Chapter 4 : Solutions of the Exercises

Exercise 4.1 (Solution) a) Without cyclic prefix, condition to avoid intersymbol interference (ISI) and inter carrier interference (ICI) is that the joint response of transmitter filter, receiver filter and the equivalent complex baseband response of the channel, $d(t)$, sampled at T/N is a delta function

$$d[m] = (g(t) * h_{eq}(t) * g(-t))\Big|_{t=m\frac{T}{N}} = \delta[m].$$

b) Equivalent discrete channels are

$$p_{k.i}[n] = \frac{N}{T} D[k]\ \delta[n]\ \delta[k-i],$$

therefore there is no ISI nor ICI. Coefficients $D[k]$ are

$$D[0] = D[2] = \frac{4}{3}, \quad D[1] = D[3] = \frac{2}{3}$$

Exercise 4.2 (Solution) a) Equivalent discrete channels are

$$p_{k.i}[n] = \frac{N}{T} D[k]\ \delta[n]\ \delta[k-i],$$

therefore there is no ISI nor ICI. Coefficients $D[k]$ are

$$D[0] = 0{,}4, \quad D[1] = 1 + j0{,}6, \quad D[2] = 1{,}6, \quad D[3] = 1 - j0{,}6$$

b) Signal to noise ration in each carrier is (expression for carrier of index k)

$$\frac{\left(\frac{N}{T}\right)^2 |D[k]|^2 E_s}{\sigma_z^2}$$

c) Probability of error is the average of the probability of error in each carrier. For a QPSK constellation with normalized levels and without ISI the probability or error is

$$P_e = 2\,Q\left(\frac{1}{\sqrt{N_0/2}}\right) - \left[Q\left(\frac{1}{\sqrt{N_0/2}}\right)\right]^2$$

Therefore, now

$$P_e = \frac{1}{4}\sum_{k=0}^{3} 2\,Q\left(\frac{\frac{N}{T}|D[k]|}{\sqrt{N_0/2}}\right) - \left[Q\left(\frac{\frac{N}{T}|D[k]|}{\sqrt{N_0/2}}\right)\right]^2$$

Exercise 4.3 (Solution) a) Without cyclic prefix

I) Condition is $p_{i,i}[n] = \delta[n]$

II) Condition is $p_{k,i}[n] = 0$ for $k \neq i$.

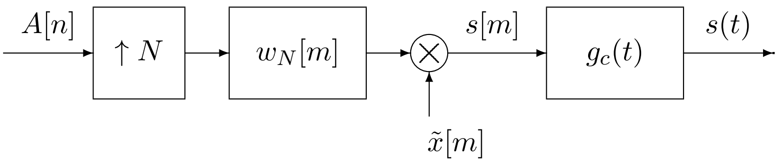
b) With cyclic extension

I) If memory of channel $d[m]$ is K_d, length of the cyclic prefix has to be $C \geq K_d$.

II) Loss of efficiency, in terms of transmission rate, using a cyclic prefix of C samples is

$$\eta = \frac{N}{N+C}.$$

Exercise 4.4 (Solution) a) Block diagram for the transmitter is



Samples a processed by blocks of size N. Samples of the n-th block are

$$s^{(n)}[m] = A[n] \times x[m], \quad m \in \{0, 1, \cdots, N-1\},$$

where $s^{(n)}[m] = s[nN + m]$.

