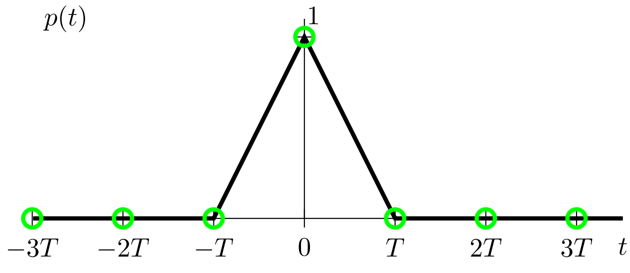
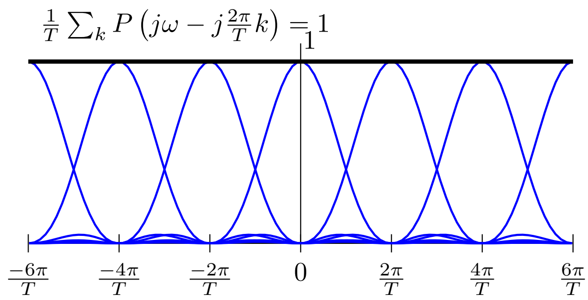
Chapter 1 : Solutions of the Exercises

Exercise 1.1 (Solution) a) In the time domain, $p[n] = p(nT) = \delta[n]$. Although this is enough to demonstrate it, and it is the simplest option in this case, it can be also demonstrated in the frequency domain

$$\frac{1}{T}\sum_k P\left(j\omega - j\frac{2\pi}{T}k\right) = \sum_k \text{sinc}^2\left(\frac{\left(\omega - \frac{2\pi}{T}k\right)T}{2\pi}\right) = 1.$$

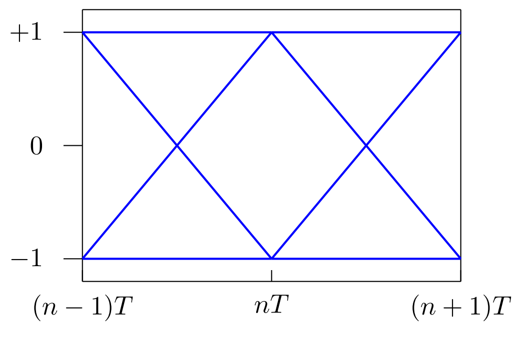




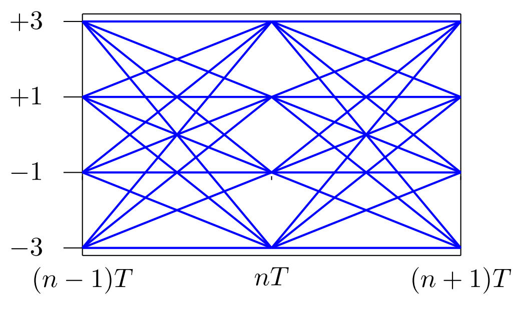
b) Power spectral density of $q(t)$ is

$$S_q(j\omega) = E_s\, T\, \text{sinc}^4\left(\frac{\omega T}{2\pi}\right)$$

c) Eye diagram for 2-PAM

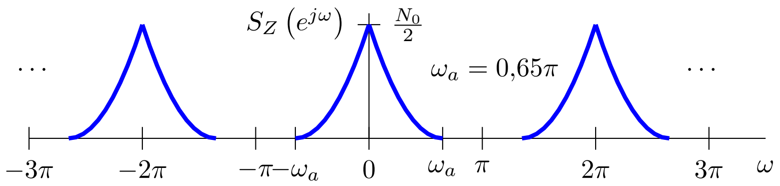


d) Eye diagram for 4-PAM



Exercise 1.2 (Solution) Power spectral density of discrete time noise

$$S_Z\left(e^{j\omega}\right) = \frac{N_0}{2}\frac{1}{T}\sum_k \left|F\left(j\frac{\omega}{T} - j\frac{2\pi}{T}k\right)\right|^2$$



Exercise 1.3 (Solution)

$$P_e = \frac{1}{2}Q\left(\frac{3/4}{\sigma_z}\right) + \frac{1}{2}Q\left(\frac{5/4}{\sigma_z}\right).$$

Exercise 1.4 (Solution) a) Relationship between power spectral densities and filter

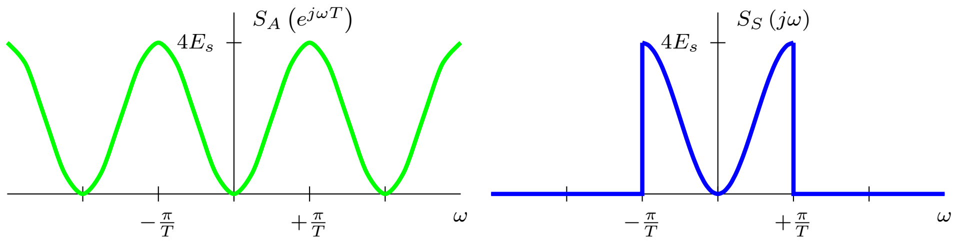
$$S_A\left(e^{j\omega}\right) = S_{A_b}\left(e^{j\omega}\right)\ \left|C\left(e^{j\omega}\right)\right|^2 = E_s\ \left|C\left(e^{j\omega}\right)\right|^2$$

b) Power spectral densities are

I) In this case

$$S_S(j\omega) = \begin{cases} E_s\left[2 - 2\cos(\omega T)\right] & \text{si } |\omega| \le \frac{\pi}{T} \\ 0 & \text{si } |\omega| > \frac{\pi}{T} \end{cases}$$

