N \left\{ v^{(n)}[m] \right\}_{m=0}^{N-1}$$

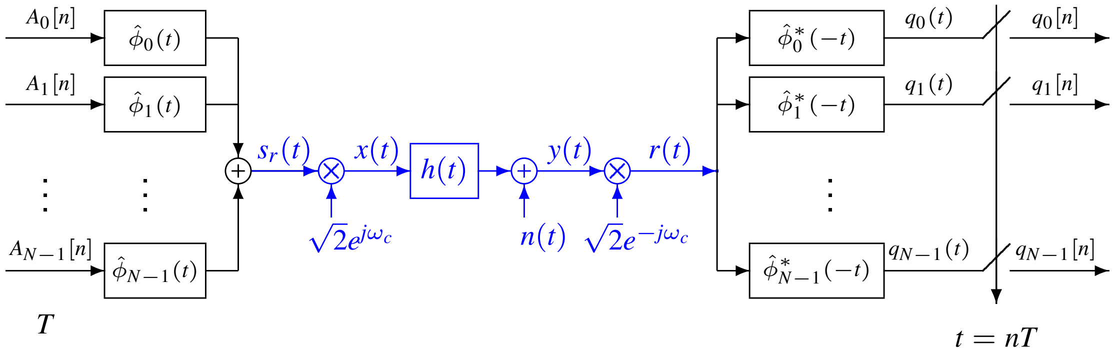
Receiver for discrete-time OFDM

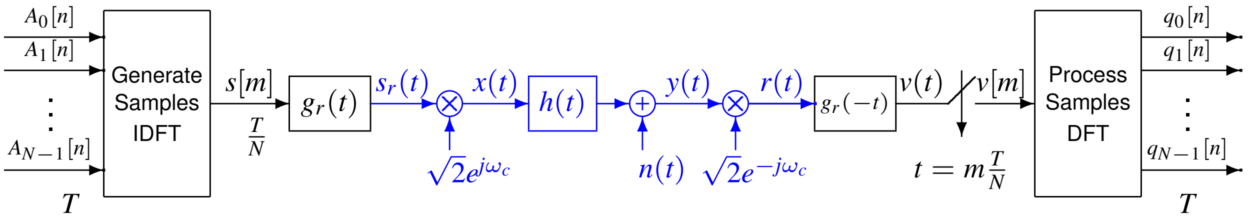
Generation of observations

$$\left\{ \begin{array}{ccccc} \overbrace{v[0], v[1], \cdots, v[N-1]}^{v^{(0)}[m]} & \overbrace{v[N], v[N+1], \cdots, v[2N-1]}^{v^{(1)}[m]} & \cdots & \overbrace{v[nN], v[nN+1], \cdots, v[nN+N-1]}^{v^{(n)}[m]} & \cdots \\ \downarrow & \downarrow & \cdots & \downarrow & \cdots \\ \frac{1}{\sqrt{T}} \times \text{DFT}_N & \frac{1}{\sqrt{T}} \times \text{DFT}_N & \cdots & \frac{1}{\sqrt{T}} \times \text{DFT}_N & \cdots \\ \downarrow & \downarrow & \cdots & \downarrow & \cdots \\ \underbrace{q_0[0], q_1[0], \cdots, q_{N-1}[0]}_{q_k[0]} & \underbrace{q_0[1], q_1[1], \cdots, q_{N-1}[1]}_{q_k[1]} & \cdots & \underbrace{q_0[n], q_1[n], \cdots, q_{N-1}[n]}_{q_k[n]} & \cdots \end{array} \right\}$$



Transmission of discrete time OFDM





- Channel between input of index i ($A_i[n]$) and output of index k ($q_k[n]$)

$$\boxed{p_{k,i}[n] = p_{k,i}(t)\big|_{t=nT}, \text{ with } p_{k,i}(t) = \hat{\phi}_i(t) * h_{eq}(t) * \hat{\phi}_k^*(-t)}$$