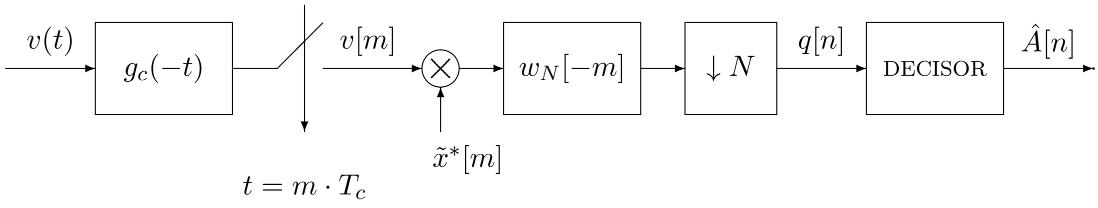
In this case

$$\begin{aligned}
s[0] &= A[0] \times x[0] = +1 \times +1 = +1\\
s[1] &= A[0] \times x[1] = +1 \times -1 = -1\\
s[2] &= A[0] \times x[2] = +1 \times +1 = +1\\
s[3] &= A[0] \times x[3] = +1 \times -1 = -1\\
s[4] &= A[1] \times x[0] = -1 \times +1 = -1\\
s[5] &= A[1] \times x[1] = -1 \times -1 = +1\\
s[6] &= A[1] \times x[2] = -1 \times +1 = -1\\
s[7] &= A[1] \times x[3] = -1 \times -1 = +1\\
s[8] &= A[2] \times x[0] = -1 \times +1 = -1\\
s[9] &= A[2] \times x[1] = -1 \times -1 = +1\\
s[10] &= A[2] \times x[2] = -1 \times +1 = -1\\
s[11] &= A[2] \times x[3] = -1 \times -1 = +1
\end{aligned}$$

b) Samples $v[m]$ are

$$\begin{array}{llll}
v[0] = +\frac{3}{2}, & v[1] = -\frac{3}{2}, & v[2] = +\frac{3}{2}, & v[3] = -\frac{3}{2}\\
v[4] = -\frac{1}{2}, & v[5] = +\frac{1}{2}, & v[6] = -\frac{3}{2}, & v[7] = +\frac{3}{2}\\
v[8] = -\frac{3}{2}, & v[9] = +\frac{3}{2}, & v[10] = -\frac{3}{2}, & v[11] = +\frac{3}{2}
\end{array}$$

c) Block diagram for the receiver is



The corresponding analytical expression is

$$q[n] = \sum_{m=0}^{N-1} x^*[m] \times v[nN+m].$$

In this case

$$q[0] = +6, \quad q[1] = -4, \quad q[2] = -6.$$

Exercise 4.5 (Solution) a) Modulated signal is plotted in Fig. 4.1.

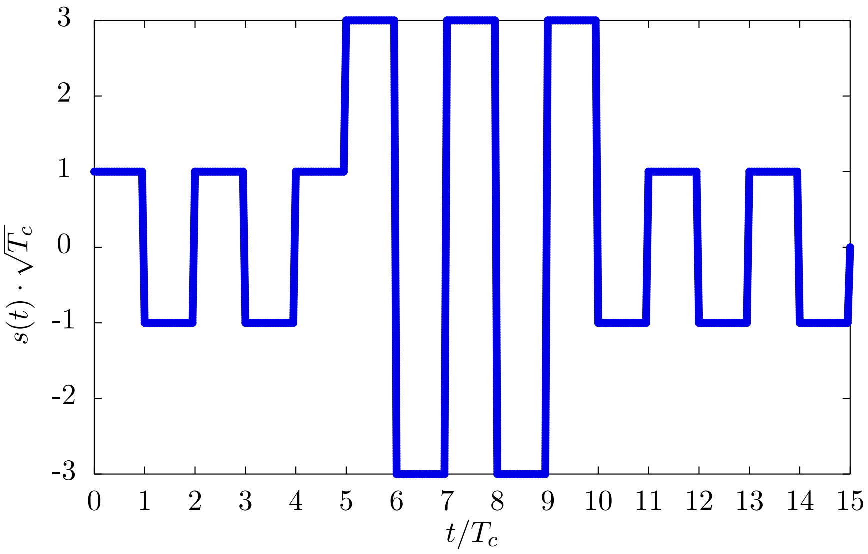


Figura 4.1: Modulated signal.

b) Power spectral density is

$$S_s(j\omega) = \frac{1}{T}\, S_A(e^{j\omega T})\, |X(e^{j\omega T_c})|^2\, |G_c(j\omega)|^2$$

$$S_s(j\omega) = \frac{E_s}{N}\, \left(1 - 2\, \cos(\omega T_c) + 2\, \cos(2\omega T_c)\right)^2\, \Pi\left(\frac{\omega T_c}{2\pi}\right).$$

c) Observations are

$$q[0] = 5, \quad q[1] = 15, \quad q[2] = -5.$$

Exercise 4.6 (Solution) a) Rates are

$$R_{s|min} = 400 \text{ baudios}.$$

$$R_{s|max} = 1 \text{ kbaudio}.$$

b) Constellation order for each user

$$M_0 = 256 \text{ símbolos}, \quad M_1 = 16 \text{ símbolos}, \quad M_2 = 4 \text{ símbolos}, \quad M_3 = 2 \text{ símbolos}.$$

c) ISI and ICI are avoided for all users if

$$C = 2 \text{ samples}.$$

For users 0 y 1, $C = 1$ sample would be enough.

d) Receiver is an standard OFDM receiver, but each user is interested in recovering only its data sequece, as it is shown in Fig. 4.2.