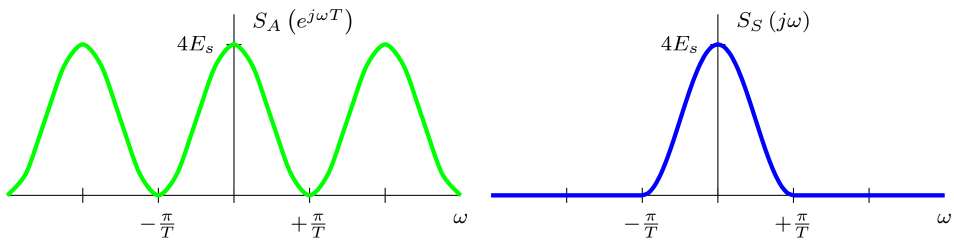
This function is plotted below



II) Now

$$S_S(j\omega) = \begin{cases} E_s\left[2 + 2\cos(\omega T)\right] & \text{si } |\omega| \le \frac{\pi}{T} \\ 0 & \text{si } |\omega| > \frac{\pi}{T} \end{cases}$$

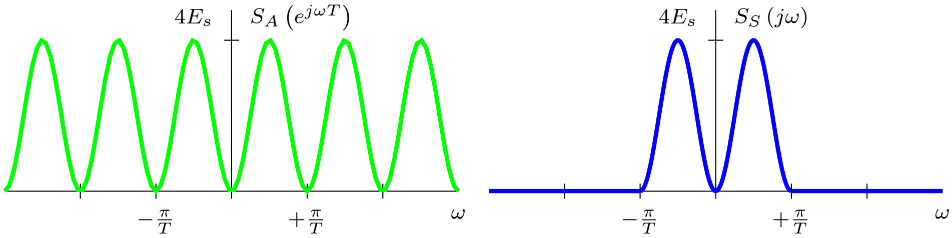
This is plotted in the figure



III) Finally

$$S_S(j\omega) = \begin{cases} E_s\left[2 - 2\cos(2\omega T)\right] & \text{si } |\omega| \le \frac{\pi}{T} \\ 0 & \text{si } |\omega| > \frac{\pi}{T} \end{cases}$$

The function is plotted below



c) Power of modulated signal is the same one in all cases

$$P_S = 2\ E_s \times R_s \text{ Watts.}$$

Exercise 1.5 (Solution) a) A 4-PSK or 4-QAM modulation is transmitted through a Gaussian channel. *Noise is seen in the scattering diagram.*

b) The *power of noize in the quadrature component is higher* than the power of noise in the in-phase component.

c) Receiver is non-coherent, there is a *phase shift* between carriers at the transmitter and at the receiver.

d) It looks like there is a *phase shift* between carriers at the transmitter and at the receiver that *evolves cyclicaly with time*. This happens when frequencies are different in the carriers of transmitter and receiver.

Exercise 1.6 (Solution)

- Carrier frequency: $\omega_c = 2{,}5\,\text{kHz}$.
- Encoder (constellation): 16-QAM.
- Transmitter filter: *Root-raised cosine* with roll-off factor $\alpha = 0{,}25$.

Exercise 1.7 (Solution) a) Equivalent discrete channel is

$$p[n] = p(t)\Big|_{t=nT} = \frac{1}{2}\delta[n] - \frac{1}{2}\delta[n-1]$$

b) Power spectran density of discrete time noise is

$$S_z\left(e^{j\omega}\right) = \frac{N_0}{2} \times \mathcal{E}\{f(t)\} = \frac{N_0}{2}$$

because $r_f(t)$ sampled at symbol rate is a delta function. If this condition is not satisfied, power spectral density would be

$$S_z\left(e^{j\omega}\right) = \frac{N_0}{2} \times \frac{1}{T}\sum_k R_f\left(\frac{\omega}{T} - \frac{2\pi}{T}k\right) = \frac{N_0}{2} \times \frac{1}{T}\sum_k \left|F\left(\frac{\omega}{T} - \frac{2\pi}{T}k\right)\right|^2$$

c) Probability of error is

$$P_e = \frac{1}{2}P_{e|A[n]=+1} + \frac{1}{2}P_{e|A[n]=-1} = \frac{1}{4} + \frac{1}{2}Q\left(\frac{1}{\sqrt{N_0/2}}\right)$$

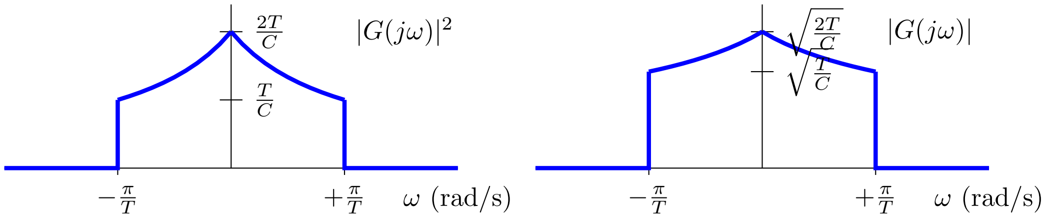
Exercise 1.8 (Solution) a) The system uses matched filters, which have to satisfy the following conditions:

- Ambiguity function of $g(t)$, $r_g(t) = g(t) * g(-t)$, satisfies the Nyquist conditions for zero ISI. For instance, *root-raised cosine filters*.
- Bandwidth has to be lower than channel bandwidth:

$$W \leq 2\pi \times 9 \times 10^3 \,\text{rad/s or } B \leq 9\,\text{kHz}$$

b) If matched filters are used:

$$G(j\omega) = \sqrt{\frac{P(j\omega)}{H(j\omega)}}$$



Here $\frac{\pi}{T} = 2\pi\ 9 \times 10^3$ rad/s.

c) By choosing a receiver filter $f(t)$ having an ambiguity function $r_f(t)$ satisfying Nyquist conditions for zero ISI. For instance, a *root-raised cosine* filter, $f(t) = h_{RRC}^{\alpha,T}(t)$.