- Channel for transmission of samples at $\frac{T}{N}$

$$\boxed{d[m] = d(t)\big|_{t=m\frac{T}{N}}, \text{ with } d(t) = g_r(t) * h_{eq}(t) * g_r(-t) = r_{g_r}(t) * h_{eq}(t)}$$

OFDM seen as a block-based process

- Process of blocks of size N
- Transmitter: samples for the n-th block

$$\text{Block of index } n: s^{(n)}[m] = s[nN+m],\ m = \{0, 1, \cdots, N-1\}$$

$$s^{(n)}[m] \text{ for } m = 0, \cdots, N-1 \text{ are given by } N \text{ values of } \text{IDFT}_N(A_k[n])$$

- Equivalent discrete channel for samples $s[m]$ (at $\frac{T}{N}$)

$$d[m] = d(t)\Big|_{t=m\frac{T}{N}}, \text{ with } d(t) = g_r(t) * h_{eq}(t) * g_r(-t)$$

- Transmission through $d[m]$

$$v[m] = s[m] * d[m] + z[m]$$

- Demodulation
  - Sequence $v[m]$ is splitted in blocks of N samples

$$v^{(n)}[m] = v[nN+m],\ m = \{0, 1, \cdots, N-1\}$$

  - Each block is processed to obtain the N observations at n

$$q_k[n] \text{ para } k = 0, \cdots, N-1 \text{ son los } N \text{ valores de } \text{DFT}_N(v^{(n)}[m])$$

Equivalent baseband discrete channel

- Output of the matched filter (before sampling)

$$q_k(t) = \sum_{i=0}^{N-1} \sum_{\ell} A_i[\ell]\, p_{k,i}(t - \ell T) + z_k(t), \qquad k = 0, \cdots, N-1$$

- Joint response of i-th transmitter, k-th receiver and equivalent baseband channel

$$\text{Continuous time OFDM: } p_{k,i}(t) = \phi_i(t) * h_{eq}(t) * \phi_k^*(-t)$$

$$\text{Discrete time OFDM: } p_{k,i}(t) = \hat{\phi}_i(t) * h_{eq}(t) * \hat{\phi}_k^*(-t)$$

- Sampled output

$$q_k[n] = \sum_{i=0}^{N-1} \sum_{\ell} A_i[\ell]\, p_{k,i}[n - \ell] + z_k[n], \qquad k = 0, \cdots, N-1$$

$$\boxed{q_k[n] = \sum_{i=0}^{N-1} A_i[n] * p_{k,i}[n] + z_k[n], \qquad k = 0, \cdots, N-1}$$

  - N^2 equivalent discrete channels are defined

$$\boxed{p_{k,i}[n],\ i \in \{0, 1, \cdots, N-1\},\ k \in \{0, 1, \cdots, N-1\}}$$

connecting all N inputs (index i) with all N outputs (index k)

Generalization of Nyquist ISI criterion

- Inter-symbol Interference (ISI)
  - Contribution in $q_k[n]$ of symbols in $A_k[n-j]$ for $j \neq 0$
- Inter-carrier interference (ICI)
  - Contribution in $q_k[n]$ of symbols in $A_i[n]$ for $i \neq k$
- Condition for avoiding intersymbol interference (ISI)

$$\boxed{p_{i,i}[n] = C\ \delta[n]}$$

- Condition for avoiding intercarrier interference (ICI)

$$\boxed{p_{k,i}[n] = 0,\ \text{for } k \neq i,\ \forall n}$$

- Both conditions together

$$\boxed{p_{k,i}[n] = C\ \delta[n]\ \delta[k-i]}$$

- Generalization of Nyquist ISI criterion in frequency domain

$$\boxed{\mathbf{P}(e^{j\omega}) = C\ \mathbf{I}_{N\times N}}$$

$P_{k,i}(e^{j\omega})$: Fourier transform of $p_{k,i}[n]$

$\mathbf{P}(e^{j\omega})$: matrix with elements $P_{k,i}(e^{j\omega})$ (row k, column i)

  - Difficult to fulfill all constraints: N^2 constraints, N degrees of freedom

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Particularization to discrete time OFDM

