Chapter 5 : Solution of the Exercises

Exercise 5.1 (Solution) Coding rate is

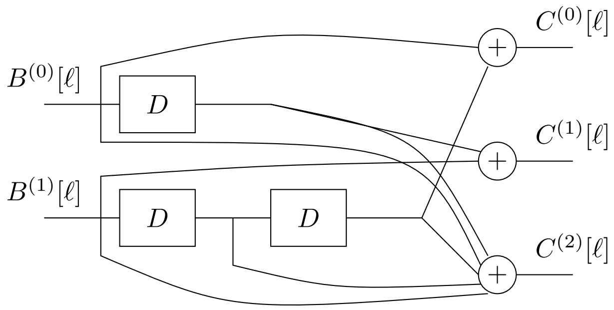
$$R = \frac{1}{2}.$$

Minimum distance $d_{min} = 4$, and syndrome table

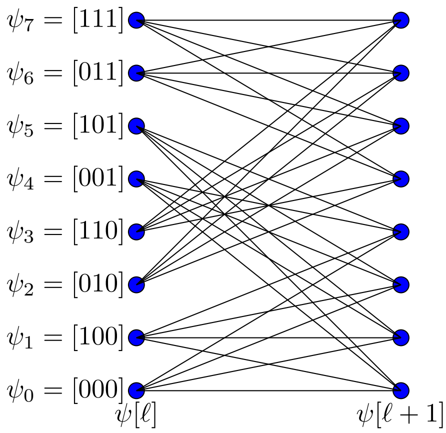
$\boldsymbol{e}$	**s**
0 0 0 0 0 0 0 0	0 0 0 0
1 0 0 0 0 0 0 0	1 1 0 1
0 1 0 0 0 0 0 0	1 0 1 1
0 0 1 0 0 0 0 0	1 1 1 0
0 0 0 1 0 0 0 0	0 1 1 1
0 0 0 0 1 0 0 0	1 0 0 0
0 0 0 0 0 1 0 0	0 1 0 0
0 0 0 0 0 0 1 0	0 0 1 0
0 0 0 0 0 0 0 1	0 0 0 1
0 0 0 0 0 0 1 1	0 0 1 1
0 0 0 0 0 1 0 1	0 1 0 1
0 0 0 0 0 1 1 0	0 1 1 0
0 0 0 0 1 0 0 1	1 0 0 1
0 0 0 0 1 0 1 0	1 0 1 0
0 0 0 0 1 1 0 0	1 1 0 0
1 0 0 0 0 0 1 0	1 1 1 1

Exercise 5.2 (Solution) a) Codign rate is $R = 2/3$

b) Schematic represetantion is plotted in the figure



c) Trellis diagram



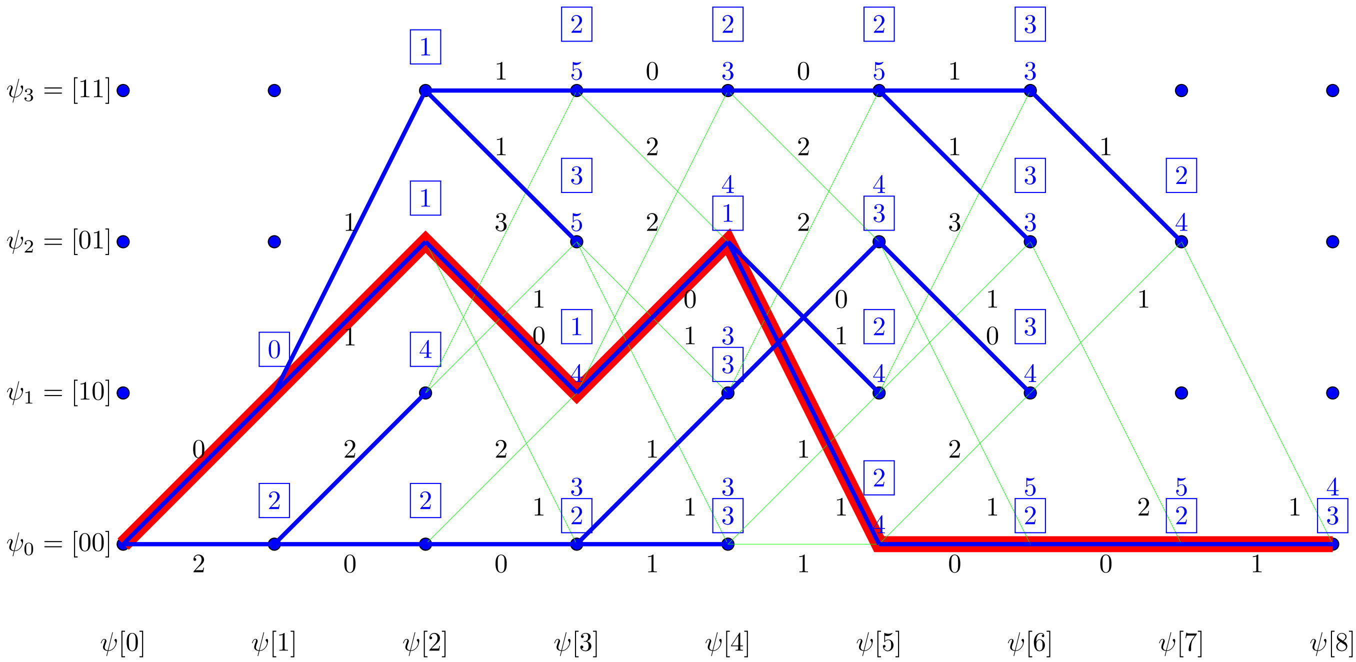
$\psi[\ell]$	$\psi[\ell+1]$	Etiquetas
ψ_7	ψ_7	11|001
	ψ_6	01|100
	ψ_5	10|010
	ψ_4	00|111
ψ_6	ψ_7	11|010
	ψ_6	01|111
	ψ_5	10|001
	ψ_4	00|100
ψ_5	ψ_3	11|000
	ψ_2	01|101
	ψ_1	10|011
	ψ_0	00|110
ψ_4	ψ_3	11|011
	ψ_2	01|110
	ψ_1	10|000
	ψ_0	00|101