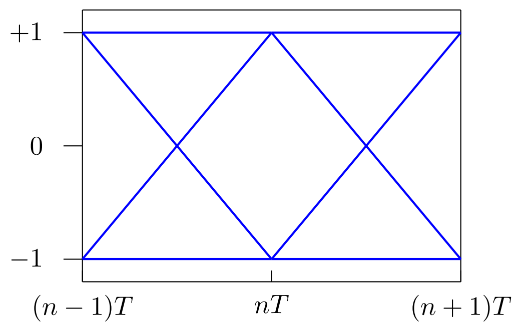
d) $R_s|_{\text{máx}} = 18$ kbauds.

Exercise 1.9 (Solution) a) $f(t) = g(-t)$ and $S_z(e^{jw}) = \frac{N_0}{2}$.

b) $p(t) = r_g(t) * h(t) = r_g(t) - \frac{1}{10} r_g(t-2T)$, therefore, $p[n] = p(t)\big|_{t=nT} = p(nT) = \delta[n] - \frac{1}{10}\delta[n-2]$

c) There exists ISI, because $p[n] \neq C\,\delta[n]$.

d) Eye diagram



Exercise 1.10 (Solution) a) Conditions that have to be satisfied simultaneously are: $r_f(t)$ must satisfy Nyquist conditions for the noise $z[n]$ to be white, and $p(t)$ has to satisfy Nyquist conditions to avoid ISI.

- With first channel, it is possible, because the channel has an ideal behavior in the passband, and therefore in this band is equivalent to a Gaussian channel. It is valid to use any transmitter filter $g(t)$ whose ambiguity function, $r_g(t)$, satisfies Nyquist conditions for zero ISI, for instance a root-raised cosine filter, as long as its bandwidth is lower or equal to $B = 4$ MHz (in band pass, equivalent to 2 baseband MHz). Moreover, given that in this case $r_f(t) = r_g(t)$, sampled noise $z[n]$ is white.

$$R_{s|\max} = B = 4\,\text{Mbauds}$$

- With second channel, behavior is not ideal in the bandpass, and therefore it produces linear distortion, therefore it is a lineal channel. It is known, that under linear channels, it is not possible to satisfy simultaneously both conditions with matched filters, designed to transmit at the conventional rates given by channel bandwidth.
  - However, the specific shape of the frequency response of this channel allows to satisfy both conditions by using an atypical design: transmitting using a sinc function as trasmitter filter, with a bandwidth equal to the channel bandwidth, but to transmit at half of the symbol rate usually associated to that bandwidth, i.e.

$$g(t) = h_{RRC}^{0,T/2}(t) = \frac{\sqrt{2}}{\sqrt{T}}\text{sinc}\left(\frac{2}{T}\right),$$

to transmit at $R_s = \frac{1}{T} = 2$ Mbauds. This is quite an specific anomaly, not the general case considered in a realistic digital communication system.

b) $m_{min} = \lceil \frac{R_b}{R_{s|max}} \rceil = 3$ bits/symbol, and $M_{min} = 8$ symbols.

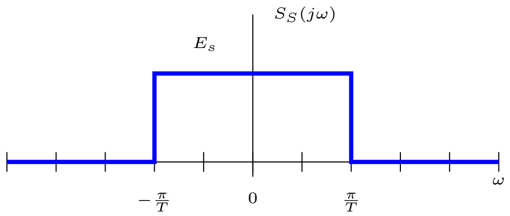
c) $B = R_s(1+\alpha)$, therefore $\alpha = 0{,}2$. We look for the highest possible value of α to minimize the sensitivity of the receiver to synchronism in the sampler.

Exercise 1.11 (Solution) a) Noise is white if $r_f(t)$ satisfies the conditions of the Nyquits criterion for zero ISI.

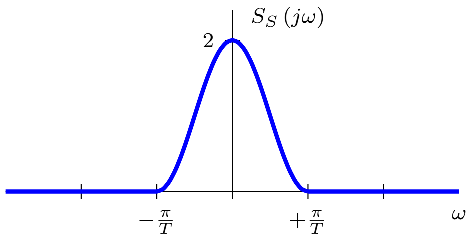
b) $R_{s|max} = 2B = 8$ kbauds, which is obtained with $\alpha = 0$.

c) PSD:

I) If $A[n]$ is white



II) Si $S_A(e^{j\omega}) = 1 + \cos(w)$:



d) $R_s = \frac{2B}{1+\alpha} = 6{,}4$ kbauds, therefore $m = \frac{R_b}{R_s} = 3$ bits/symbolo and $M = 2^m = 8$ symbols.

Exercise 1.12 (Solution) a) $p(t) = g(t) * g(-t) * h(t) = r_g(t) + \frac{1}{4}r_g(t-2T)$, therefore $p[n] = \delta[n] + \frac{1}{4}\delta[n-2]$.

b) Transmission with a M-PAM (baseband):

I) $R_{s|max} = 2B$, $m_{min} = \lceil \frac{R_b}{R_{s|max}} \rceil = 3$ bits/symbol and $M = 2^m = 8$ symbols.

II) $R_s = \frac{R_b}{m} = 18$ kbauds.

III) $\alpha = \frac{2B}{R_b} - 1 = 0{,}11$.

c) Transmission with M-QAM (bandpass):

I) $R_{s|max} = B$, $m_{min} = \lceil \frac{R_b}{R_{s|max}} \rceil = 6$ bits/symbol and $M = 2^m = 64$ symbols.

II) $R_s = \frac{R_b}{m} = 9$ kbauds.

III) $\alpha = \frac{B}{R_b} - 1 = 0{,}11$.