- Joint input-channel-output responses

$$p_{k,i}(t) = \frac{N}{T}\sum_m \sum_\ell \xi_i[m]\ \xi_k^*[\ell]\ \left[g_r\left(t - m\frac{T}{N}\right) * h_{eq}(t) * g_r\left(-t - \ell\frac{T}{N}\right)\right]$$

- Equivalent discrete channels are $p_{k,i}[n] = p_{k,i}(nT)$

$$\boxed{p_{k,i}[n] = \frac{1}{T}\sum_{m=0}^{N-1}\sum_{\ell=0}^{N-1} e^{j\frac{2\pi i}{N}\ m}\ e^{-j\frac{2\pi k}{N}\ \ell}\ d[nN + \ell - m]}$$

$d[m]$: samples of joint response of reconstruction filter, baseband equivalent channel and receiver (matched) filter at $\frac{T}{N}$

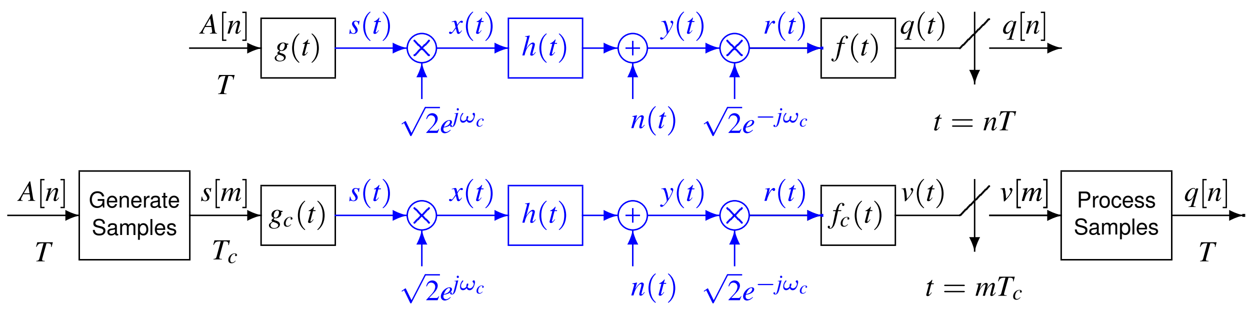
$$\boxed{d[m] = \left(g_r(t) * h_{eq}(t) * g_r(-t)\right)\Big|_{t=m\frac{T}{N}}}$$