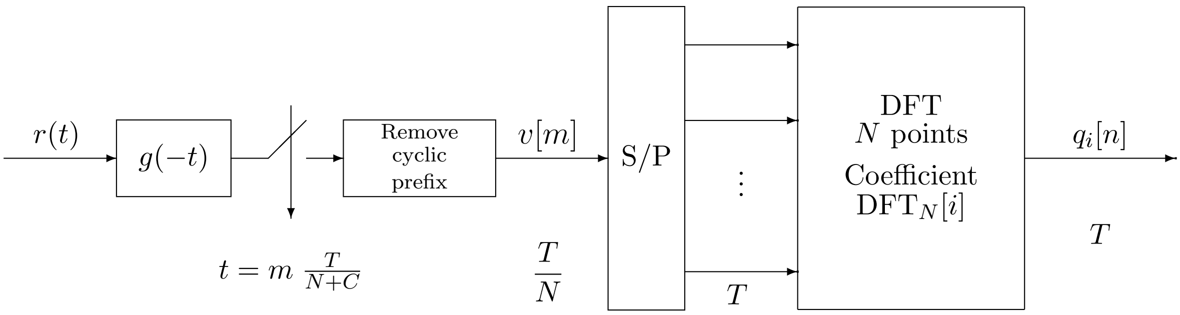


Figura 4.2: OFDM receiver user of index i.

The analytical expression is

$$q_i[n] = \frac{1}{\sqrt{T}} \sum_{m=0}^{N-1} e^{j\frac{2\pi i}{N}m}\, v[nN+m] = \frac{1}{\sqrt{T}} \text{DFT}_N[i]\left\{ \{v^{(n)}[m]\}_{m=0}^{N-1} \right\}.$$

Exercise 4.7 (Solution) a) In this case

$$a = -1,\ b = +1,\ c = -1.$$

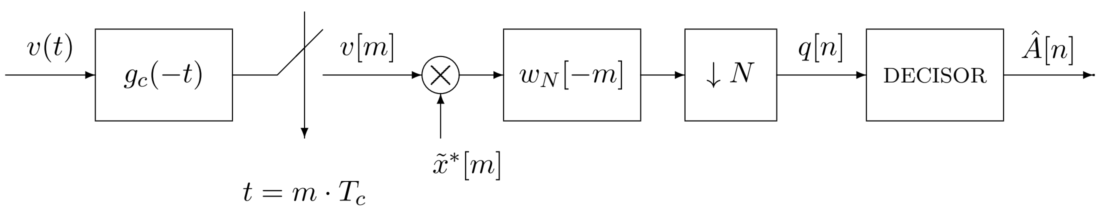
$$A[0] = +1,\ A[1] = -3,\ A[2] = -1.$$

b) Received signal is

$$v(t) = \sqrt{T} \times s(t),$$

which is precisely the figure shown in the exercise

Blcok diagram of the receiver



Observations at symbol rate

$$q[0] = +4, \quad q[1] = -12, \quad q[2] = -4.$$

Exercise 4.8 (Solution) a) Observation $q[n]$ in this case is

$$q[0] = q[1] = q[2[= 0.$$

b) Using $x[m]$

$$p[n] = 6\delta[n] + 2\delta[n-1].$$

There is ISI in the system. The probability of error is

$$P_e = \frac{1}{2}Q\left(\frac{4}{\sqrt{2N_0}}\right) + \frac{1}{2}Q\left(\frac{8}{\sqrt{2N_0}}\right)$$

If alternative sequence $x_r[m]$ is used at the receiver

$$p[n] = 2\delta[n] + 2\delta[n-1].$$

Probability of error is now

$$P_e = \frac{1}{4} + \frac{1}{2}Q\left(\frac{4}{\sqrt{2N_0}}\right)$$

c) In this case the receiver uses sequence

$$x_r[m] = a\,\delta[m] + b\,\delta[m-1] + c\,\delta[m-2] + d\,\delta[m-3].$$

To avoid ISI it is enough to have $a = b$. For instance, using the sequence given by

$$a = -1,\ b = -1,\ c = +1,\ d = -1,$$

produces equivalent discrete channel

$$p[n] = 2\delta[n].$$

Exercise 4.9 (Solution) a) Transmitter filter at symbol rate, $g(t)$, is plotted in Fig. 4.3.