Exercise 1.13 (Solution) a) In the first scenario, with $f(t) = g(-t)$

$$p[n] = \frac{3}{4}\delta[n] - \frac{1}{4}\delta[n-1].$$

In the second one, with $f(t) = m(-t)$

$$p[n] = \frac{1}{2}\delta[n].$$

b) Given that $f(t) = m(-t)$ lasts less than T seconds, its autocorrelation function $r_f(t) = f(t) * f(-t)$ has a support lower than $[-T, T]$, and therefore it satisfies that $r_f[n] = r_f(nT) = C\delta[n]$, in this case with $C = 2$. Therefore, the sampled noise $z[n]$ is white.

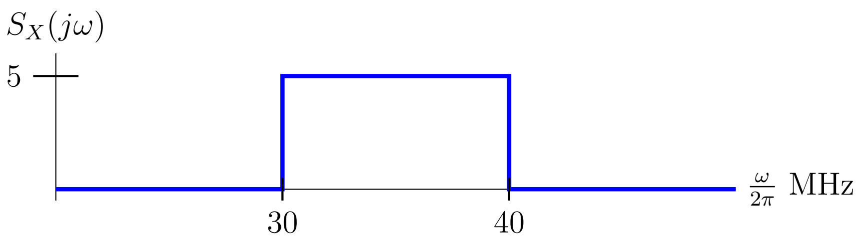
c) In the second case there is no ISI. As in both cases the noise is white (if $f(t) = g(-t)$, again $r_f[n] = r_f(nT) = \delta[n]$), second option is better than first option.

Exercise 1.14 (Solution) a) Maximum rate, which is achieved with $\alpha = 0$, is $R_{s|max}$=10 Mbauds.

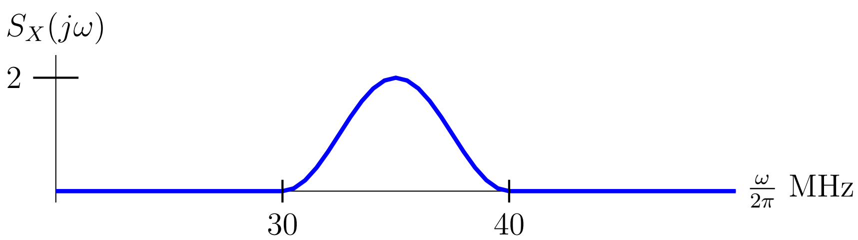
b) Minimum order for constellation is $M = 16$ symbols.

c) Only the part corresponding to positive frequencies is plotted for the power spectral densities

I) If $A[n]$ is white



II) For $A[n]$ having the specified $R_A[n]$ function



Exercise 1.15 (Solution) a) In the case of an ideal channel

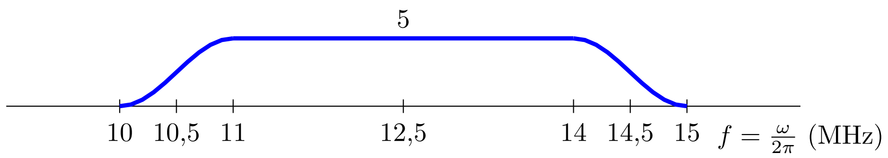
I) Maximum transmission rates are

$$R_{s|max} = 4 \text{ Mbaudios}, \qquad R_{b|max} = 16 \text{ Mbits/s}.$$

II) Power spectral density for a white data sequence $A[n]$ is

$$S_X(j\omega) = \frac{E_s}{T}\frac{1}{2}\left[H_{RC}^{\alpha,T}(j\omega - j\omega_c) + H_{RC}^{\alpha,T}(j\omega + j\omega_c)\right],$$

where $E_s = 10$ J. This response is shown for positive frequencies.

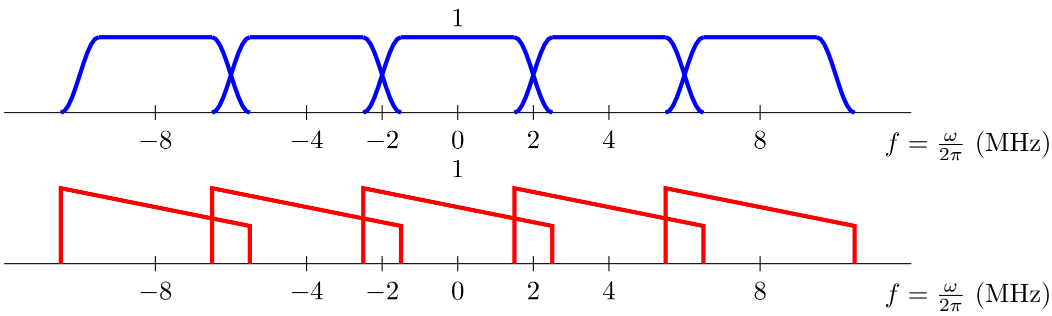


b) For the specified channel

I) There exists ISI. For that channel, it is simpler to demonstrate this in the frequency domain, where replicas of joint transmitter-channel-receiver response, $P(j\omega)$, do not add a constant value

$$\frac{1}{T}\sum_k P\left(j\omega - j\frac{2\pi}{T}k\right) \neq 1\ (\times C)$$

To illustrate this, and having into account that $P(j\omega) = H_{RC}^{\alpha,T}(j\omega)\ H_{eq}(j\omega)$, below you can find plotted separately $\frac{1}{T}\sum_k H_{RC}^{\alpha,T}\left(j\omega - j\frac{2\pi}{T}k\right)$, in blue line, and $\sum_k H_{eq}\left(j\omega - j\frac{2\pi}{T}k\right)$, in red line. The addition $\frac{1}{T}\sum_k P\left(j\omega - j\frac{2\pi}{T}k\right)$ is the product of both components.



