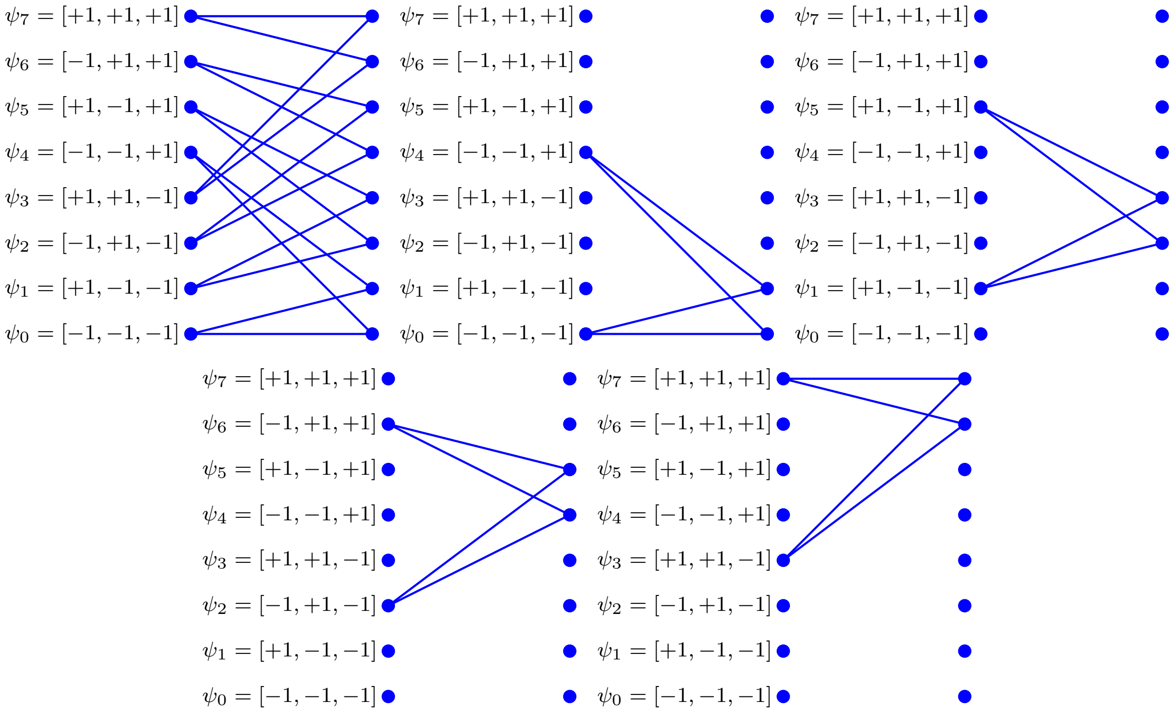
Chapter 2 : Solutions of the Exercises

Exercise 2.1 (Solution) a) The trellis diagram and the four butterflies are plotted in the figure



In this particular case, a butterfly connects the states with the following shape

$$[a, b, x] \to [y, a, b]$$

where a and b are fixed values, and $x, y \in \{\pm 1\}$.

b) In the general case, a butterfly connects the states

$$[A[n-1], A[n-2], \cdots, A[n-(K_p-1)], x] \to [y, A[n-1], A[n-2], \cdots, A[n-(K_p-1)]]$$

where $[A[n-1], A[n-2], \cdots, A[n-(K_p-1)]$ is fixed and with x and y taking all possible values of the alphabet of the transmitted constellation (the M possible values of $A[n-K_p]$ for x, and the M possible values of $A[n]$ for y).

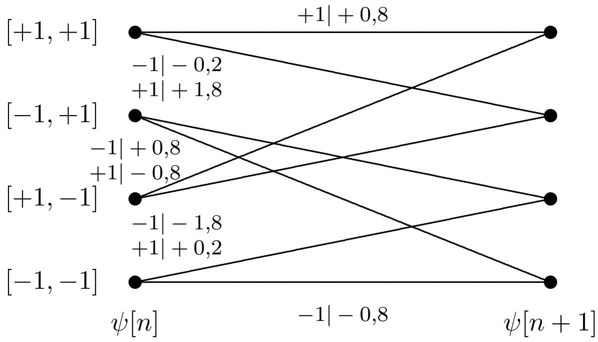
c) There are M nodes at the input (one for each value of x) and M nodes at the output (one for each value of y).

d) The number of butterflies is

$$\frac{M^{K_p}}{M} = M^{K_p-1}$$

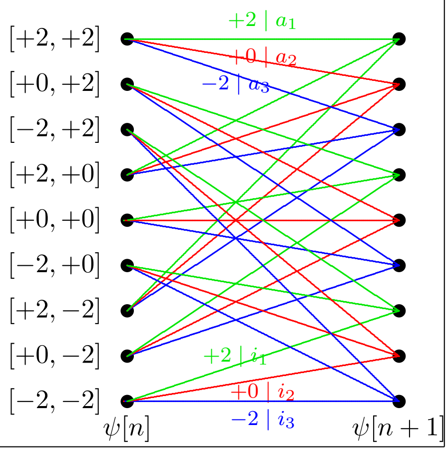
Exercise 2.2 (Solution) a) Noiseless output: $o[n] = \frac{1}{2}\, A[n] - \frac{1}{2}\, A[n-1] + 0{,}8\, A[n-2]$

$A[n]$	$A[n-1]$	$A[n-2]$	$o[n]$
$+1$	$+1$	$+1$	$+0{,}8$
-1	$+1$	$+1$	$-0{,}2$
$+1$	-1	$+1$	$+1{,}8$
-1	-1	$+1$	$+0{,}8$
$+1$	$+1$	-1	$-0{,}8$
-1	$+1$	-1	$-1{,}8$
$+1$	-1	-1	$+0{,}2$
-1	-1	-1	$-0{,}8$



b) For the constellation of errors $\xi[n] = A_i[n] - A_j[n]$
In this case $\xi[n] \in \{+2, 0, -2\}$ Now $o[n] = \frac{1}{2}\, \xi[n] - \frac{1}{2}\, \xi[n-1] + 0{,}8\, \xi[n-2]$

$\xi[n]$	$\xi[n-1]$	$\xi[n-2]$	$o[n]$
$+2$	$+2$	$+2$	$a_1 \equiv +1{,}6$
$+0$	$+2$	$+2$	$a_2 \equiv +0{,}6$
-2	$+2$	$+2$	$a_3 \equiv -0{,}4$
$+2$	$+0$	$+2$	$b_1 \equiv +2{,}6$
$+0$	$+0$	$+2$	$b_2 \equiv +1{,}6$
-2	$+0$	$+2$	$b_3 \equiv +0{,}6$
$+2$	-2	$+2$	$c_1 \equiv +3{,}6$
$+0$	-2	$+2$	$c_2 \equiv +2{,}6$
-2	-2	$+2$	$c_3 \equiv +1{,}6$
$+2$	$+2$	$+0$	$d_1 \equiv +0$
$+0$	$+2$	$+0$	$d_2 \equiv -1$
-2	$+2$	$+0$	$d_3 \equiv -2$
$+2$	$+0$	$+0$	$e_1 \equiv +1$
$+0$	$+0$	$+0$	$e_2 \equiv +0$
-2	$+0$	$+0$	$e_3 \equiv -1$
$+2$	-2	$+0$	$f_1 \equiv +2$
$+0$	-2	$+0$	$f_2 \equiv +1$
-2	-2	$+0$	$f_3 \equiv +0$
$+2$	$+2$	-2	$g_1 \equiv -1{,}6$
$+0$	$+2$	-2	$g_2 \equiv -2{,}6$
-2	$+2$	-2	$g_3 \equiv -3{,}6$
$+2$	$+0$	-2	$h_1 \equiv -0{,}6$
$+0$	$+0$	-2	$h_2 \equiv -1{,}6$
-2	$+0$	-2	$h_3 \equiv -2{,}6$
$+2$	-2	-2	$i_1 \equiv +0{,}4$
$+0$	-2	-2	$i_2 \equiv -0{,}6$
-2	-2	-2	$i_3 \equiv -1{,}6$



c) Minimum distance $D_{min} = \sqrt{3{,}92} = 1{,}9799$

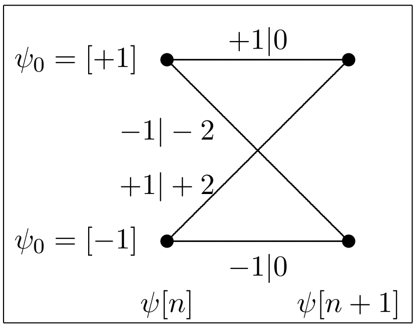
Exercise 2.3 (Solution) a) Noiseless constellation: $o[n] = A[n] * p[n] = A[n] - A[n-1]$

$A[n]$	$A[n-1]$	$o[n]$
$+1$	$+1$	0
-1	$+1$	-2
$+1$	-1	$+2$
-1	-1	0

Probability of error

$$P_e = \frac{1}{4} + \frac{1}{2} Q\left(\frac{2}{\sqrt{N_0/2}}\right)$$

b) Trellis diagram



c) Probability of error for sequence detector

$$P_e \approx k_0\, Q\left(\frac{\sqrt{2}}{\sqrt{N_0/2}}\right)$$

d) ZF equalizer without constraints

$$W\left(e^{j\omega}\right) = \frac{e^{-j\omega d}}{1-e^{-j\omega}}$$

$$P_e = Q(0) = \frac{1}{2}.$$

e) MMSE equalizer without constraints

$$W\left(e^{j\omega}\right) = \frac{(1-e^{j\omega})e^{-j\omega d}}{2{,}01 - 2\cos(\omega)}$$

$$P_e \approx Q\left(\frac{1}{0{,}2235}\right)$$

f) MMSE equalizer with 3 coefficients

$$\mathbf{w}_d^{MMSE} = \begin{bmatrix} w[0] \\ w[1] \\ w[2] \end{bmatrix} = \underbrace{\left(\mathbf{P}^H\mathbf{P} + \lambda\mathbf{I}\right)^{-1}\mathbf{P}^H}_{\mathbf{P}_\lambda^{\#}}\ \mathbf{c}_d$$

where $\lambda = 0{,}01$, and the matrices are

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{c}_d = \begin{bmatrix} c[0] \\ c[1] \\ c[2] \\ c[3] \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} p[0] & 0 & 0 \\ p[1] & p[0] & 0 \\ 0 & p[1] & p[0] \\ 0 & 0 & p[1] \end{bmatrix} = \begin{bmatrix} +1 & 0 & 0 \\ -1 & +1 & 0 \\ 0 & -1 & +1 \\ 0 & 0 & -1 \end{bmatrix}$$