REMARK: with this definition $v[m] = s[m] * d[m] + z[m]$

- Conditions of generalized Nyquist ISI criterion are fulfilled if

$$d[m] = K\ \delta[m]$$

Transmission of DSSS

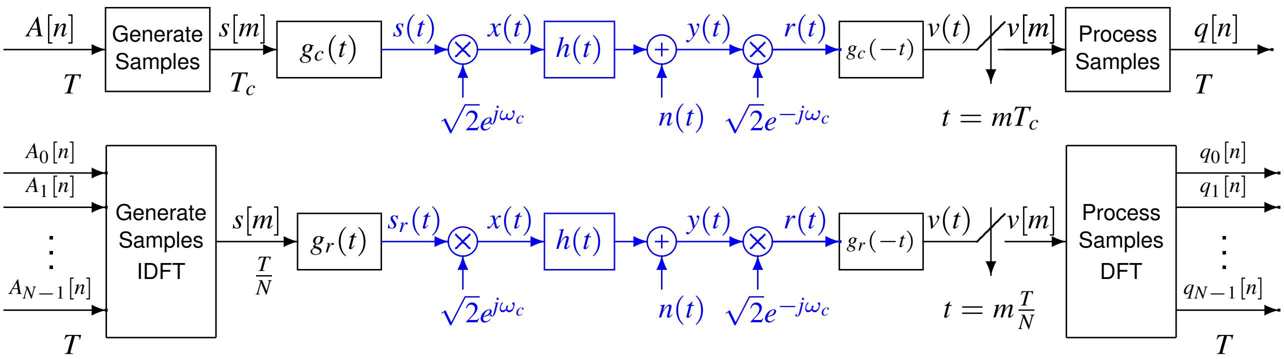


- Receiver filters - matched filters: $f(t) = g^*(-t), \;\; f_c(t) = g_c(-t)$
- Joint transmitter/receiver/channel responses
  - Transmission of $A[n]$ at symbol rate: $p(t) = g(t) * h_{eq}(t) * f(t) = r_g(t) * h_{eq}(t)$
  - Transmission of $s[m]$ at chip rate: $d(t) = g_c(t) * h_{eq}(t) * f_c(t) = r_{g_c}(t) * h_{eq}(t)$
- Equivalent discrete channels
  - At symbol rate

    $p[n] = p(t)\big|_{t=nT} = p(nT)$ relates $q[n]$ with $A[n]$: $q[n] = A[n] * p[n] + z[n]$
  - At chip rate

    $d[m] = d(t)\big|_{t=mT_c} = d(mT_c)$ relates $v[m]$ with $s[m]$: $v[m] = s[m] * d[m] + z_c[m]$

Comparison DSSS/OFDM



$$T_c \equiv \frac{T}{N} \qquad g_c(t) \equiv g_r(t)$$

$$g(t) = \sum_{m=0}^{N-1} x[m] g_c(t - mT_c) \equiv \hat{\phi}_i(t) = \sum_{m=0}^{N-1} \frac{1}{\sqrt{T}} e^{j\frac{2\pi i}{N}m} g_r\left(t - m\frac{T}{N}\right) \qquad x[m] \equiv \frac{1}{\sqrt{T}} e^{j\frac{2\pi i}{N}m}$$

$$g^*(-t) = \sum_{\ell=0}^{N-1} x^*[\ell] g_c(-t - \ell T_c) \equiv \hat{\phi}_k^*(-t) = \sum_{\ell=0}^{N-1} \frac{1}{\sqrt{T}} e^{-j\frac{2\pi k}{N}\ell} g_r\left(-t - \ell\frac{T}{N}\right) \qquad x^*[\ell] \equiv \frac{1}{\sqrt{T}} e^{-j\frac{2\pi k}{N}\ell}$$

Comparison DSSS/OFDM (II)

- DSSS

$$s^{(n)}[m] = A[n]\; x[m]$$

$$q[n] = \sum_{\ell=0}^{N-1} x^*[\ell]\; v[nN+\ell] = \sum_{\ell=0}^{N-1} x^*[\ell]\; v^{(n)}[\ell]$$

$$p[n] = \sum_{m=0}^{N-1}\sum_{\ell=0}^{N-1} x[m]\; x^*[\ell]\; d[nN+\ell-m]$$

- OFDM

$$s^{(n)}[m] = \frac{1}{\sqrt{T}} \sum_{k=0}^{N-1} A_k[n] e^{j\frac{2\pi k}{N}m} = \frac{N}{\sqrt{T}} \mathsf{IDFT}\{A_k[n]\}$$

$$q_k[n] = \frac{1}{\sqrt{T}} \sum_{\ell=0}^{N-1} e^{-j\frac{2\pi k}{N}\ell}\; v[nN+\ell] = \frac{1}{\sqrt{T}} \sum_{\ell=0}^{N-1} e^{-j\frac{2\pi k}{N}\ell}\; v^{(n)}[\ell] = \frac{1}{\sqrt{T}} \mathsf{DFT}\{v^{(n)}[m]\}$$

$$p_{k,i}[n] = \frac{1}{T} \sum_{m=0}^{N-1}\sum_{\ell=0}^{N-1} e^{j\frac{2\pi i}{N}\, m}\; e^{-j\frac{2\pi k}{N}\, \ell}\; d[nN+\ell-m]$$