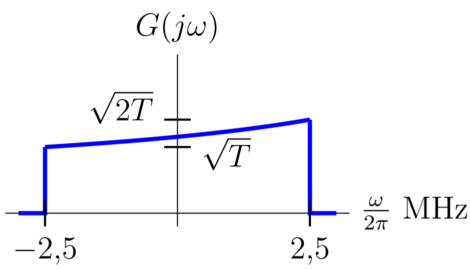
II) Noise is white because the time ambiguity function of receiver filter, sampled at symbol rate is a delta functions (or equivalently, the addition of replicas of its frequency response each $\frac{2\pi}{T}$ rad/s add a constant value). This is because given that $f(t) = h_{RRC}^{\alpha,T}(t)$, the ambiguity function is $r_f(t) = h_{RC}^{\alpha,T}(t)$.

c) If now matched filters are used

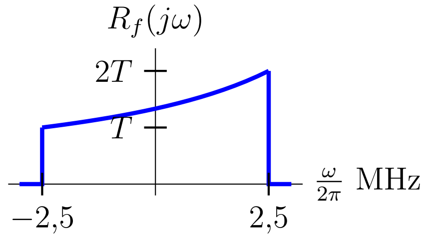
I) In the frequency domain

$$G(j\omega) = \sqrt{\frac{H_{RC}^{\alpha,T}(j\omega)}{H_{eq}(j\omega)}}$$

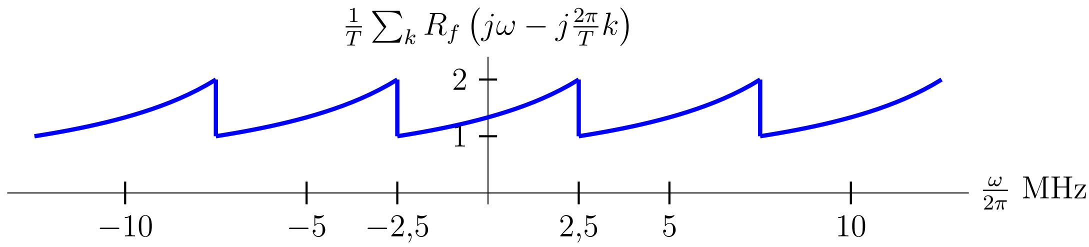
For $\alpha = 0$, this response is plotted in the next figure



II) In this case noise is not white, because replicas of the fourier transform of the ambiguity function of the receiver filter shifted multiples of $\frac{2\pi}{T}$ do not add a constant value. Now, given that $f(t) = g(-t)$, $R_f(j\omega) = |F(j\omega)|^2 = |G(j\omega)|^2$, respnse that is shown in the next figure.

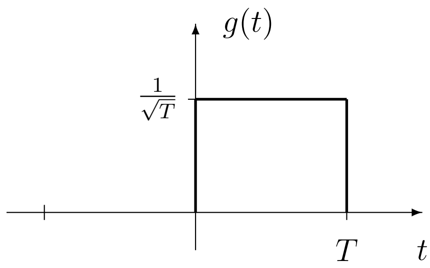


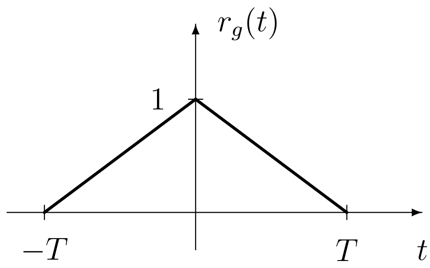
The addition of replicas of this response shifted multiples of $\frac{2\pi}{T}$ rad/s is shown below



Exercise 1.16 (Solution) a) Joint transmitter-receiver response, in this case, is equal to the ambiguity function of the transmitter filter

$$p(t) = r_g(t) = g(t) * g(-t).$$





Equivalent discrete channel is obtained sampling at symbol rate this joint response, $p(t)$, i.e.

$$p[n] = p(t)\big|_{t=nT} = \delta[n],$$

which means that conditions to avoid ISI are satisfied.

b) Equivalent discrete channel is

$$p[n] = \left(1 + j\frac{1}{2}\right)\delta[n] + j\frac{1}{2}\delta[n-1].$$

Values of the received constellation are in this case

$A[n]$	$A[n-1]$	$o[n]$
$+1$	$+1$	$+1+j$
$+1$	$+j$	$+\frac{1}{2}+j\frac{1}{2}$
$+j$	$+1$	$-\frac{1}{2}+j\frac{3}{2}$
$+j$	$+j$	$-1+j$

c) Now, the equivalent discrete channel is

$$p[n] = j\delta[n-1],$$

which means that there is not ISI, although there is a delay and a scaling factor in the received sequence. In particular, there is a delay of a symbol and at the receiver the transmitted sequence is multiplied by j, therefore $o[n] \in \{+j, -1\}$.

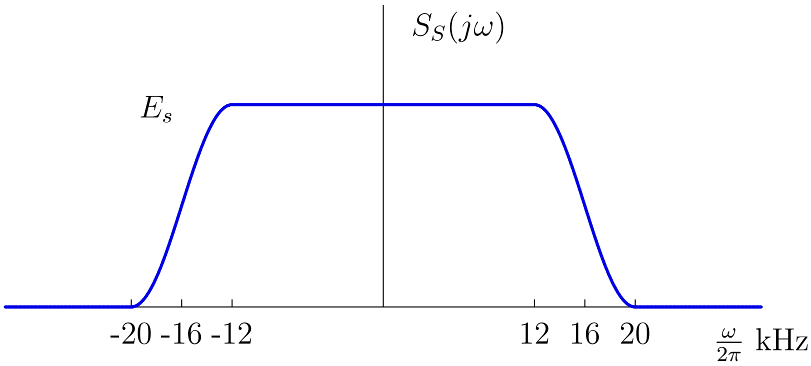
Exercise 1.17 (Solution) a) For $\alpha = 0$ in both cases:

I) $R_s = 40$ kbaudios.