Solving the system coefficients are

$$\mathbf{w}_d^{MMSE} = \begin{bmatrix} w[0] \\ w[1] \\ w[2] \end{bmatrix} = \begin{bmatrix} -0{,}2512 \\ +0{,}4951 \\ +0{,}2463 \end{bmatrix}$$

Coefficients of the joint channel-equalize response are

$$\mathbf{c} = \begin{bmatrix} -0{,}2512 \\ +0{,}7463 \\ -0{,}2488 \\ -0{,}2463 \end{bmatrix}$$

$$\sigma_{z'}^2 = \sigma_z^2 \times 0{,}3689 = 0{,}003689, \quad \sigma_{ISI}^2 = E_s \times 0{,}1857 = 0{,}1857$$

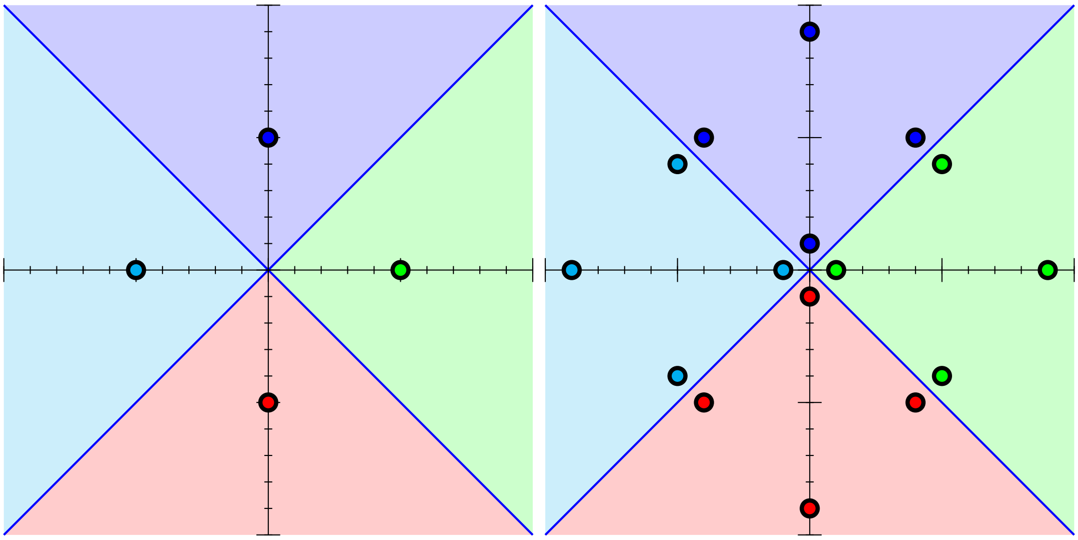
$$P_e \approx Q\left(\frac{0{,}7463}{\sqrt{0{,}1893}}\right)$$

Exercise 2.4 (Solution) a) A memoriless symbol-by-symbol detector

I) The noiseless output is

$A[n]$	$A[n-1]$	$o[n]$	$A[n]$	$A[n-1]$	$o[n]$
$+1$	$+1$	$+1+j0{,}8$	$+j$	$+1$	$+j1{,}8$
$+1$	-1	$+1-j0{,}8$	$+j$	-1	$+j0{,}2$
$+1$	$+j$	$+0{,}2$	$+j$	$+j$	$-0{,}8+j$
$+1$	$-j$	$+1{,}8$	$+j$	$-j$	$+0{,}8+j$
-1	$+1$	$-1+j0{,}8$	$-j$	$+1$	$-j0{,}2$
-1	-1	$-1-j0{,}8$	$-j$	-1	$-j1{,}8$
-1	$+j$	$-1{,}8$	$-j$	$+j$	$-0{,}8-j$
-1	$-j$	$-0{,}2$	$-j$	$-j$	$+0{,}8-j$

Received constellation, along with the trasmitted one, is shown in the figure



II) The probability of error is

$$P_e = \frac{1}{4}P_{e1} + \frac{1}{4}P_{e2} + \frac{1}{2}P_{e3}$$

with

$$P_{e1} = 2Q\left(\frac{0{,}2/\sqrt{2}}{\sqrt{N_0/2}}\right) - Q^2\left(\frac{0{,}2/\sqrt{2}}{\sqrt{N_0/2}}\right)$$

$$P_{e2} = 2Q\left(\frac{0{,}8/\sqrt{2}}{\sqrt{N_0/2}}\right) - Q^2\left(\frac{0{,}8/\sqrt{2}}{\sqrt{N_0/2}}\right)$$

$$P_{e3} = Q\left(\frac{0{,}2/\sqrt{2}}{\sqrt{N_0/2}}\right) + Q\left(\frac{0{,}8/\sqrt{2}}{\sqrt{N_0/2}}\right) - Q\left(\frac{0{,}2/\sqrt{2}}{\sqrt{N_0/2}}\right)\, Q\left(\frac{0{,}8/\sqrt{2}}{\sqrt{N_0/2}}\right)$$

b) ZF linear equalizer without constraints

I) Equalizer response

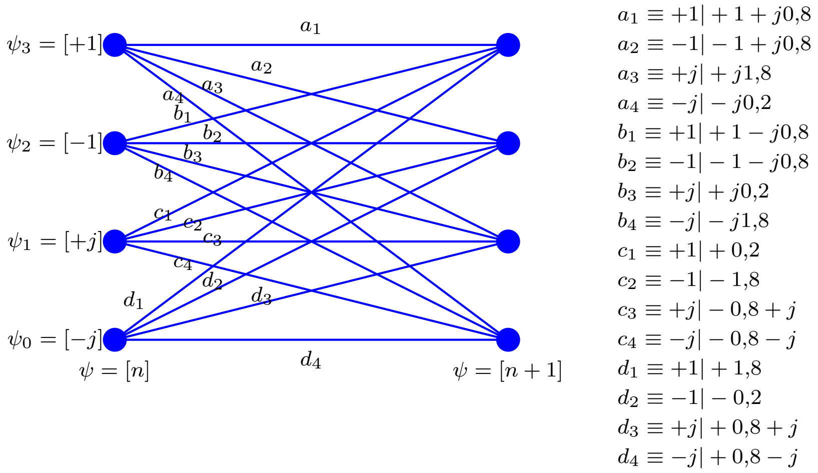
$$W\left(e^{j\omega}\right) = \frac{e^{-j\omega d}}{1+j0{,}8e^{-j\omega}}$$

II) Approximation for the probability of error: as $|P(e^{j\omega}|^2 = 1{,}64 + 1{,}6\,\mathrm{sen}(\omega)$ (squared real part plus squared imaginary part)

$$P_e \approx 2Q\left(\frac{\sqrt{2}}{2\sqrt{2{,}7778 N_0/2}}\right)$$

c) Maximum likelihood sequence detector

I) Trellis diagram is shown in the figure



Minimum euclidean distance between the noiseless outputs generated by two different sequences is

$$D_{min} = \sqrt{3{,}28}$$

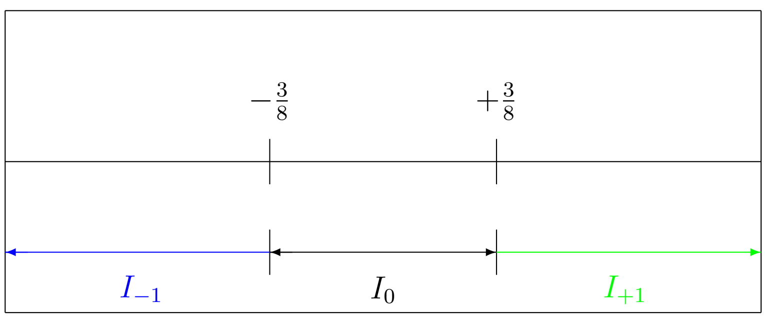
II) The approximated probability of error is

$$P_e \approx k_0 Q\left(\frac{\sqrt{3{,}28}}{2\sqrt{N_0/2}}\right)$$

Exercise 2.5 (Solution) a) Equivalent discrete channel

$$p[n] = \frac{3}{4}\,\delta[n] - \frac{1}{4}\,\delta[n-1].$$

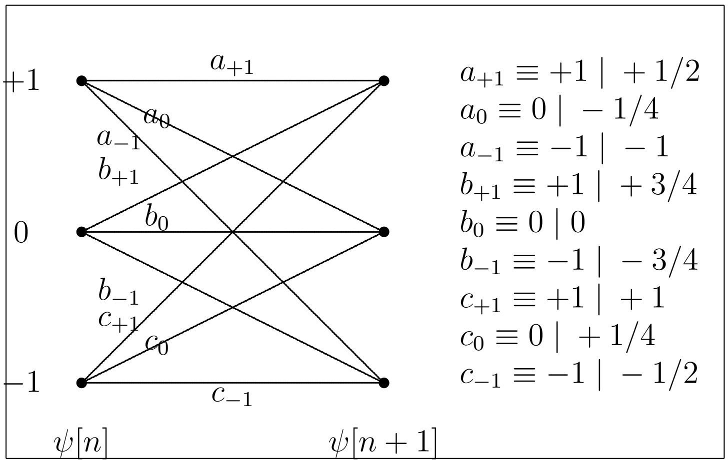
b) Memoryless symbol-by-symbol detector: delay $d = 0$, and decision regions



Probability of error

$$P_e = \frac{4}{9} Q\left(\frac{1/8}{\sqrt{N_0/2}}\right) + \frac{4}{9} Q\left(\frac{3/8}{\sqrt{N_0/2}}\right) + \frac{4}{9} Q\left(\frac{1/8}{\sqrt{N_0/2}}\right).$$