Avoidance of ISI and ICI - Cyclic prefix

- Asumption: response $d[m]$ is causal and with finite length $K_d + 1$
  - Channel $d[m]$ has memory K_d lags (samples)
- With a cyclic extension of the samples $s[m]$, including a cyclic prefic of C samples, such that $C \geq K_d$, equivalent channels are

$$p_{k,i}[n] = \frac{N}{T}\; \delta[n]\; \delta[k-i]\; D[k]$$

- ISI and ICI are completely removed
- Observation for carrier of index k, $q_k[n]$, is now

$$q_k[n] = \frac{N}{T}\; A_k[n]\; D[k] + z_k[n]$$

$D[k]$: coefficient of index k of the DFT of N points of $d[m]$

- Different signal to noise ratio for each carrier (gain factor $D[k]$)

REMARK: with the extension, $N+C$ samples will be transmitted each T seconds, which means that the new definition of $d[m]$ is

$$d[m] = d(t)\Big|_{t=m\frac{T}{N+C}}$$

Cyclic prefix seen as a block-based process

- Samples of OFDM signal a processed in blocks
- Samples for block of index n

$$s^{(n)}[m] \text{ for } m=0,\cdots,N-1 \text{ are given by } N \text{ values of IDFT}_N(A_k[n])$$

- Cyclic extension for each block - Cyclic prefix

$$\tilde{s}^{(n)}[m] = \begin{cases} s^{(n)}[m+N] & m=-C,\cdots,-1\\ s^{(n)}[m] & m=0,\cdots,N-1\end{cases}$$

- Transmission through $d[m]$ (at $\frac{T}{T+C}$)

$$\tilde{v}^{(n)}[m] = \tilde{s}^{(n)}[m]*d[m]+\tilde{z}^{(n)}[m]$$

- Elimination of cyclic prefix samples at the receiver

$$v^{(n)}[m]=\tilde{v}^{(n)}[m]\; w_N[m]$$

- Demodulation

$$q_k[n] \text{ for } k=0,\cdots,N-1 \text{ is now given by the } N \text{ values of DFT}_N(v^{(n)}[m])$$

Cyclic prefix seen as a block-based process (II)

$$\left\{\begin{array}{ccccccc}
\underbrace{A_0[0],A_1[0],\cdots,A_{N-1}[0]}_{A_k[0]} & \underbrace{A_0[1],A_1[1],\cdots,A_{N-1}[1]}_{A_k[1]} & \cdots & \underbrace{A_0[n],A_1[n],\cdots,A_{N-1}[n]}_{A_k[n]} & \cdots \\
\downarrow & \downarrow & \cdots & \downarrow & \cdots \\
\frac{N}{\sqrt{T}}\times \text{IDFT}_N & \frac{N}{\sqrt{T}}\times \text{IDFT}_N & \cdots & \frac{N}{\sqrt{T}}\times \text{IDFT}_N & \cdots \\
\downarrow & \downarrow & \cdots & \downarrow & \cdots \\
\overbrace{s[0],s[1],\cdots,s[N-1]}^{s^{(0)}[m]} & \overbrace{s[N],s[N+1],\cdots,s[2N-1]}^{s^{(1)}[m]} & \cdots & \overbrace{s[nN],s[nN+1],\cdots,s[nN+N-1]}^{s^{(n)}[m]} & \cdots \\
\downarrow & \downarrow & \cdots & \downarrow & \cdots \\
\tilde{s}^{(0)}[m] & \tilde{s}^{(1)}[m] & \cdots & \tilde{s}^{(n)}[m] & \cdots
\end{array}\right\}$$

$$\overbrace{\underbrace{s[N-C],\cdots,s[N-2],s[N-1]}_{C},\underbrace{s[0],s[1],\cdots,\underbrace{s[N-C],\cdots,s[N-2],s[N-1]}_{C}}_{N}}^{\tilde{s}^{(0)}[m]}$$

$$\overbrace{\underbrace{s[2N-C],\cdots,s[2N-2],s[2N-1]}_{C},\underbrace{s[N],s[N+1],\cdots,\underbrace{s[2N-C],\cdots,s[2N-2],s[2N-1]}_{C}}_{N}}^{\tilde{s}^{(1)}[m]}$$

$$\overbrace{\underbrace{s[(n+1)N-C],\cdots,s[(n+1)N-2],s[(n+1)N-1]}_{C},\underbrace{s[nN],s[nN+1],\cdots,\underbrace{s[(n+1)N-C],\cdots,s[(n+1)N-1]}_{C}}_{N}}^{\tilde{s}^{(n)}[m]}$$