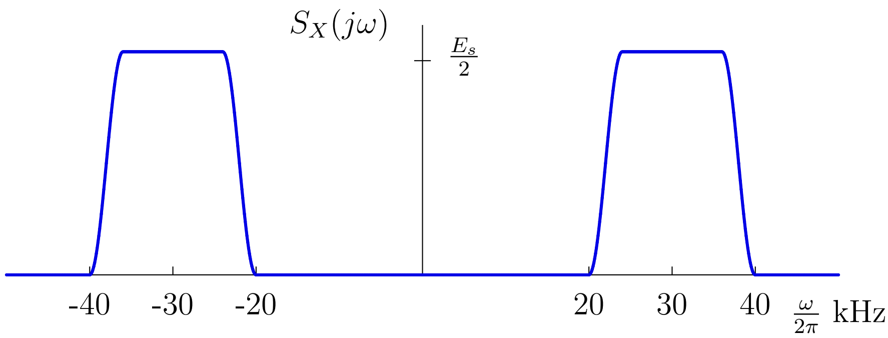
II) $R_s = 20$ kbaudios.

b) For $\alpha = 0{,}25$:

I) Baseband, 2-PAM: With $E_s = 1$ J



II) Bandpass, 4-QAM: With $E_s = 2$ J



c) I) $\omega_c = 2\pi \times 30 \cdot 10^3$ rad/s.

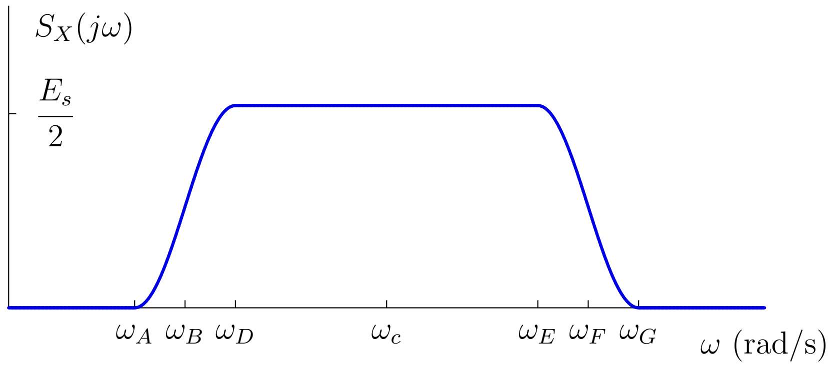
II) Constellation order: $M = 16$ symbols.

III) Symbol rate: $R_s = 16$ kbauds.

Exercise 1.18 (Solution) a)

- Carrier frequency: $\omega_c = 2\pi \cdot 875$ Mrad/s.
- Symbol rate: $R_s = 120$ Mbauds.
- Binary rate: $R_b = R_s \cdot m = 480$ Mbits/s.

b) Power spectral density $S_X(j\omega)$ is

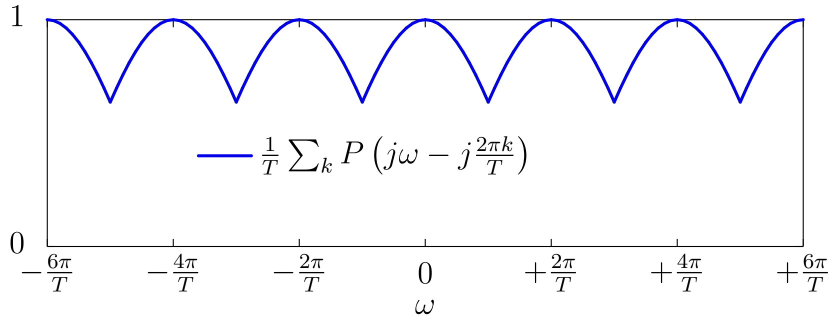


where

$$\omega_A = 2\pi \cdot 800\text{Mrad/s},\ \omega_B = 2\pi \cdot 815\text{Mrad/s},$$
$$\omega_D = 2\pi \cdot 830\text{Mrad/s},\ \omega_E = 2\pi \cdot 920\text{Mrad/s},$$
$$\omega_F = 2\pi \cdot 935\text{Mrad/s},\ \omega_G = 2\pi \cdot 950\text{Mrad/s}.$$

c) I) $p(t) = r_g(t) = r_f(t)$ is a raised cosine, which satisfies the Nyquist conditions for zero ISI. Therefore, DO NOT exists ISI, and sampled noise $z[n]$ IS white.

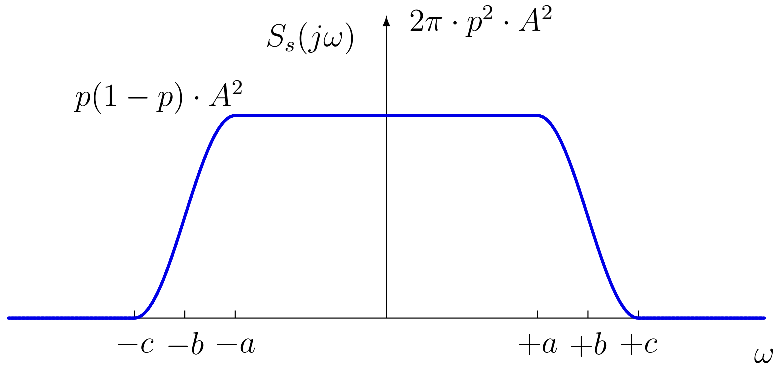
II) There exists ISI, because



Sampled noise $z[n]$ is white because the ambiguity function of $f(t) = f_b(t)$ is a triangle with support between $-T$ anx T, which sampled at symbol rate is a delta function.