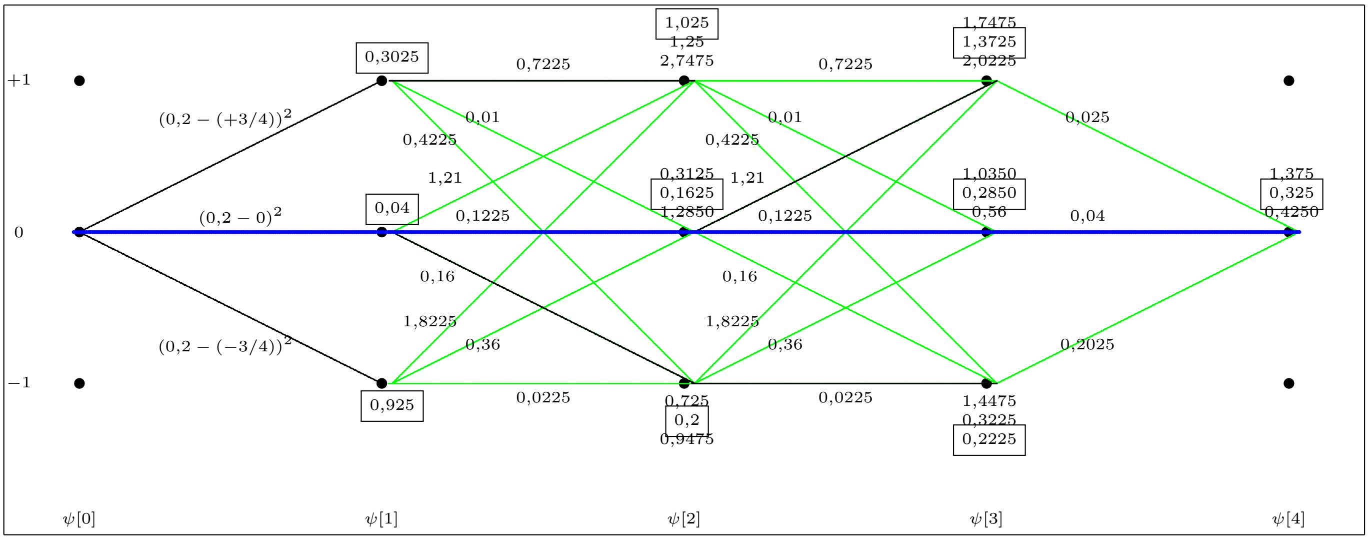
c) Trellis diagram



Approximated probability of error

$$p_e \approx k_0 \, Q\left(\frac{0{,}3953}{\sqrt{N_0/2}}\right).$$

d) Viterbi's algorithm



Decoded sequence

$$\hat{A}[0] = 0,\ \hat{A}[1] = 0,\ \hat{A}[2] = 0.$$

Exercise 2.6 (Solution) a) ZF linear equalizer without constraints and performance

$$W\left(e^{j\omega}\right) = \frac{e^{-j\omega d}}{\frac{1}{2} - e^{-j\omega} + \frac{1}{2}e^{-j2\omega}}$$

$$P_e = Q(0) = \frac{1}{2}.$$

b) ZF equalizer with 2 coefficients

$$\mathbf{w}_d^{ZF} = \begin{bmatrix} w[0] \\ w[1] \end{bmatrix} = \underbrace{\left(\mathbf{P}^H\mathbf{P}\right)^{-1}\mathbf{P}^H}_{\mathbf{P}^{\#}} \, \mathbf{c}_d$$

where

$$\mathbf{c}_d = \begin{bmatrix} c[0] \\ c[1] \\ c[2] \\ c[3] \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} 1/2 & 0 \\ -1 & 1/2 \\ 1/2 & -1 \\ 0 & 1/2 \end{bmatrix}$$

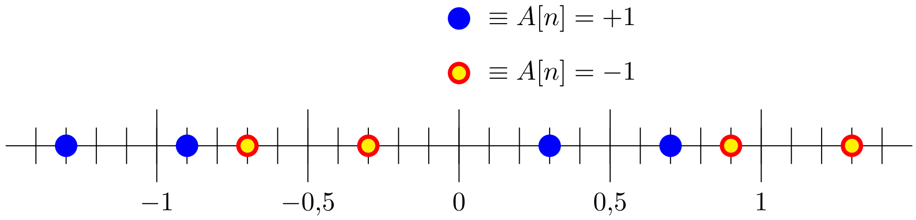
Solving the system, the coefficients are

$$\mathbf{w}_d^{ZF} = \begin{bmatrix} w[0] \\ w[1] \end{bmatrix} = \begin{bmatrix} -0{,}8 \\ -0{,}2 \end{bmatrix}$$

Exercise 2.7 (Solution) a) Memoryless symbol-by-symbol detector

i) Noiseless output constellation

$A[n]$	$A[n-1]$	$A[n-2]$	$o[n]$
$+1$	$+1$	$+1$	$+0{,}3$
$+1$	$+1$	-1	$+0{,}7$
$+1$	-1	$+1$	$-1{,}3$
$+1$	-1	-1	$-0{,}9$
-1	$+1$	$+1$	$+0{,}9$
-1	$+1$	-1	$+1{,}3$
-1	-1	$+1$	$-0{,}7$
-1	-1	-1	$-0{,}3$



ii) Conditional power spectral density of the observation for $A[n] = +1$

$$f_{q[n]|A[n]=+1} = \frac{1}{4}\mathcal{N}(+0{,}3, \sigma_z^2) + \frac{1}{4}\mathcal{N}(+0{,}7, \sigma_z^2) + \frac{1}{4}\mathcal{N}(-0{,}9, \sigma_z^2) + \frac{1}{4}\mathcal{N}(-1{,}3, \sigma_z^2)$$

iii) Conditional power spectral density of the observation for $A[n] = -1$

$$f_{q[n]|A[n]=-1} = \frac{1}{4}\mathcal{N}(-0{,}3, \sigma_z^2) + \frac{1}{4}\mathcal{N}(-0{,}7, \sigma_z^2) + \frac{1}{4}\mathcal{N}(+0{,}9, \sigma_z^2) + \frac{1}{4}\mathcal{N}(+1{,}3, \sigma_z^2)$$

iv) Deciding $\hat{A}[n]$, the probability of error is

$$P_e = \frac{1}{4}Q\left(\frac{+0{,}3}{\sigma_z}\right) + \frac{1}{4}Q\left(\frac{+0{,}7}{\sigma_z}\right)\frac{1}{4}\left[1 - Q\left(\frac{+0{,}9}{\sigma_z}\right)\right] + \frac{1}{4}\left[1 - Q\left(\frac{+1{,}3}{\sigma_z}\right)\right]$$

Deciding $\hat{A}[n-1]$, now the probability of error is

$$P_e = \frac{1}{4}Q\left(\frac{+0{,}3}{\sigma_z}\right) + \frac{1}{4}Q\left(\frac{+0{,}7}{\sigma_z}\right)\frac{1}{4}Q\left(\frac{+0{,}9}{\sigma_z}\right) + \frac{1}{4}Q\left(\frac{+1{,}3}{\sigma_z}\right)$$

Performance is better (lower probability of error) deciding $\hat{A}[n-1]$ because $d = 1$ is the optimal delay for this channel (the maximum value of $|p[n]|$ is at $n = 1$). Moreover, for a delay $d = 0$, given that $p[0]$ is negative, to obtain the best performance it is necessary to invert the decision regions.

b) Linear equalizer

I) The coefficients of the equalizer are obtained solving the system

$$\mathbf{w}_d^{ZF} = \begin{bmatrix} w[0] \\ w[1] \\ w[2] \end{bmatrix} = \underbrace{\left(\mathbf{P}^H\mathbf{P}\right)^{-1}\mathbf{P}^H}_{\mathbf{P}^\#}\ \mathbf{c}_d$$

where the channel matrix does not depend on the delay

$$\mathbf{P} = \begin{bmatrix} -0{,}3 & 0 & 0 \\ +0{,}8 & -0{,}3 & 0 \\ -0{,}2 & +0{,}8 & -0{,}3 \\ 0 & -0{,}2 & +0{,}8 \\ 0 & 0 & -0{,}2 \end{bmatrix}$$

and the joint channel-equalizer vector are

$$\mathbf{c}_d = \begin{bmatrix} c[0] \\ c[1] \\ c[2] \\ c[3] \\ c[4] \end{bmatrix}, \text{ with } \mathbf{c}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ for } d = 0, \text{ and } \mathbf{c}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ for } d = 1.$$

If the system is solved, coefficients are

$$\mathbf{w}_d^{ZF} = \begin{bmatrix} w[0] \\ w[1] \\ w[2] \end{bmatrix}, \text{ con } \mathbf{w}_0^{ZF} = \begin{bmatrix} -0{,}5732 \\ -0{,}3761 \\ -0{,}1507 \end{bmatrix} \text{ y } \mathbf{w}_1^{ZF} = \begin{bmatrix} 1{,}1525 \\ 0{,}2225 \\ 0{,}0258 \end{bmatrix}$$

II) Power of filtered noise is

$$\sigma_{z'}^2 = \sigma_z^2 \times 1{,}3713$$

Power of the residual ISI is

$$\sigma_{ISI}^2 = 0{,}1234$$

Exercise 2.8 (Solution) a) Optimal delay $d = 1$, and probability of error

$$P_e = \frac{1}{4}Q\left(\frac{3/2}{\sqrt{N_0/2}}\right) + \frac{1}{2}Q\left(\frac{1}{\sqrt{N_0/2}}\right) + \frac{1}{4}Q\left(\frac{1/2}{\sqrt{N_0/2}}\right)$$

b) Unconstrained ZF equalizer

$$W\left(e^{j\omega}\right) = \frac{e^{-j\omega d}}{\frac{1}{4} + e^{-j\omega} - \frac{1}{4}e^{-j2\omega}}$$

Parameter d is the minimum necessary delay to make causal the inverse Fourier transform of the equalizer response. Probability of error (approximation) is

$$P_e \approx= Q\left(\frac{1}{0{,}08944}\right).$$

c) Constrained ZF equalizer

$$\mathbf{w}_d^{ZF} = \begin{bmatrix} w[0] \\ w[1] \\ w[2] \end{bmatrix} = \underbrace{\left(\mathbf{P}^H\mathbf{P}\right)^{-1}\mathbf{P}^H}_{\mathbf{P}^\#} \cdot \mathbf{c}_d.$$

Constrained MMSE equalizer

$$\mathbf{w}_d^{MMSE} = \begin{bmatrix} w[0] \\ w[1] \\ w[2] \end{bmatrix} = \underbrace{\left(\mathbf{P}^H\mathbf{P} + \lambda\mathbf{I}\right)^{-1}\mathbf{P}^H}_{\mathbf{P}_\lambda^{\#}} \cdot \mathbf{c}_d,$$

where in this case $\lambda = 2 \times 10^{-3}$ and where

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \mathbf{c}_d = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{P} = \begin{bmatrix} +1/4 & 0 & 0 \\ +1 & +1/4 & 0 \\ -1/4 & +1 & +1/4 \\ 0 & -1/4 & +1 \\ 0 & 0 & -1/4 \end{bmatrix}$$

Exercise 2.9 (Solution) a) Received constellation for the two channels are, respectively