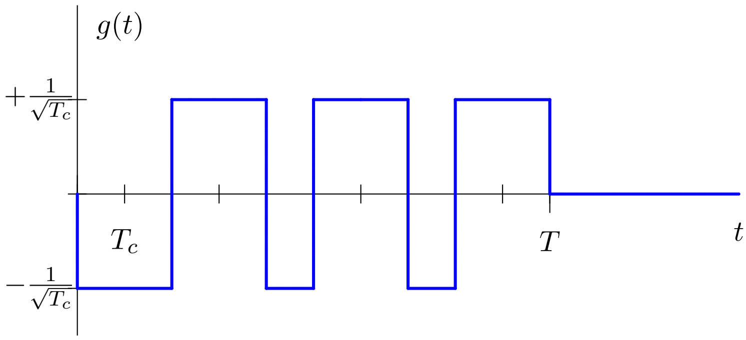


Figura 4.3: Filter $g(t)$.

b) Equivalent discrete channel at symbol rate, $p[n]$, is

$$p[n] = 10\,\delta[n] - \delta[n-1]$$

Probability of error

$$P_e = \frac{1}{2}\,Q\left(\frac{11}{\sqrt{5N_0}}\right) + \frac{1}{2}\,Q\left(\frac{9}{\sqrt{5N_0}}\right).$$

Exercise 4.10 (Solution) a) Samples at T_c

$$s[0] = +1,\ s[1] = -1,\ s[2] = +1$$

$$s[3] = -3,\ s[4] = +3,\ s[5] = -3$$

$$s[6] = +1,\ s[7] = -1,\ s[8] = +1.$$

Bandwidth of the modulated signal

$$B = 7{,}5 \text{ kHz},$$

b) OFDM: it will be taken into account that

$$s[0]=\frac{1}{\sqrt{T}}\left(\underbrace{A_0[0]}_{+1}\underbrace{e^{j\frac{2\pi\,0}{4}\,0}}_{e^{j0}=1}+\underbrace{A_1[0]}_{-3}\underbrace{e^{j\frac{2\pi\,1}{4}\,0}}_{e^{j0}=1}+\underbrace{A_2[0]}_{+1}\underbrace{e^{j\frac{2\pi\,2}{4}\,0}}_{e^{j0}=1}+\underbrace{A_3[0]}_{+3}\underbrace{e^{j\frac{2\pi\,3}{4}\,0}}_{e^{j0}=1}\right)=\frac{2}{\sqrt{T}}.$$

$$s[1]=\frac{1}{\sqrt{T}}\left(\underbrace{A_0[0]}_{+1}\underbrace{e^{j\frac{2\pi\,0}{4}\,1}}_{e^{j0}=+1}+\underbrace{A_1[0]}_{-3}\underbrace{e^{j\frac{2\pi\,1}{4}\,1}}_{e^{j\pi/2}=+j}+\underbrace{A_2[0]}_{+1}\underbrace{e^{j\frac{2\pi\,2}{4}\,1}}_{e^{j\pi}=-1}+\underbrace{A_3[0]}_{+3}\underbrace{e^{j\frac{2\pi\,3}{4}\,1}}_{e^{j3\pi/2}=-j}\right)=\frac{-j6}{\sqrt{T}}.$$

$$s[2]=\frac{1}{\sqrt{T}}\left(\underbrace{A_0[0]}_{+1}\underbrace{e^{j\frac{2\pi\,0}{4}\,2}}_{e^{j0}=+1}+\underbrace{A_1[0]}_{-3}\underbrace{e^{j\frac{2\pi\,1}{4}\,2}}_{e^{j\pi}=-1}+\underbrace{A_2[0]}_{+1}\underbrace{e^{j\frac{2\pi\,2}{4}\,2}}_{e^{j2\pi}=+1}+\underbrace{A_3[0]}_{+3}\underbrace{e^{j\frac{2\pi\,3}{4}\,2}}_{e^{j3\pi}=-1}\right)=\frac{2}{\sqrt{T}}.$$