Exercise 1.19 (Solution) a) $E_s = p \cdot A^2$.

b) $S_s(j\omega)$ is plotted in the figure:



Frequencies have the following values

$$a = \frac{\pi}{T} \cdot (1-\alpha),\ b = \frac{\pi}{T},\ c = \frac{\pi}{T} \cdot (1+\alpha),$$

Bandwidth of the modulated signal is

$$W = \pi \times 1{,}25 \text{ krad/s},$$

or alternatively

$$B = 625 \text{ Hz}.$$

Exercise 1.20 (Solution) a) Nyquist criterion for zero ISI:

$$p[n] = p(t)\big|_{t=nT} = p(nT) = C \times \delta[n] \leftrightarrow \frac{1}{T}\sum_k P\left(j\omega - j\frac{2\pi}{T}k\right) = C.$$

- Scenario (a):

$$T = \frac{t_0}{2} \text{ y para } T \geq t_0$$

$$R_{s|max} = \frac{1}{T_{min}} = \frac{2}{t_0} \text{ bauds.}$$

- Scenario (b): replicas of $P(j\omega)$ shifted $\frac{2\pi}{T}$ have to add a constant value.

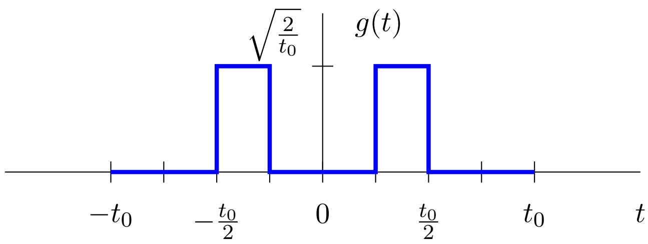
$$\frac{2\pi}{T} = \frac{W}{2} \rightarrow R_s = \frac{1}{T} = \frac{W}{4\pi} \text{ bauds.}$$

b) With an ideal channel, $h(t) = \delta(t)$:

I) Shaping pulse:

$$p(t) = g(t) * h(t) * f(t) = g(t) * \delta(t) * f(t) = g(t) * f(t) = g(t) * g(-t) \equiv r_g(t).$$

Therefore, the transmitter filter is the filter whose ambiguity function is the joint response $p(t)$ given in the figure, which is plotted below (it is valid this function, as well as any shifted version).

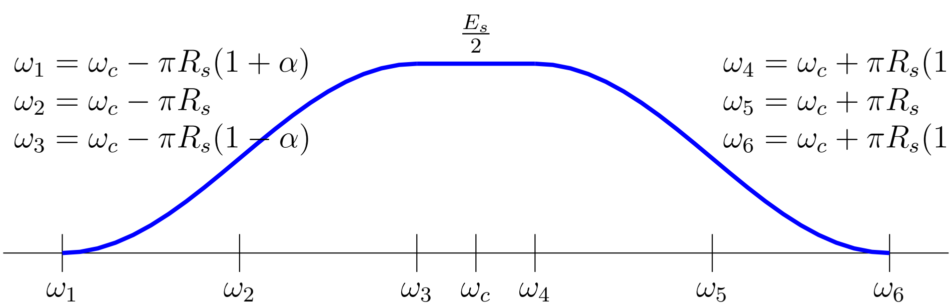


II) Sampled noise: $r_f[n] = C\,\delta[n]$, is satisfied for the values of T given in the first section.

Exercise 1.21 (Solution) a) I) If $\alpha = 0$: $f_c = 7{,}5$ kHz, $P_X = R_s \cdot E_s = 10$ kWatts, $B = R_s = 5$ kHz, and $M = 4$ symbols.

II) If $\alpha = 0{,}75$: $f_c = 7{,}5$ kHz, $P_X = R_s \cdot E_s = 25$ kWatts, $B = R_s(1+\alpha) = 4{,}375$ kHz, and $M = 16$ symbols.

b) Power spectral densitiy is plotted in the next figure:



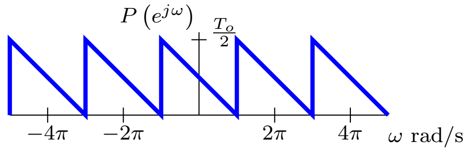
The equivalent frecuencies expressed in kHz are

$$f_1 = 5{,}3125,\ f_2 = 6{,}25,\ f_3 = 7{,}1875,\ f_c = 7{,}5,\ f_4 = 7{,}8125,\ f_5 = 8{,}75,\ f_6 = 9{,}6875 \text{ kHz}$$

c) There exists ISI. In this case, it is simpler to work in the frequency domain. In this domain que equivalent discrete channel is

$$P\left(e^{j\omega}\right)=\frac{1}{T}\sum_{k=-\infty}^{\infty}P\left(j\frac{\omega}{T}-j\frac{2\pi}{T}k\right)\neq C$$

where $P(j\omega)$ is Fourier transform of the joint transmitter-channel-receiver response.



Exercise 1.22 (Solution) a) Carrier frequency is

$$f_c = 11 \text{ MHz}$$

Transmitter filter is a root-raised cosine with roll-off factor $\alpha=0$, i.e., a sinc function

$$g(t)=h_{RRC}^{\alpha=0,T}(t)=\frac{1}{\sqrt{T}}\text{sinc}\left(\frac{t}{T}\right),$$

with $T=\frac{1}{R_s}=\frac{1}{B}=5\times10^{-7}$, i.e. $T=0{,}5\mu s$. Binary rate is

$$R_b = 8 \text{ Mbits/s}.$$

Noise $z[n]$ is white, because the ambiguity funciton of the receiver filter is

$$r_f(t)=r_g(t)=h_{RC}^{\alpha=0,T}(t)=\text{sinc}\left(\frac{t}{T}\right)$$

is afunction that sampled at symbol rate is a delta function, $r_f[n]=\delta[n]$.