For $p_1[n]$

$A[n]$	$A[n-1]$	$o[n]$
+1	+1	+0,1
−1	+1	−0,3
+1	−1	+0,3
−1	−1	−0,1

and for $p_2[n]$

$A[n]$	$A[n-1]$	$A[n-2]$	$o[n]$
+1	+1	+1	+0,1
−1	+1	+1	−0,3
+1	−1	+1	+0,7
−1	−1	+1	+0,3
+1	+1	−1	−0,3
−1	+1	−1	−0,7
+1	−1	−1	+0,3
−1	−1	−1	−0,1

b) Probability of error for $p_1[n]$ is

$$P_e = \frac{1}{2}Q\left(\frac{+0{,}1}{\sigma_z}\right) + \frac{1}{2}Q\left(\frac{+0{,}3}{\sigma_z}\right)$$

with $\sigma_z = \sqrt{N_0/2}$, and for $p_2[n]$

$$P_e = \frac{1}{4}Q\left(\frac{+0{,}1}{\sigma_z}\right)+\frac{1}{4}Q\left(\frac{+0{,}3}{\sigma_z}\right)+\frac{1}{4}Q\left(\frac{+0{,}7}{\sigma_z}\right)+\frac{1}{4}\left[1-Q\left(\frac{0{,}3}{\sigma_z}\right)\right] = \frac{1}{4}+\frac{1}{4}Q\left(\frac{+0{,}1}{\sigma_z}\right)+\frac{1}{4}Q\left(\frac{+0{,}7}{\sigma_z}\right)$$

Performance is worse for second channel, because for this channel the optimal delay for decisions is $d = 1$.

c) Channels are

$$p_1[n] = 0{,}2\delta[n] - 0{,}1\delta[n-1], \qquad p_2[n] = 0{,}2\delta[n] - 0{,}3\delta[n-1] + 0{,}2\delta[n-2].$$

d) Approximated probability of error is

$$P_e \approx k_0 Q\left(\frac{D_{min}}{2\sigma_z}\right),$$

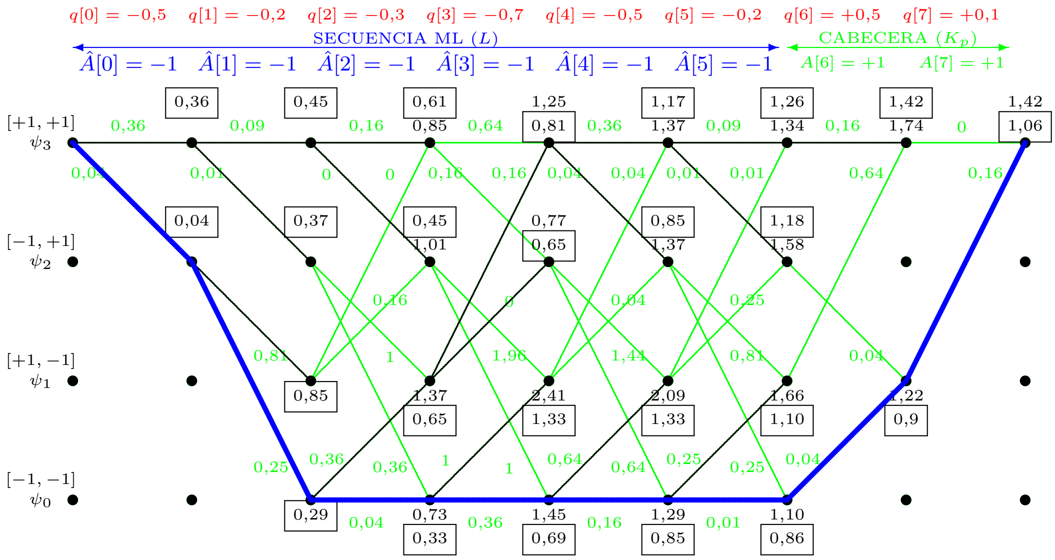
where $D_{min} = 0{,}4472$ for first channel, and $D_{min} = 0{,}6325$ for the second channel. Using a maximum likelihood sequence detector, performance will be better in the second channel.

e) Viterbi's algorithm is applied. The estimated sequence is

$$\hat{A}[0] = -1, \quad \hat{A}[1] = -1, \quad \hat{A}[2] = -1, \quad \hat{A}[3] = -1, \quad \hat{A}[4] = -1, \quad \hat{A}[5] = -1.$$

Details of the application of Viterbi's algorithm are shown in the figure:

- Green: branches to be considered during decoding (and corresponding branch metric)
- Black: survival paths for every state
- Blue: path associated to the maximum likelihood sequence



Exercise 2.10 (Solution) a) Equivalent discrete channel is

$$p[n] = \delta[n] - 2\,\delta[n-1] + \frac{1}{2}\,\delta[n-2].$$

b) Probability of error is

$$P_e = \frac{1}{4}Q\left(\frac{+0{,}5}{\sqrt{N_0/2}}\right) + \frac{1}{4}Q\left(\frac{1{,}5}{\sqrt{N_0/2}}\right) + \frac{1}{4}Q\left(\frac{2{,}5}{\sqrt{N_0/2}}\right) + \frac{1}{4}Q\left(\frac{3{,}5}{\sqrt{N_0/2}}\right).$$

c) ZF equalizer with 3 coefficients is obtained solving the system

$$\mathbf{w}_d^{ZF} = \begin{bmatrix} w[0] \\ w[1] \\ w[2] \end{bmatrix} = \underbrace{\left(\mathbf{P}^H\mathbf{P}\right)^{-1}\mathbf{P}^H}_{\mathbf{P}^{\#}}\ \mathbf{c}_d$$

where channel matrix and joint channel-equalizer response are

$$\mathbf{P} = \begin{bmatrix} +1 & 0 & 0 \\ -2 & +1 & 0 \\ +1/2 & -2 & +1 \\ 0 & +1/2 & -2 \\ 0 & 0 & +1/2 \end{bmatrix},\quad \mathbf{c}_d = \begin{bmatrix} c[0] \\ c[1] \\ c[2] \\ c[3] \\ c[4] \end{bmatrix},\ \text{con } \mathbf{c}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\ \text{para } d = 1.$$

Solving the system (it is not necessary) coefficients are

$$\mathbf{w}_d^{ZF} = \begin{bmatrix} w[0] \\ w[1] \\ w[2] \end{bmatrix},\ \text{con } \mathbf{w}_1^{ZF} = \begin{bmatrix} -0{,}3951 \\ -0{,}0205 \\ +0{,}0259 \end{bmatrix}$$

d) The MMSE equalizer is now

$$\mathbf{w}_d^{MMSE} = \begin{bmatrix} w[0] \\ w[1] \\ w[2] \end{bmatrix} = \underbrace{\left(\mathbf{P}^H\mathbf{P} + \lambda\mathbf{I}\right)^{-1}\mathbf{P}^H}_{\mathbf{P}_\lambda^{\#}}\ \mathbf{c}_d$$

where channel matrix is the same one as in previous section and now

$$\mathbf{c}_d = \begin{bmatrix} c[0] \\ c[1] \\ c[2] \\ c[3] \\ c[4] \end{bmatrix}, \text{ con } \mathbf{c}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ para } d=3,\ \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and regularization parameter is $\lambda = 0{,}01$.

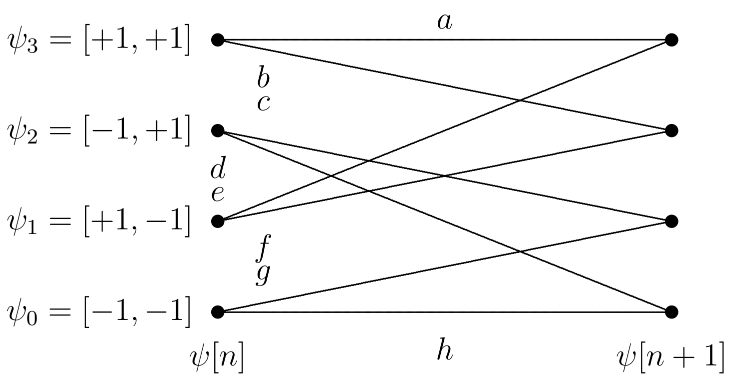
e) Approximated probability of error is

$$P_e \approx Q\left(\frac{1}{\sqrt{0{,}0966}}\right)$$

Exercise 2.11 (Solution) a) Channel coefficients are:

$$p[0] = +\frac{1}{2},\ p[1] = -0{,}7,\ p[2] = +0{,}1.$$

The trellis diagram is



$a \equiv +1| - 0{,}1$
$b \equiv -1| - 1{,}1$
$c \equiv +1| + 1{,}3$
$d \equiv -1| + 0{,}3$
$e \equiv +1| - 0{,}3$
$f \equiv -1| - 1{,}3$
$g \equiv +1| + 1{,}1$
$h \equiv -1| + 0{,}1$

b) The values of symbols for sequences are

$$\mathbf{A}_1 = \{+1,\ -1,\ +1,\ +1\} \qquad \mathbf{A}_2 = \{-1,\ -1,\ -1,\ -1\}.$$

c) The probability of error, for a delay$d = 0$

$$P_e = \frac{1}{4}\left(1 - Q\left(\frac{0{,}1}{\sqrt{N_0/2}}\right)\right) + \frac{1}{4}Q\left(\frac{1{,}3}{\sqrt{N_0/2}}\right) + \frac{1}{4}\left(1 - Q\left(\frac{0{,}3}{\sqrt{N_0/2}}\right)\right) + \frac{1}{4}Q\left(\frac{1{,}1}{\sqrt{N_0/2}}\right).$$

d) Equalizer is obtained by solving

$$\mathbf{w}_d^{ZF} = \mathbf{P}^{\#}\mathbf{c}_d,$$

where operator $^{\#}$ denotes pseudo-inverse of a matrix, which is defined as

$$\mathbf{P}^{\#} = \left(\mathbf{P}^H\mathbf{P}\right)^{-1}\mathbf{P}^H,$$

where ^{H} denotes the hermitian operator (transposition + conjugation). In this particular problem, the values are

$$\mathbf{P} = \begin{bmatrix} p[0] & 0 \\ p[1] & p[0] \\ p[2] & p[1] \\ 0 & p[2] \end{bmatrix},\ \mathbf{c}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ y } \mathbf{c}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Solving the system

$$\mathbf{P}^H\mathbf{P} = \begin{bmatrix} +0{,}75 & -0{,}42 \\ -0{,}42 & 0{,}75 \end{bmatrix},\ \left(\mathbf{P}^H\mathbf{P}\right)^{-1} = \begin{bmatrix} 1{,}9425 & 1{,}0878 \\ 1{,}0878 & 1{,}9425 \end{bmatrix},$$

$$\mathbf{P}^\# = \begin{bmatrix} +0{,}9713 & -0{,}8159 & -0{,}5672 & +0{,}1088 \\ +0{,}5439 & +0{,}2098 & -1{,}2510 & +0{,}1943 \end{bmatrix},$$

and therefore

$$\mathbf{w}_0^{ZF} = \begin{bmatrix} +0{,}9713 \\ +0{,}5439 \end{bmatrix} \text{ for } d=0, \text{ and } \mathbf{w}_1^{ZF} = \begin{bmatrix} -0{,}8159 \\ +0{,}2098 \end{bmatrix} \text{ for } d=1.$$

e) For the specified values of n and with coefficients obtained for $d = 1$

$$u[1] = o[1]w[0] + o[0]w[1] = 0{,}8765$$

$$u[2] = o[2]w[0] + o[1]w[1] = 0{,}8765$$

It can be seen that the exact values of the transmitted sequence are not recovered. This happens because with only two coefficients, the joint channel-equalizer response is different from the ideal one. In particular, for $d = 1$, coefficients of this joint response are

$$\mathbf{c} = \mathbf{P}\mathbf{w} = \begin{bmatrix} -0{,}4079 \\ +0{,}6760 \\ -0{,}2284 \\ +0{,}0210 \end{bmatrix},$$

which means the the level of residual ISI is relatively high.