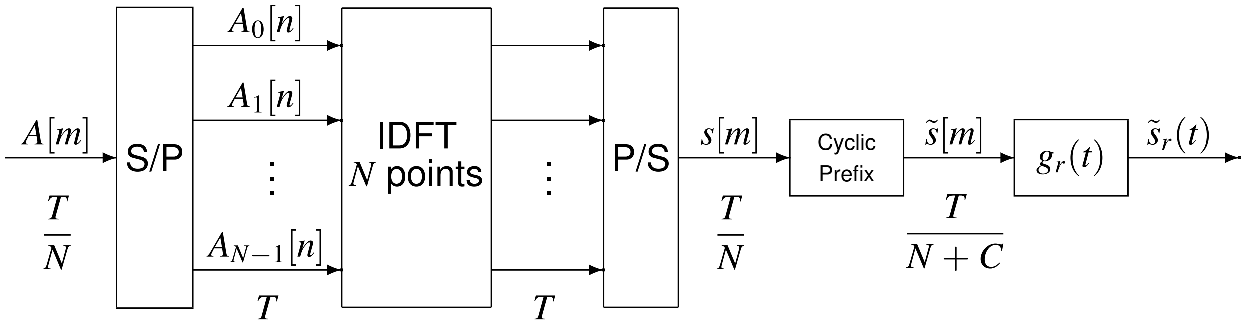
Cyclic prefix seen as a block-based process (III)

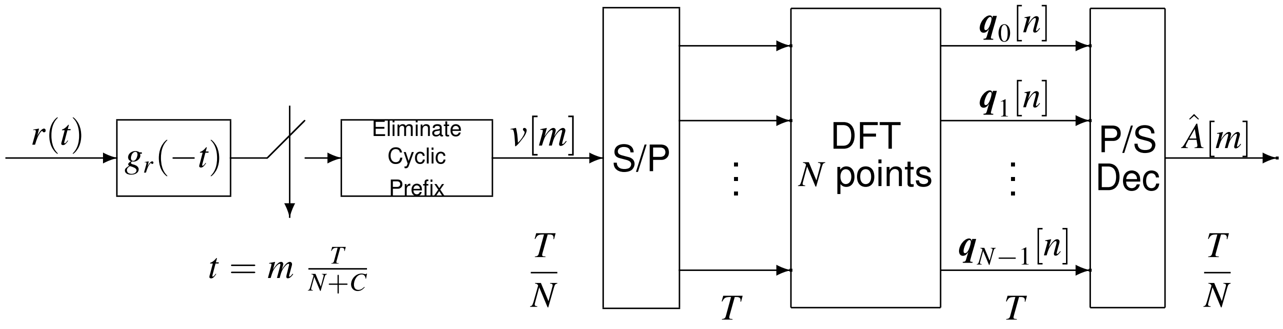
- $q_k[n]$: is obtained from DFT of N points of $v^{(n)}[m]$
- Cyclic prefix is introduced to simulate a circular convolution
  - Linear convolution of $\tilde{s}^{(n)}[m]$ with $d[m]$ is equivalent to circular convolution of $s^{(n)}[m]$ with $d[m]$
- Useful because of the DFT property of being multiplicative under circular convolution

  If $\boxed{z[n] = x[n] \circledast y[n]}$ then $\boxed{\text{DFT}_N(z[n]) = \text{DFT}_N(x[n]) \times \text{DFT}_N(y[n])}$
- Having this into account, without noise and abusing of notation