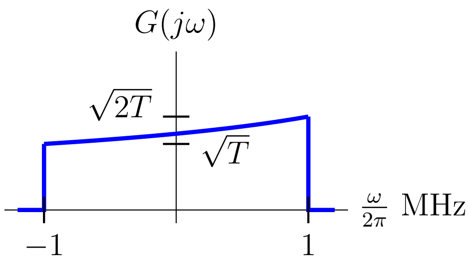
b) Carrier filter is

$$f_c = 11 \text{ MHz}$$

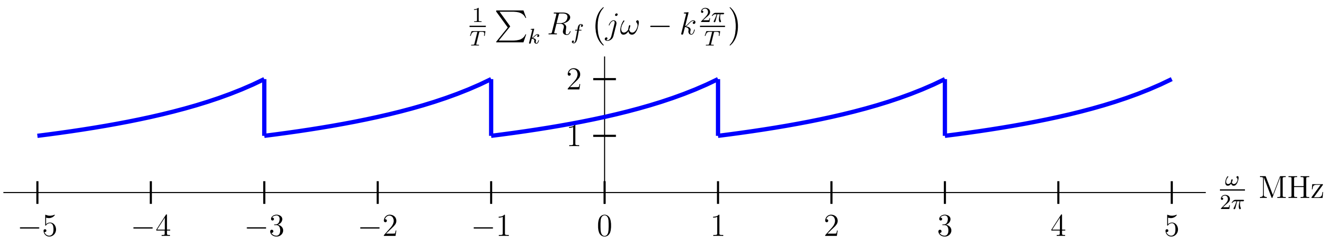
Transmitter filter is

$$g(t)=h_{RRC}^{\alpha=0{,}6,T}(t),\ \text{with } T=\frac{1}{R_s}=\frac{1}{1{,}25\times10^6}=0{,}8\ \mu s.$$

c) Transmitter filter, in the frequency domain, is the one shown in the figure



Noise $z[n]$ is not white, because



Exercise 1.23 (Solution) a) Minimum constellation order is $M = 8$ symbols (8-PSK constellation). Roll-off factor is

$$\alpha = 0{,}125.$$

Power is

$$P_S = 448 \text{ kWatt}$$

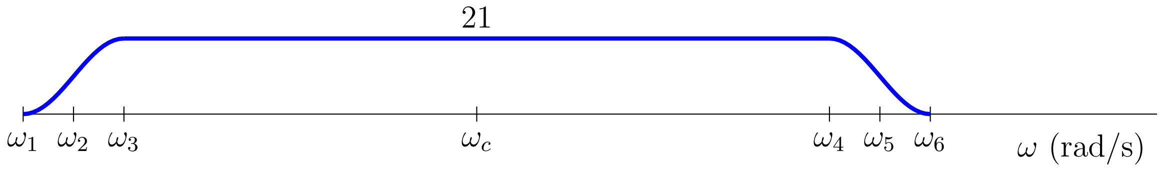
b) Carrier frequency

$$f_c = 56 \text{ kHz}$$

Minumum constellation order is $M = 64$ symbols. Roll-off factor

$$\alpha = 0{,}125.$$

Power spectral density



The equivalent frequencies in kHz are

$$f_1 = 50,\ f_2 = 50{,}666,\ f_3 = 51{,}333,\ f_c = 56,\ f_4 = 60{,}666,\ f_5 = 61{,}333,\ f_6 = 62 \text{ kHz}$$

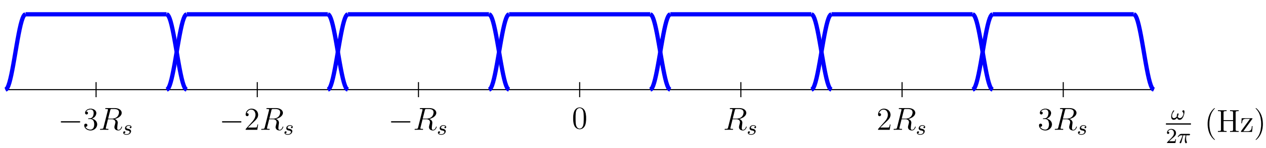
c) Noise $z[n]$ is white, because

$$r_f(t) = r_g(t) = h_{RC}^{\alpha,T}(t)$$

is a function that sampled at symbol rate is $r_f[n] = \delta[n]$, or equivalently

$$\frac{1}{T}\sum_k R_f\left(j\omega - \frac{2\pi}{T}k\right) = 1$$

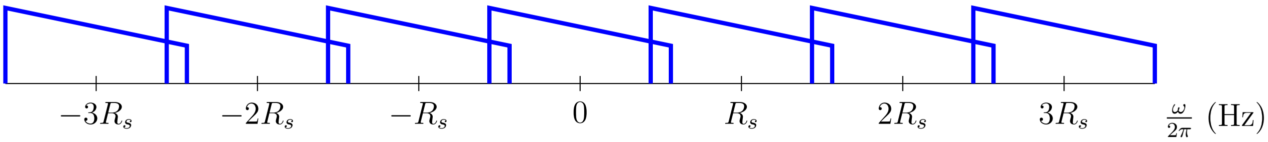
as illustrated in the figure



Intersymbol interference is present, because

$$\frac{1}{T}\sum_k P\left(j\omega - j\frac{2\pi}{T}k\right)$$

is the addition of the elements shown in the previous picture multiplied with the elements shown in the figure below


$-3R_s$
$-2R_s$
$-R_s$
0
R_s
$2R_s$
$3R_s$
$\frac{\omega}{2\pi}$ (Hz)