Exercise 2.12 (Solution) a) Optimal delay is $\boxed{d=1}$, and the probability of error is

$$P_e == \frac{1}{2}Q\left(\frac{1/2}{\sqrt{N_0/2}}\right) + \frac{1}{2}Q\left(\frac{2}{\sqrt{N_0/2}}\right).$$

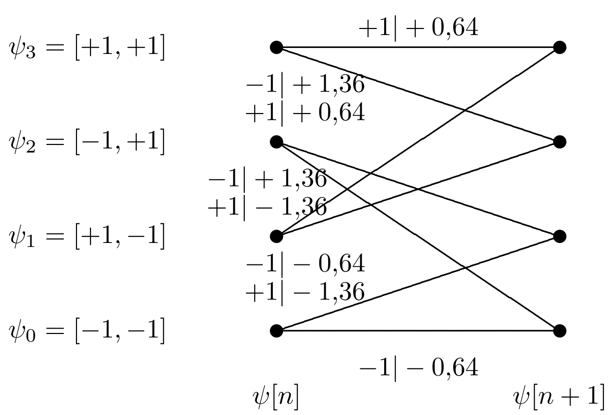
b) Now optimal delay is $\boxed{d=2}$. The exact probabiilty of error is

$$P_e == \frac{1}{2}Q\left(\frac{0{,}64}{\sigma_{z'}}\right) + \frac{1}{2}Q\left(\frac{1{,}36}{\sigma_{z'}}\right).$$

where

$$\sigma_{z'}^2 = 0{,}8704 \times \frac{N_0}{2}.$$

c) Trellis diagram is shown in the following figure

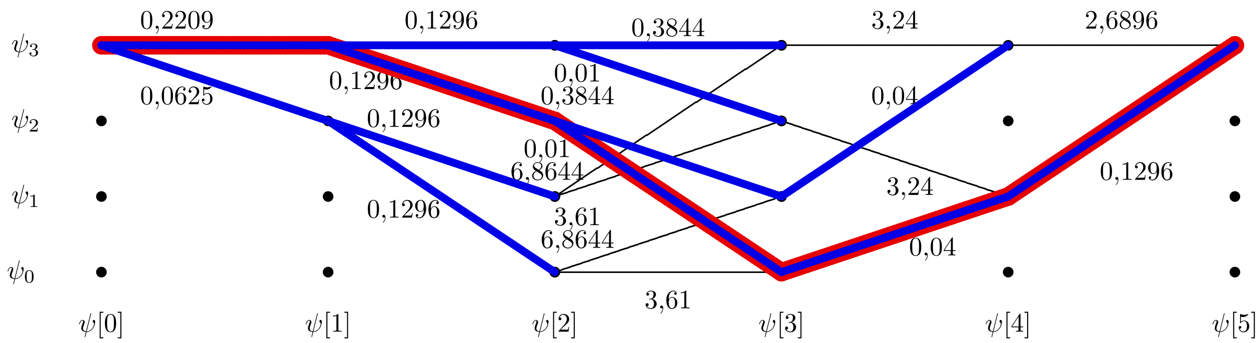


Probability of error can be approximated by

$$P_e \approx k_0 \, Q\left(\frac{D_{min}}{\sqrt{N_0/2}}\right),$$

where $D_{min} = sqrt4{,}5184$

d) To decode this sequence of observations, Viterbi's algorithm is used. The figure show the branch metrics, the survival paths and the path for maximum likelihood sequence. Accumulated metrics for each state are also provided. Boldface remarks the metric of the survival path.



	$\psi[1]$	$\psi[2]$	$\psi[3]$	$\psi[4]$	$\psi[5]$
ψ_3	**0.2209**	**0.3505**	**0.7349**/7.0565	3.9749/**0.7749**	3.4645/**0.5301**
ψ_2	**0.0625**	**0.3525**	**0.3605**/3.9605		
ψ_1		**0.1921**	**0.7349**/7.0565	3.6005/**0.4005**	
ψ_0		**0.1921**	**0.3605**/3.9605		

Therefore, the solution (maximum likelihood sequence) is:

$$\hat{A}[0] = +1, \; \hat{A}[1] = -1, \; \hat{A}[2] = -1.$$

Exercise 2.13 (Solution) a) Taking into account the values of the two equivalent discrete channels, the noiseless output is

$$o[n] = A_1[n] * p_1[n] + A_2[n] * p_2[n] = 0{,}9 \; A_1[n] - 0{,}1 \; A_1[n-1] + 0{,}8 \; A[n] - 0{,}2 \; A_2[n-1]$$

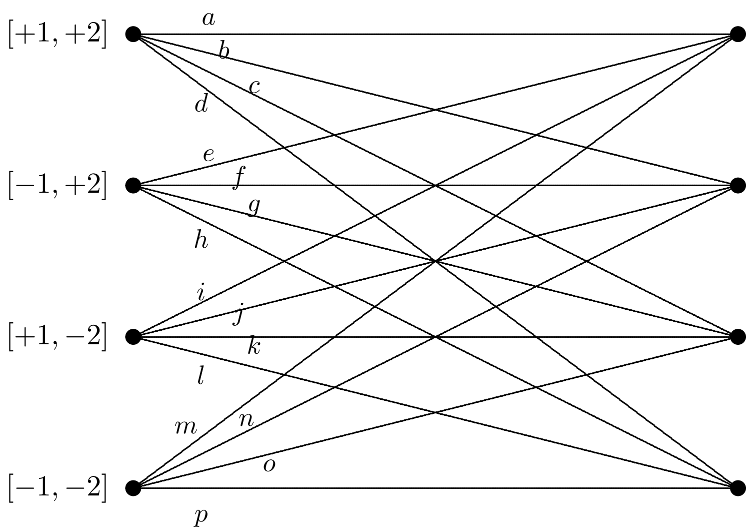
This output depends on the values of the two transmitted data sequences. The 16 possible values are shown in the table

$A_1[n]$	$A_2[n]$	$A_1[n-1]$	$A_2[n-1]$	$o[n]$	$A_1[n]$	$A_2[n]$	$A_1[n-1]$	$A_2[n-1]$	$o[n]$
+1	+2	+1	+2	$a \equiv +2{,}0$	+1	+2	+1	−2	$i \equiv +2{,}8$
−1	+2	+1	+2	$b \equiv +0{,}2$	−1	+2	+1	−2	$j \equiv +1{,}0$
+1	−2	+1	+2	$c \equiv -1{,}2$	+1	−2	+1	−2	$k \equiv -0{,}4$
−1	−2	+1	+2	$d \equiv -3{,}0$	−1	−2	+1	−2	$l \equiv -2{,}2$
+1	+2	−1	+2	$e \equiv +2{,}2$	+1	+2	−1	−2	$m \equiv +3{,}0$
−1	+2	−1	+2	$f \equiv +0{,}4$	−1	+2	−1	−2	$n \equiv +1{,}2$
+1	−2	−1	+2	$g \equiv -1{,}0$	+1	−2	−1	−2	$o \equiv -0{,}2$
−1	−2	−1	+2	$h \equiv -2{,}8$	−1	−2	−1	−2	$p \equiv -2{,}0$

If both sequences are decoded simultaneously, the system has two inputs, $A_1[n]$ and $A_2[n]$ (columns 1 and 2 in the table), and one output, $o[n]$, being the state of the system

$$\psi[n] = [A_1[n-1], A_2[n-1]]$$

which means that $\psi[n]$ corresponds with columns 3 and 4 in the table, and that $\psi[n+1]$ corresponds with columns 1 and 2. Therefore, the trellis diagram is shown in the figure



$a \equiv +1, +2| + 2{,}0$
$b \equiv -1, +2| + 0{,}2$
$c \equiv +1, -2| - 1{,}2$
$d \equiv -1, -2| - 3{,}0$

$e \equiv +1, +2| + 2{,}2$
$f \equiv -1, +2| + 0{,}4$
$g \equiv +1, -2| - 1{,}0$
$h \equiv -1, -2| - 2{,}8$

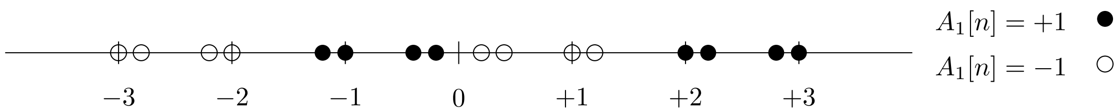
$i \equiv +1, +2| + 2{,}8$
$j \equiv -1, +2| + 1{,}0$
$k \equiv +1, -2| - 0{,}4$
$l \equiv -1, -2| - 2{,}2$

$m \equiv +1, +2| + 3{,}0$
$n \equiv -1, +2| + 1{,}2$
$o \equiv +1, -2| - 0{,}2$
$p \equiv -1, -2| - 2{,}0$

b) To evaluate the probability of error for user 1, with a delay $d = 0$, the conditional probabilities of error are averaged

$$P_{e1} = \frac{1}{2} P_{e|A_1[n]=+1} + \frac{1}{2} P_{e|A_1[n]=-1}.$$

To compute these conditional probabilities, it is necessary to know which points of the extended constellation are obtained given that $A_1[n] = +1$ or $A_1[n] = -1$. This points are shown in the figure



Conditional probability of error is

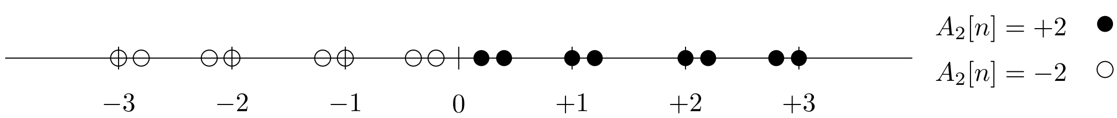
$$P_{e|A_1[n]=+1} = \frac{1}{8} \Big[(1 - Q(0{,}2)) + (1 - Q(0{,}4)) + (1 - Q(1)) + (1 - Q(1{,}2)) + Q(2) + Q(2{,}2) + Q(2{,}8) + Q(3) \Big]$$

In this case $P_{e|A_1[n]=-1} = P_{e|A_1[n]=+1}$, and therefore $P_{e1} = P_{e|A_1[n]=+1}$.