$$s[3]=\frac{1}{\sqrt{T}}\left(\underbrace{A_0[0]}_{+1}\underbrace{e^{j\frac{2\pi\,0}{4}\,3}}_{e^{j0}=+1}+\underbrace{A_1[0]}_{-3}\underbrace{e^{j\frac{2\pi\,1}{4}\,3}}_{e^{j3\pi/2}=-j}+\underbrace{A_2[0]}_{+1}\underbrace{e^{j\frac{2\pi\,2}{4}\,3}}_{e^{j3\pi}=-1}+\underbrace{A_3[0]}_{+3}\underbrace{e^{j\frac{2\pi\,3}{4}\,3}}_{e^{j9\pi/2}=+j}\right)=\frac{+j6}{\sqrt{T}}.$$

$$s[4]=\frac{1}{\sqrt{T}}\left(\underbrace{A_0[1]}_{+1}\underbrace{e^{j\frac{2\pi\,0}{4}\,0}}_{e^{j0}=1}+\underbrace{A_1[1]}_{-1}\underbrace{e^{j\frac{2\pi\,1}{4}\,0}}_{e^{j0}=1}+\underbrace{A_2[1]}_{+1}\underbrace{e^{j\frac{2\pi\,2}{4}\,0}}_{e^{j0}=1}+\underbrace{A_3[1]}_{-1}\underbrace{e^{j\frac{2\pi\,3}{4}\,0}}_{e^{j0}=1}\right)=0.$$

$$s[5]=\frac{1}{\sqrt{T}}\left(\underbrace{A_0[1]}_{+1}\underbrace{e^{j\frac{2\pi\,0}{4}\,1}}_{e^{j0}=+1}+\underbrace{A_1[1]}_{-1}\underbrace{e^{j\frac{2\pi\,1}{4}\,1}}_{e^{j\pi/2}=+j}+\underbrace{A_2[1]}_{+1}\underbrace{e^{j\frac{2\pi\,2}{4}\,1}}_{e^{j\pi}=-1}+\underbrace{A_3[1]}_{-1}\underbrace{e^{j\frac{2\pi\,3}{4}\,1}}_{e^{j3\pi/2}=-j}\right)=0.$$

$$s[6]=\frac{1}{\sqrt{T}}\left(\underbrace{A_0[1]}_{+1}\underbrace{e^{j\frac{2\pi\,0}{4}\,2}}_{e^{j0}=+1}+\underbrace{A_1[1]}_{-1}\underbrace{e^{j\frac{2\pi\,1}{4}\,2}}_{e^{j\pi}=-1}+\underbrace{A_2[1]}_{+1}\underbrace{e^{j\frac{2\pi\,2}{4}\,2}}_{e^{j2\pi}=+1}+\underbrace{A_3[1]}_{-1}\underbrace{e^{j\frac{2\pi\,3}{4}\,2}}_{e^{j3\pi}=-1}\right)=\frac{4}{\sqrt{T}}.$$

$$s[7]=\frac{1}{\sqrt{T}}\left(\underbrace{A_0[1]}_{+1}\underbrace{e^{j\frac{2\pi\,0}{4}\,3}}_{e^{j0}=+1}+\underbrace{A_1[1]}_{-1}\underbrace{e^{j\frac{2\pi\,1}{4}\,3}}_{e^{j3\pi/2}=-j}+\underbrace{A_2[1]}_{+1}\underbrace{e^{j\frac{2\pi\,2}{4}\,3}}_{e^{j3\pi}=-1}+\underbrace{A_3[1]}_{-1}\underbrace{e^{j\frac{2\pi\,3}{4}\,3}}_{e^{j9\pi/2}=+j}\right)=0.$$

where $T=2\times 10^{-3}$.

I) In this case, samples corresponding to the 8 initial symbols are

$$\underbrace{s[0],s[1],s[2],s[3]}_{Bloque\ 1},\underbrace{s[4],s[5],s[6],s[7]}_{Bloque\ 2}.$$

Bandwidth is

$$B=2\text{ kHz}.$$

II) Now

$$B=3\text{ kHz}$$

Samples corresponding to the 8 initial symbols

$$\underbrace{\tilde{s}[-2],\tilde{s}[-1],\tilde{s}[0],\tilde{s}[1],\tilde{s}[2],\tilde{s}[3]}_{Bloque\ 1},\underbrace{\tilde{s}[4],\tilde{s}[5],\tilde{s}[6],\tilde{s}[7],\tilde{s}[8],\tilde{s}[9]}_{Bloque\ 2},$$