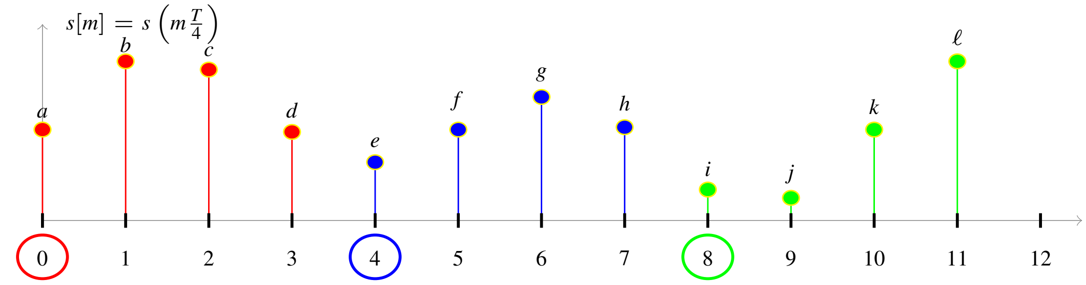
$$\boxed{\begin{aligned} q_k[n] =& \frac{1}{\sqrt{T}}\text{DFT}_N(\tilde{s}^{(n)}[m] * d[m]) \\ =& \frac{1}{\sqrt{T}}\text{DFT}_N(s^{(n)}[m] \circledast d[m]) \\ =& \frac{1}{\sqrt{T}}\text{DFT}_N(s^{(n)}[m]) \times \text{DFT}_N(d[m]) \\ =& \frac{1}{\sqrt{T}}\text{DFT}_N\left(\frac{N}{\sqrt{T}}\text{IDFT}_N(A_k[n])\right) \times \text{DFT}_N(d[m]) \\ =& A_k[n] \times \frac{N}{T}D[k] \end{aligned}}$$

Modulator/demodulator for OFDM with cyclic prefix





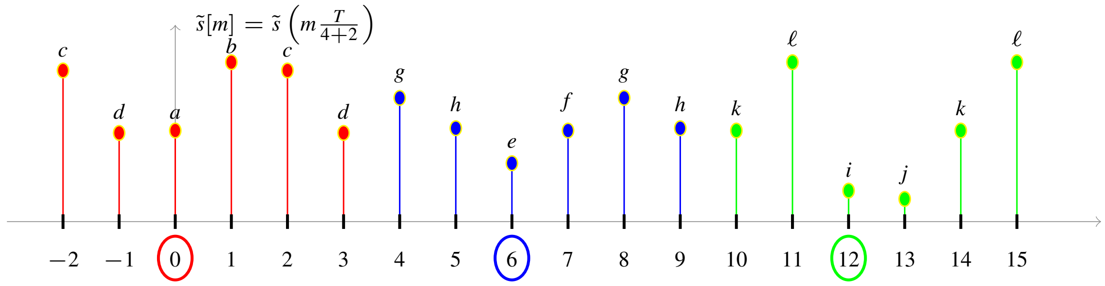
Cyclic prefix: $N = 4$, $C = 2$



$$\{a, b, c, d\} = \frac{N}{\sqrt{T}} \text{IDFT}_4\{A_0[0], A_1[0], A_2[0], A_3[0]\}$$

$$\{e, f, g, h\} = \frac{N}{\sqrt{T}} \text{IDFT}_4\{A_0[1], A_1[1], A_2[1], A_3[1]\}$$

$$\{i, j, k, \ell\} = \frac{N}{\sqrt{T}} \text{IDFT}_4\{A_0[2], A_1[2], A_2[2], A_3[2]\}$$



Avoidance of ISI and ICI - Analysis

- Asumption: response $d[m]$ is causal with length $K_d + 1$
  - Channel $d[m]$ has memory K_d lags (samples)
- New discrete time basis functions (length is extended C samples)

$$\boxed{\tilde{\xi}_k[m] = \frac{1}{\sqrt{N}}\ e^{j\frac{2\pi k}{N}\ m}\ w_{N+C}[m+C], \qquad k = 0, \cdots, N-1}$$

  - Non-null values for $m \in [-C, N-1]$ (en lugar de $m \in [0, N-1]$)
  - Condition to avoid ISI and ICI:

$$C \geq K_d$$

- Samples of the OFDM signals are now

$$\boxed{\tilde{s}[m] = \sqrt{\frac{N}{T}} \sum_n \sum_{k=0}^{N-1} A_k[n]\ \tilde{\xi}_k[m - n(N+C)]}$$