Probability of error for user 2, with delay $d = 0$, is

$$P_{e2} = \frac{1}{2} P_{e|A_2[n]=+2} + \frac{1}{2} P_{e|A_2[n]=-2}.$$

To compute these conditional probabilities, it is necessary to know which points of the extended constellation are obtained given that $A_2[n] = +2$ or $A_2[n] = -2$. This points are shown in the figure



Conditional probability of error is

$$P_{e|A_2[n]=+2} = \frac{1}{8}\left[Q\left(0{,}2\right)+Q\left(0{,}4\right)+Q\left(1\right)+Q\left(1{,}2\right)+Q\left(2\right)+Q\left(2{,}2\right)+Q\left(2{,}8\right)+Q\left(3\right)\right]$$

In this case $P_{e|A_2[n]=-2} = P_{e|A_2[n]=+2}$, and therefore $P_{e2} = P_{e|A_2[n]=+2}$.

Clearly, $P_{e1} > P_{e2}$, because the energy of the constellation is higher for sequence $A_2[n]$, which makes higher its contribution to the noiseless output.

c) Now the ZF equalizer with 2 coefficients is obtained by solving

$$\mathbf{w} = \underbrace{(\mathbf{P}_1^H\mathbf{P}_1)^{-1}\mathbf{P}_1^H}_{\mathbf{P}_1^\sharp}\ \mathbf{c}_{d1},$$

where

$$\mathbf{P}_1 = \begin{bmatrix} p_1[0] & 0 \\ p_1[1] & p_1[0] \\ 0 & p_1[1] \end{bmatrix} = \begin{bmatrix} +0{,}9 & 0 \\ -0{,}1 & +0{,}9 \\ 0 & -0{,}1 \end{bmatrix},\ \mathbf{c}_{d1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The solution is

$$\mathbf{w} = \begin{bmatrix} +1{,}1109 \\ +0{,}1219 \end{bmatrix}.$$

Now, joint response between channel $p_1[n]$ and the equalizer is

$$c[n] = p_1[n] * w[n] = \sum_{k=0}^{K_w} w[k]\ p_1[n-k].$$

In vectorial notation

$$\mathbf{c}_1 = \mathbf{P}_1\ \mathbf{w} = \begin{bmatrix} +1 \\ -0{,}0014 \\ -0{,}0122 \end{bmatrix}.$$

It can be seen that residual ISI is relatively low.

Taking into account that channel matrix for user 2 is

$$\mathbf{P}_2 = \begin{bmatrix} p_2[0] & 0 \\ p_2[1] & p_2[0] \\ 0 & p_2[1] \end{bmatrix} = \begin{bmatrix} +0{,}8 & 0 \\ -0{,}2 & +0{,}8 \\ 0 & -0{,}2 \end{bmatrix},$$

joint response between channel$p_2[n]$ and equalizer is

$$\mathbf{c}_2 = \mathbf{P}_2\ \mathbf{w} = \begin{bmatrix} +0{,}8888 \\ -0{,}1246 \\ -0{,}0244 \end{bmatrix}.$$

When there is a single user

$$u[n] = \underbrace{c[d]}_{\text{gain}}\ A[n-d] + \underbrace{\sum_{\substack{k=0 \\ k\neq d}}^{K+K_w} c[k]\ A[n-k]}_{\text{residual ISI}} + \underbrace{\sum_{k=0}^{K_w} w[k]\ z[n-k]}_{\text{filtered noise } z'[n]}$$

The approximated proabbility of error is

$$P_e \approx k\ Q\left(\frac{d_{min}\ |c[d]|}{2\sqrt{\sigma_{z'}^2+\sigma_{ISI}^2}}\right)$$

where

$$\sigma_{z'}^2 = \sigma_z^2 \sum_{k=0}^{K_w} |w[k]|^2$$

and

$$\sigma_{ISI}^2 = E_s \sum_{\substack{k=0\\k\neq d}}^{K+K_w} |c[k]|^2 .$$

Now, with two users, and considering the contribution of user 2 as ICI for user 1, the output of the equalizer is

$$u[n] = \underbrace{c_1[d]}_{\text{gain}}\; A_1[n-d] + \underbrace{\sum_{\substack{k=0\\k\neq d}}^{K+K_w} c_1[k]\; A_1[n-k]}_{\text{residual ISI}} + \underbrace{\sum_{k=0}^{K+K_w} c_2[k]\; A_2[n-k]}_{\text{ICI user 2}} + \underbrace{\sum_{k=0}^{K_w} w[k]\; z[n-k]}_{\text{filtered noise } z'[n]} .$$

Therefore, the approximation is now

$$P_{e1} \approx k\; Q\left(\frac{d_{min1}\;\; |c_1[d]|}{2\sqrt{\sigma_{z'}^2 + \sigma_{ISI}^2 + \sigma_{ICI}^2}}\right)$$

where

$$\sigma_{z'}^2 = \sigma_z^2 \sum_{k=0}^{K_w} |w[k]|^2 ,$$

$$\sigma_{ISI}^2 = E_{s1} \sum_{\substack{k=0\\k\neq d}}^{K+K_w} |c_1[k]|^2 ,$$

and

$$\sigma_{ICI}^2 = E_{s2} \sum_{k=0}^{K+K_w} |c_2[k]|^2 .$$

Involved values are, $d_{min1} = 2$, $E_{s1} = 1$, $E_{s2} = 4$, $\sigma_{z'}^2 = 1{,}2491$, $\sigma_{ISI}^2 = 1{,}5 \times 10^{-4}$, $\sigma_{ICI}^2 = 3{,}2241$, and therefore

$$P_{e1} \approx Q(0{,}4727) = 0{,}3182.$$

d) Now, it is necessary to consider that the ICI introduced by user 2 has to be cancelled. It is convenient to remember that for user 1

$$\mathbf{c}_1 = \mathbf{P}_1\; \mathbf{w},$$

and that for user 2

$$\mathbf{c}_2 = \mathbf{P}_2\; \mathbf{w}$$

when all this coefficients are desired to be zero.

Therefore, the system to be solved is

$$\mathbf{c}_{dT} = \mathbf{P}_T\; \mathbf{w}, \quad \text{with } \mathbf{P}_T = \begin{bmatrix} p_1[0] & 0 \\ p_1[1] & p_1[0] \\ 0 & p_1[1] \\ p_2[0] & 0 \\ p_2[1] & p_2[0] \\ 0 & p_2[1] \end{bmatrix} = \begin{bmatrix} +0{,}9 & 0 \\ -0{,}1 & +0{,}9 \\ 0 & -0{,}1 \\ +0{,}8 & 0 \\ -0{,}2 & +0{,}8 \\ 0 & -0{,}2 \end{bmatrix}, \; \mathbf{c}_{dT} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solution is given by

$$\mathbf{w} = \underbrace{(\mathbf{P}_T^H \mathbf{P}_T)^{-1} \mathbf{P}_T^H}_{\mathbf{P}_T^\sharp} \mathbf{c}_{dT}.$$

Solving the system

$$\mathbf{w} = \begin{bmatrix} +0{,}6171 \\ +0{,}1029 \end{bmatrix}.$$

Joint response betwen channel $p_1[n]$ and equalizer is

$$\mathbf{c}_1 = \mathbf{P}_1 \ \mathbf{w} = \begin{bmatrix} +0{,}5554 \\ +0{,}0309 \\ -0{,}0103 \end{bmatrix},$$

and joint response betwen channel $p_2[n]$ and equalizer is

$$\mathbf{c}_2 = \mathbf{P}_2 \ \mathbf{w} = \begin{bmatrix} +0{,}4937 \\ -0{,}0411 \\ -0{,}0206 \end{bmatrix}.$$

Involved values are, $d_{min1} = 2$, $E_{s1} = 1$, $E_{s2} = 4$, $\sigma_{z'}^2 = 0{,}3914$, $\sigma_{ISI}^2 = 0{,}0011$, $\sigma_{ICI}^2 = 0{,}9835$, and therefore

$$P_{e1} \approx Q(0{,}4735) = 0{,}3179.$$

Exercise 2.14 (Solution) a) Optimal delay is

$$d = 1$$

Probability of error

$$P_e = \frac{1}{8} + \frac{1}{2} Q\left(\frac{1}{\sqrt{N_0/2}}\right) + \frac{1}{4} Q\left(\frac{2}{\sqrt{N_0/2}}\right).$$

b) Now a channel equalizer is used

I) The MMSE equalizer is obtained solving the following system

$$\mathbf{w}_d^{MMSE} = \begin{bmatrix} w[0] \\ w[1] \\ w[2] \end{bmatrix} = \underbrace{\left(\mathbf{P}^H \mathbf{P} + \lambda \mathbf{I}\right)^{-1} \mathbf{P}^H}_{\mathbf{P}_\lambda^\#} \mathbf{c}_d,$$

where in this case $\lambda = \frac{\sigma_z^2}{E_s} = \frac{0{,}01}{1}$ and matrix $\mathbf{I}$ is