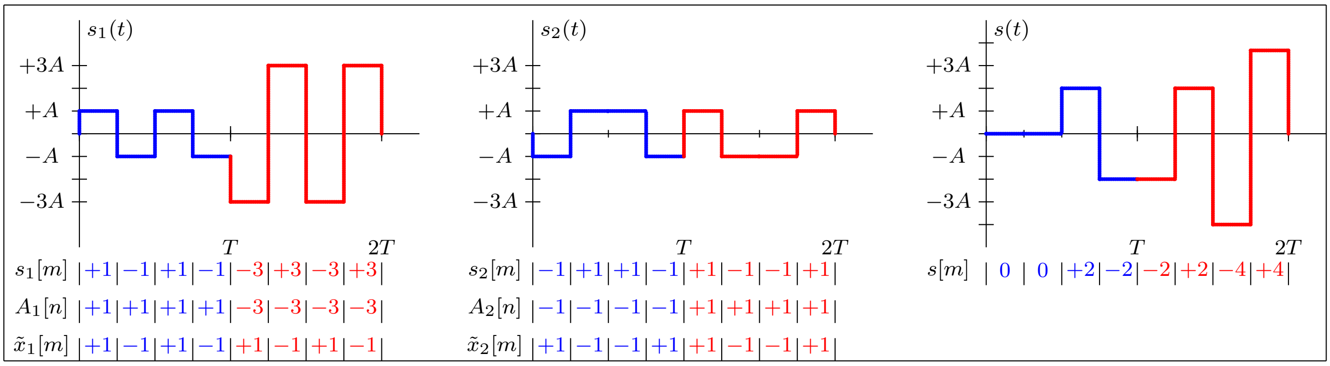
which in this case are

$$\underbrace{s[2],s[3],s[0],s[1],s[2],s[3]}_{Bloque\ 1},\underbrace{s[5],s[6],s[4],s[5],s[6],s[7]}_{Bloque\ 2}.$$

Exercise 4.11 (Solution) a) Transmitter pulses for the two users have to be orthogonal, therefore the choice is

$$x_2[m] = x_b[m].$$

b) The modulated signal, along with the contribution for each user, is shown in the figure



c) Demodulated observations are

$$\begin{aligned} q_1[0] =& v[0] \times x_1[0] + v[1] \times x_1[1] + v[2] \times x_1[2] + v[3] \times x_1[3] \\ =& 0 \times (+1) + 0 \times (-1) + 2 \times (+1) - 2 \times (-1) = +4. \end{aligned}$$

$$\begin{aligned} q_1[1] =& v[4] \times x_1[0] + v[5] \times x_1[1] + v[6] \times x_1[2] + v[7] \times x_1[3] \\ =& -2 \times (+1) + 2 \times (-1) - 4 \times (+1) + 4 \times (-1) = -12. \end{aligned}$$

Exercise 4.12 (Solution) a) Without cyclic prefix, ISI and ICI are avoided only if

$$d[m] = K\,\delta[m].$$

In this case equivalent channel at $\frac{T}{N}$ is

$$d[m] = d(t)|_{t=m\frac{T}{N}} = d(t)|_{t=m\frac{T}{4}} = \delta[m] + 0{,}25\,\delta[m-1].$$

Since $d[m]$ is not null at $m = 1$ (and at $m = 0$), this system will have ISI and ICI.

To avoid both ISI and ICI it is necessary to use a cyclic prefix of length $C \geq K_d$. To have the maximum spectral efficiency, $C = K_d$, because efficiency is

$$\eta = \frac{N}{N+C}.$$

If $C = 1$ is tried, equivalent discrete channel $d[m]$ is now

$$d[m] = d(t)|_{t=m\frac{T}{N+C}} = d(t)|_{t=m\frac{T}{5}} = \delta[m] + 0{,}5\,\delta[m-1]$$

whose memory is $K_d = 1$. Therefore, $C = 1$ is enough to avoid ISI and ICI.

b) In this case equivalent discrete channels $p_{k,i}[n]$ are

$$p_{k,i}[n] = \frac{N}{T}\delta[n]\delta[k-i]D[i],$$

with coefficient $D[i]$

$$D[0] = 1{,}5, \quad D[1] = 1 - 0{,}5j, \quad D[2] = 0{,}5, \quad D[3] = 1 + 0{,}5j.$$