  - There are now $N + C$ samples per symbol interval (T s)
- Signal at the demodulator is obtained as

$$\boxed{q_k[n] = \frac{1}{\sqrt{T}} \sum_{m=0}^{N-1} \xi_k^*[m]\ v[n(N+C)+m]}$$

New equivalent discrete channels

- With this modifications the equivalent discrete channels are

$$\boxed{\begin{aligned}p_{k,i}[n] &= \frac{1}{T}\sum_{m=-C}^{N-1}\sum_{\ell=0}^{N-1} e^{j\frac{2\pi i}{N}\ m}\ e^{-j\frac{2\pi k}{N}\ \ell}\ d[n(N+C)+\ell-m] \\ &= \frac{1}{T}\sum_{\ell=0}^{N-1}\sum_{u=\ell-N+1}^{\ell+C} e^{-j\frac{2\pi i}{N}\ u}\ e^{j\frac{2\pi(i-k)}{N}\ \ell}\ d[n(N+C)+u] \\ &= \frac{1}{T}\sum_{u=0}^{K_d} e^{-j\frac{2\pi i}{N}\ u}\ d[u]\ \delta[n]\ \sum_{\ell=0}^{N-1} e^{j\frac{2\pi(i-k)}{N}\ \ell} \\ &= \frac{N}{T}\ \delta[n]\ \delta[k-i]\ \underbrace{\sum_{u=0}^{K_d} e^{-j\frac{2\pi i}{N}\ u}\ d[u]}_{\text{DFT of } d[m]} = \frac{N}{T}\ \delta[n]\ \delta[k-i]\ D[i]\end{aligned}}$$

$D[k]$: coefficient of index k for DFT of N points of $d[m]$

- ISI and ICI are eliminated

Spectral efficiency of OFDM with cyclic prefix

- OFDM signal is constructed from samples with interpolation filter $g_r(t)$

$$s_r(t) = \sum_m s[m]\ g_r(t - mT_s), \text{ with } g_r(t) = \text{sinc}\left(\frac{N}{T_s}t\right)$$

T_s: sampling period associated to samples $s[m]$
  - Bandwidth of corresponding bandpass modulated signal $x(t)$ is

$$\boxed{W = \frac{2\pi}{T_s} \text{ rad/s},\ B = \frac{1}{T_s} \text{ Hz}}$$

- OFDM without cyclic prefix
  - In this case samples are interpolated at $T_s = \frac{T}{N}$

$$\boxed{W = \frac{2\pi}{T} \times N \text{ rad/s},\ B = R_s \times N \text{ Hz}}$$

- OFDM with cyclic prefix
  - In this case samples are interpolated at $T_s = \frac{T}{N+C}$

$$\boxed{W = \frac{2\pi}{T} \times (N+C) \text{ rad/s},\ B = R_s \times (N+C) \text{ Hz}}$$

- Efficiency of OFDM using cyclic prefix of lenght C

$$\boxed{\eta = \frac{N}{N+C}}$$

Signal to noise ratio for each carrier

- Observations at symbol rate

$$\boxed{q_k[n] = \frac{N}{T}\, D[k]\, A_k[n] + z_k[n]}$$

- Signal to noise ratio for each carrier

$$\boxed{\left.\frac{S}{N}\right|_k = \frac{\mathcal{E}\left\{\frac{N}{T}\, D[k]\, A_k[n]\right\}}{\mathcal{E}\left\{z_k[n]\right\}} = \frac{E\left[\left|\frac{N}{T}\, D[k]\, A_k[n]\right|^2\right]}{E\left[\left|z_k[n]\right|^2\right]} = \frac{\left|\frac{N}{T}\, D[k]\right|^2 E_s}{\sigma_z^2}}$$

  - Relationship $\left.\frac{S}{N}\right|_k$ is proportional to $|D[k]|^2$
  - Performance is different for each carrier
    - The higher the value of $|D[k]|$, the better the performance of the k-th carrier