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

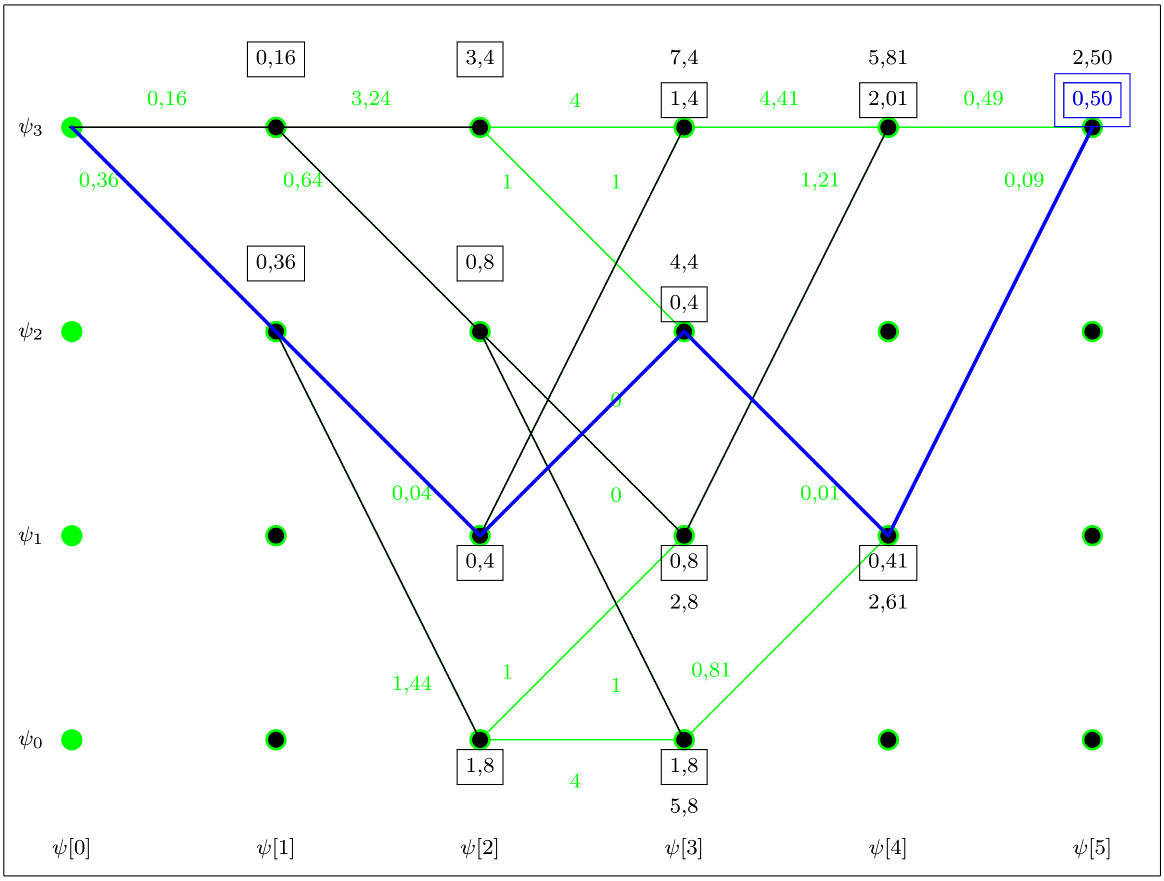
Ideal joint channel-equalizer response, and channel matrix are

$$\mathbf{c}_d = \begin{bmatrix} c[0] \\ c[1] \\ c[2] \\ c[3] \\ c[4] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{P} = \begin{bmatrix} p[0] & 0 & 0 \\ p[1] & p[0] & 0 \\ p[2] & p[1] & p[0] \\ 0 & p[2] & p[1] \\ 0 & 0 & p[2] \end{bmatrix} = \begin{bmatrix} +1/2 & 0 & 0 \\ +1 & +1/2 & 0 \\ +1/2 & +1 & +1/2 \\ 0 & +1/2 & +1 \\ 0 & 0 & +1/2 \end{bmatrix}$$

II) The approximated probability of error is

$$P_e \approx 1\; Q\left(\frac{0{,}8}{\sqrt{0{,}16+0{,}0176}}\right) = Q(1{,}8983).$$

c) Details of the application of Vitervi's algorithm are given in the figure



Therefore, the solution is

$$\hat{A}[0] = -1,\; \hat{A}[1] = +1,\; \hat{A}[2] = -1,$$

Exercise 2.15 (Solution) a) Probability of error with a null delay is

$$P_e = \frac{1}{2}Q\left(\frac{0}{\sqrt{N_0/2}}\right) + \frac{1}{2}Q\left(\frac{1}{\sqrt{N_0/2}}\right) = \frac{1}{4} + \frac{1}{2}Q\left(\frac{1}{\sqrt{N_0/2}}\right).$$

For $d = 1$

$$P_e = \frac{1}{2}.$$

b) ZF equalizer response, in the frequency domain is

$$W(e^{j\omega}) = \frac{e^{-j\omega d}}{P(e^{j\omega})} = \frac{2e^{-j\omega d}}{1-e^{-2j\omega}}.$$

Probability of error is

$$P_e \approx= \frac{1}{2}.$$

c) Now for the MMSE equalizer, frequency response is

$$W(e^{j\omega}) = \frac{\frac{1}{2}[1-e^{2j\omega}]\;e^{-j\omega d}}{\frac{1}{2}-\frac{1}{2}\cos(2\omega)+\frac{\sigma_z^2}{E_s}} = \frac{\frac{1}{2}[1-e^{2j\omega}]\;e^{-j\omega d}}{0{,}6-\frac{1}{2}\cos(2\omega)}$$

and probabiity of error

$$P_e \approx Q\left(\frac{1}{\sqrt{0{,}301}}\right)$$

d) ZF and MMSE equalizers are now obtained solving the following systems

$$\mathbf{w}_d^{ZF} = \mathbf{P}^{\#} \times \mathbf{c}_d = (\mathbf{P}^H\;\mathbf{P})^{-1}\;\mathbf{P}^H \times \mathbf{c}_d$$

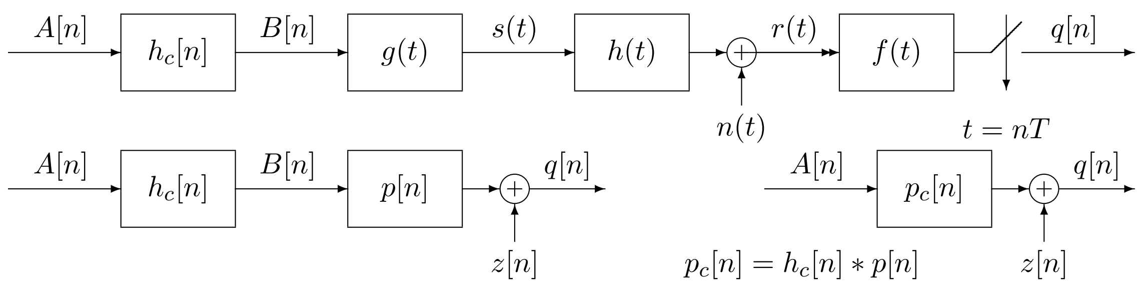
and

$$\mathbf{w}_d^{MMSE} = \mathbf{P}_\lambda^{\#} \times \mathbf{c}_d = (\mathbf{P}^H\;\mathbf{P}+\lambda\;\mathbf{I})^{-1}\;\mathbf{P}^H \times \mathbf{c}_d$$

The numerical values that are involved are

$$\mathbf{c}_d = \begin{bmatrix} 0\\0\\1\\0\\0 \end{bmatrix},\;\mathbf{P} = \begin{bmatrix} p[0] & 0 & 0\\ p[1] & p[0] & 0\\ p[2] & p[1] & p[0]\\ 0 & p[2] & p[1]\\ 0 & 0 & p[2] \end{bmatrix} = \begin{bmatrix} +\frac{1}{2} & 0 & 0\\ 0 & +\frac{1}{2} & 0\\ -\frac{1}{2} & 0 & +\frac{1}{2}\\ 0 & -\frac{1}{2} & 0\\ 0 & 0 & -\frac{1}{2} \end{bmatrix},\;\mathbf{I} = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{bmatrix},\;\lambda = \frac{\sigma_z^2}{E_s} = 0{,}1$$

Exercise 2.16 (Solution) The block diagram for this exercise is shown in the figure, inclding continuous time models, and discrete time models



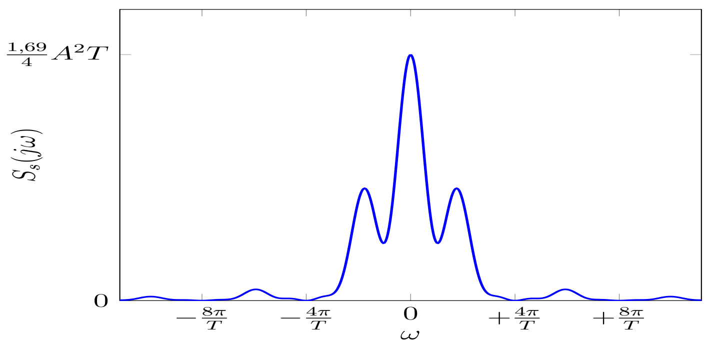
a) In this digital communications sytems ISI is avoided if

$$T \geq T_0 \Longrightarrow T_0 \leq \frac{1}{R_b} = 10^{-3}\text{ s.}$$

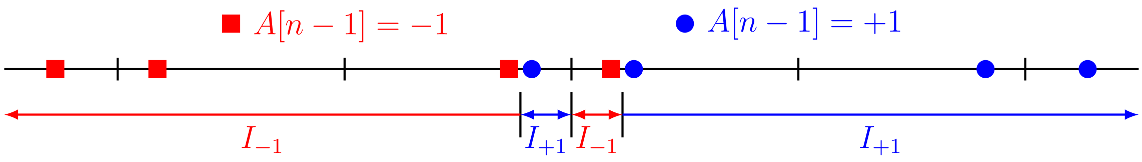
b) Power spectral density is

$$S_S(j\omega) = \frac{A^2T}{4}\,(1,09+0{,}6\cos(\omega T))\,\text{sinc}^2\left(\frac{\omega T}{4\pi}\right).$$

This response is plotted in the figure



c) Optimal delay is $d = 1$. For a MAP detector and assuming a high SNR decision regions are



d) The MMSE equalizer is obtained by solving

$$\mathbf{w}_d^{MMSE} = \mathbf{P}_\lambda^{\#} \times \mathbf{c}_d = (\mathbf{P}^H\ \mathbf{P} + \lambda\ \mathbf{I})^{-1}\ \mathbf{P}^H \times \mathbf{c}_d$$

where

$$\mathbf{c}_d = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} p_c[0] & 0 & 0 \\ p_c[1] & p_c[0] & 0 \\ p_c[2] & p_c[1] & p_c[0] \\ 0 & p_c[2] & p_c[1] \\ 0 & 0 & p_c[2] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1{,}05 & 1 & 0 \\ 0{,}225 & 1{,}05 & 1 \\ 0 & 0{,}225 & 1{,}05 \\ 0 & 0 & 0{,}225 \end{bmatrix}, \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

the regularization parameter is $\lambda = \sigma_z^2/E_s = \frac{N_0}{2E_s} = 0{,}05$ ($E_s = 1$).

e) For each value of delay d, a different vector $\mathbf{c}_d$ is used. Given this vector, the equalizer is

$$\mathbf{w}_d = \left(\mathbf{P}_c^H \mathbf{P}_c + \lambda \mathbf{I}\right)^{-1} \mathbf{P}_c^H \mathbf{c}_d,$$

Now, coefficients of the joint channel-equalizer response are

$$c_d[n] = p_c[n] * w_d[n].$$

Finally, the delary d that maximizes

$$P_e \approx kQ\left(\frac{d_{min}|c_d[d]|}{2\sqrt{\sigma_{z'}^2 + \sigma_{ISI}^2}}\right)$$

or, equivalently,

$$\frac{|c_d[d]|}{\sqrt{\sigma_{z'}^2 + \sigma_{ISI}^2}}$$

is chosen. Here

$$\sigma_{z'}^2 = \sigma_z^2 \sum_{n=0}^{2} |w_d[n]|^2,$$

$$\sigma_{ISI}^2 = E_s \sum_{\substack{n=0 \\ n\neq d}}^{4} |c_d[n]|^2.$$

Usually, the best performace is obtained when delay is located at the central positions of the joint channel-equalizer response.