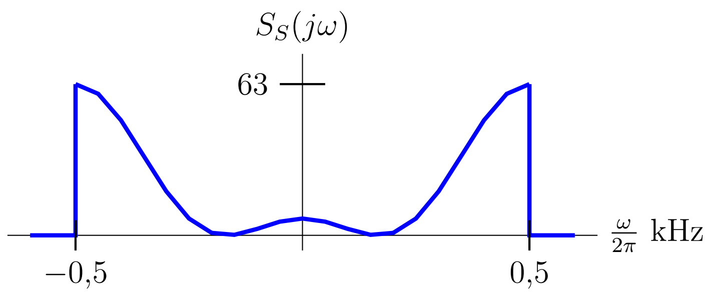
Exercise 4.13 (Solution) The power spectral density is

$$S_S(j\omega) = \frac{7}{T_c}(1 - 2\cos(\omega T_c))^2 \; H_{RC}^{\alpha,T_c}(j\omega),$$

with $T_c = 1$ ms. For $\alpha = 0$,

$$S_S(j\omega) = 7\left[1 - 2\cos\left(\omega \times 10^{-3}\right)\right]^2, \text{ para } |\omega| \leq \frac{\pi}{T_c} \text{ rad/s.}$$

This response is plotted in the figure



The bandwidth is

$$B = 0{,}5 \text{ kHz.}$$

Exercise 4.14 (Solution) a) ISI and ICI can be avoided using a cyclic prefix of appropriate length, at least equal to the memory of $d[m]$

$$C \geq K_d = 2.$$

In this case, the equivalent discrete channels $p_{k,i}[n]$ are

$$p_{k,i}[n] = \frac{N}{T}\delta[n]\delta[k-i]D[k],$$

with $N = 4$ and $T = 2$ μs, where $D[k]$ are

$$D[0] = 1{,}5,\; D[1] = 0{,}5,\; D[2] = 1{,}5,\; D[3] = 0{,}5.$$

b) The bandwidth has to be obtained for two cases

I) In this case

$$B = 3 \text{ MHz.}$$

II) Now

$$B = 2 \text{ MHz.}$$

Exercise 4.15 (Solution) a) Modulación de espectro ensanchado por secuencia directa

I) Las muestras a tiempo de chip son

m	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$s[m]$	+1	−1	−1	+1	−3	+3	+3	−3	+1	−1	−1	+1	−1	+1	+1	−1

El ancho de banda de la señal modulada es

$$B = 24 \text{ kHz.}$$

II) La única observación es
$$q[0] = 3{,}5$$

b) OFDM modulation: the rate per carrier is the total rate divided by the number of carriers, $N = 4$, i.e.
$$R_s = \frac{R_s^{TOTAL}}{N} = \frac{4}{4} = 1 \text{ baud, and therefore } T = \frac{1}{R_s} = 1 \text{ s}$$

I) Without cyclic prefix, the samples are

m	0	1	2	3
$s[m]$	-2	$-2j$	$+6$	$+2j$

The bandwidth is
$$B = 4 \text{ Hz}$$

II) With cyclic prefix the samples are

m	-1	0	1	2	3
$\tilde{s}[m]$	$+2j$	-2	$-2j$	$+6$	$+2j$

Now bandwidth is
$$B = 5 \text{ Hz}$$

Exercise 4.16 (Solution) a) In this case
$$p[n] = 10\,\delta[n-1].$$

There is no ISI, only a delay of a symbol.

b) Now
$$p[n] = 8\,\delta[n] + 2\,\delta[n-1].$$

There is intersymbol interference (with previous symbol).

c) In this case expression
$$p[n] = \sum_{m=0}^{N-1}\sum_{\ell=0}^{N-1} x[m]\,x^*[\ell]\,d[nN+\ell-m],$$

can not be simplified, with
$$d[m] = \sum_m \delta[m] \times r_{g_c}(mT_c - \frac{5}{2}T_c).$$

Respose $d[m]$ is not ideal, and the only option to avoid ISI is to have an spreading sequence satisfying
$$\sum_{m=0}^{N-1} x[m]\,x[m+k] = 0, \text{ para } -9 \le k \le 9,$$

condition that is not satidfied for some values of k. Therefore, ISI will be present in the system.