Exercise 2.17 (Solution) a) Optimal delay is

$$d = 2.$$

Probabiolity of error

$$P_e = \frac{1}{2}Q\left(\frac{1/2}{\sqrt{N_0/2}}\right) + \frac{1}{2}Q\left(\frac{3/2}{\sqrt{N_0/2}}\right) \approx 1{,}27 \times 10^{-2}$$

b) Now, equalizers without constraints and a 4-PAM constellation are used

I) For ZF equalizer, frequency response is

$$W(e^{j\omega}) = \frac{e^{-j\omega d}}{P(e^{j\omega})} = \frac{e^{-j\omega d}}{\frac{1}{2}+e^{-2j\omega}}.$$

Delay d (it is not necessary here to compute it), would be the minimal required delay to guaranty that the inverse Fourier transform, $w[n]$, be causal.

The probability of error is

$$P_e \approx 2Q\left(\frac{1}{\sqrt{0{,}0667}}\right) = 1{,}075 \times 10^{-4}.$$

II) Now, for MMSE equalizer frequency response is

$$W(e^{j\omega}) = \frac{\left(\frac{1}{2}+e^{2j\omega}\right)\ e^{-j\omega d}}{\frac{5}{4}+\cos(2\omega)+0{,}01} = \frac{\left(\frac{1}{2}+e^{2j\omega}\right)\ e^{-j\omega d}}{1{,}26+\cos(2\omega)}$$

Probability of error is

$$P_e \approx 2Q\left(\frac{1}{\sqrt{0{,}0652}}\right) = 9{,}0223 \times 10^{-5}.$$

Evidently, the MMSE equalizer has better performance than the ZF equalizer.

c) Now, a linear equalizer with 5 coefficients is considered.

I) Heuristicaly, it has been observed that typically the best performance is obtained when the delay corresponds to the central position of the joint channel-equalizer response (this strategy is called *tap centering*). Therefore, the selected delay is $d=3$.

The MMSE equalizer is obtained by solvikng

$$\mathbf{w}_d^{MMSE} = \mathbf{P}_\lambda^{\#} \times \mathbf{c}_d = (\mathbf{P}^H\ \mathbf{P} + \lambda\ \mathbf{I})^{-1}\ \mathbf{P}^H \times \mathbf{c}_d$$

In this case $\lambda = 0{,}01$ and

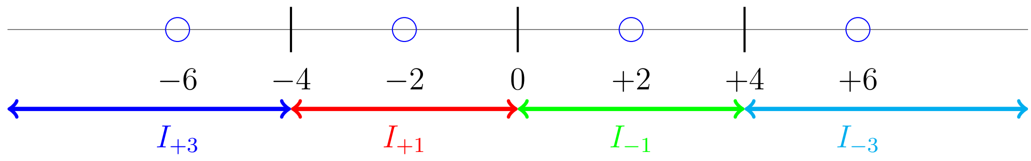
$$\mathbf{c}_d = \begin{bmatrix} 0\\0\\0\\1\\0\\0\\0 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0\\ 0 & \frac{1}{2} & 0 & 0 & 0\\ 1 & 0 & \frac{1}{2} & 0 & 0\\ 0 & 1 & 0 & \frac{1}{2} & 0\\ 0 & 0 & 1 & 0 & \frac{1}{2}\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},\ \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

II) Optimal delay is $d=3$, and the approximated probability of error is

$$P_e \approx 2\ Q\left(\frac{0{,}85}{\sqrt{0{,}0325^2+3{,}25^2}}\right) = 2Q(0{,}2615) \approx 0{,}7937$$

Exercise 2.18 (Solution) a) Memoryless symbol-by-symbol detector

I) Optimal delay is $d=1$. Decision regions are given in the figure.



The alternative option is to decide from the normalized observation

$$q_n[n] = \frac{q[n]}{p[d]} = -\frac{q[n]}{2}, \text{ where now noise variance is } \sigma_{z_n}^2 = \frac{\sigma_z^2}{|p[d]|^2} = \frac{\sigma_z^2}{4},$$

and to maintain the decision regions of a 4-PAM, $I_{-3} = (-\infty, -2)$, $I_{-1} = [-2, 0)$, $I_{+1} = [0, +2)$, $I_{+3} = [+2, +\infty)$.

II) Probability of error is

$$P_e = \frac{1}{4} Q(15) + \frac{1}{2} Q(20) + \frac{1}{4} Q(25)$$

b) Linear equalizer without constraints in the length

I) The equalizer, in the frequency domain

$$W\left(e^{j\omega}\right) = \frac{e^{-j\omega d}}{\frac{1}{4} - 2e^{-j\omega} + \frac{1}{4}e^{-j2\omega}}$$

Probability of error is

$$P_e \approx 2\, Q(10{,}055)$$

II) Delay d is the minimum integer necessary to guarantee that the inverse Fourier transform of $W\left(e^{j\omega}\right)$, i.e., $w[n]$, is causal.

c) Equalizer with the 3 given coefficients

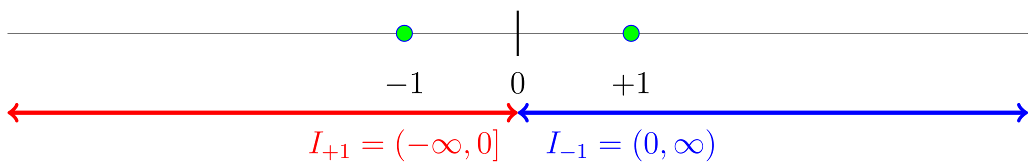
I) Delay $d = 2$

II) Probability of error

$$P_e \approx 2\, Q(3{,}7205) = 1{,}9884 \times 10^{-4}$$

Exercise 2.19 (Solution) a) Memoryless symbol-by-symbol detector

I) Optimal delay is $d = 2$. Decision regions are shown in the figure



The alternative option is to decide from the normalized observation

$$q_n[n] = \frac{q[n]}{p[d]} = -q[n], \text{ where now noise variance is } \sigma_{z_n}^2 = \frac{\sigma_z^2}{|p[d]|^2} = \sigma_z^2,$$

and to maintain the decicios regios of a 2-PAM, $I_{-1} = (-\infty, 0)$, $I_{+1} = [0, +\infty)$. In anay case, the decision are

$$\hat{A}[0] = \text{dec}(q[2]) = -1,\ \hat{A}[1] = \text{dec}(q[3]) = +1,\ \hat{A}[2] = \text{dec}(q[4]) = -1$$

$$\hat{A}[3] = \text{dec}(q[5]) = -1,\ \hat{A}[4] = \text{dec}(q[6]) = +1,\ \hat{A}[5] = \text{dec}(q[7]) = -1$$

II) Probability of error is

$$P_e = \frac{1}{2}Q\left(\frac{0{,}7}{\sqrt{0{,}2}}\right) + \frac{1}{2}Q\left(\frac{1{,}3}{\sqrt{0{,}2}}\right) = 0{,}0303$$

b) Linear equalizer without constraints in the length

I) The equalizer, in the frequency domain

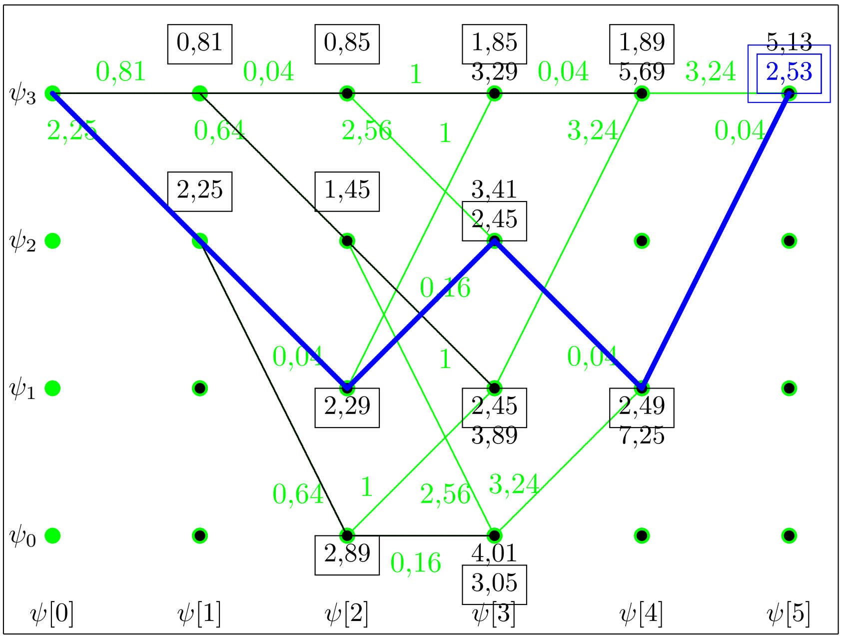
$$W\left(e^{j\omega}\right) = \frac{e^{-j\omega d}}{0{,}3 - e^{-j2\omega}}$$

Probability of error is

$$P_e \approx Q(2{,}1331) = 0{,}0165$$

II) Delay d is the minimum integer necessary to guarantee that the inverse Fourier transform of $W\left(e^{j\omega}\right)$, i.e., $w[n]$, is causal.

c) The application of Viterbi's algorithm, which is the optimal decoding algorithm in this case, is shown in the figure



$\hat{A}[0] = -1$

$\hat{A}[1] = +1$

$\hat{A}[2] = -1$