$\psi[\ell]$	$\psi[\ell+1]$	Etiquetas
ψ_3	ψ_7	11|110
	ψ_6	01|001
	ψ_5	10|111
	ψ_4	00|010
ψ_2	ψ_7	11|111
	ψ_6	01|010
	ψ_5	10|100
	ψ_4	00|001
ψ_1	ψ_3	11|101
	ψ_2	01|000
	ψ_1	10|110
	ψ_0	00|011
ψ_0	ψ_3	11|110
	ψ_2	01|011
	ψ_1	10|101
	ψ_0	00|000

d) Minimum Hamming distance $D^H_{min} = 4$.

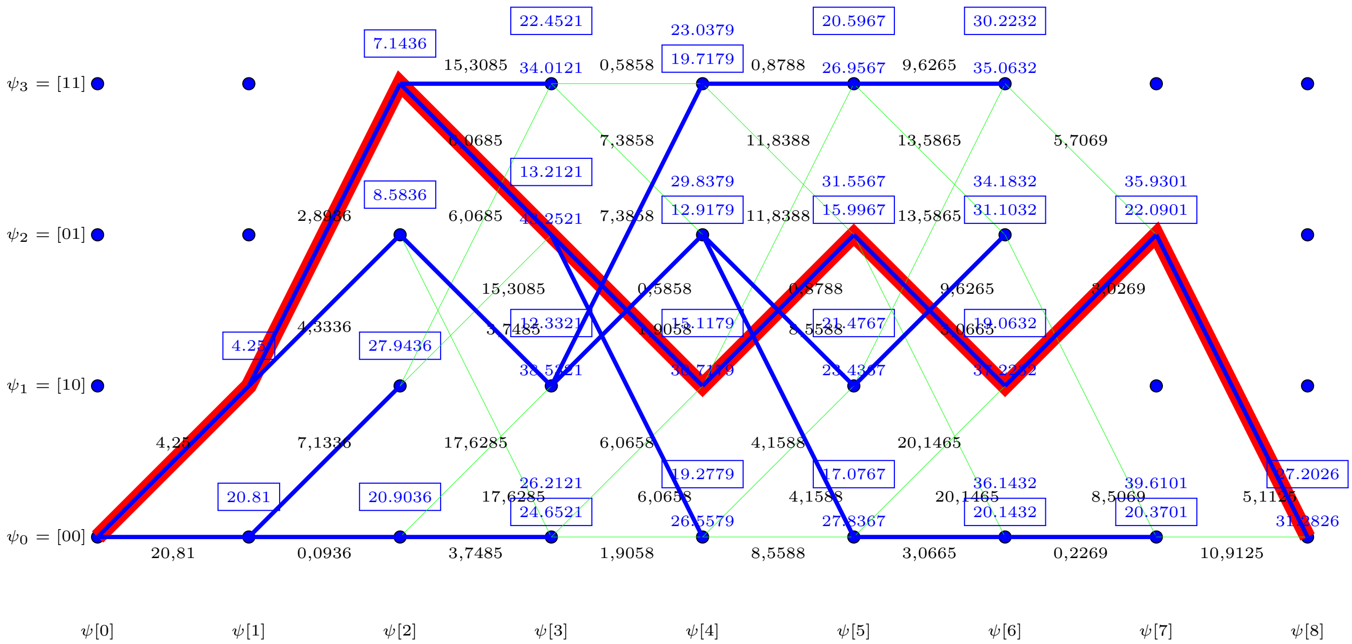
Exercise 5.3 (Solution) a) Decoded sequence (from hard output)

ℓ	0	1	2	3	4	5
$B[\ell]$	1	0	1	0	0	0



b) Decoded sequence (soft output)

ℓ	0	1	2	3	4	5
$B[\ell]$	1	1	0	1	0	1



Exercise 5.4 (Solution) a) For the block code

I) Minimum distance is $d_{min} = 2$.

II) It is possible to have two systematic matrices, one by the beginning and the other one by the end

$$\mathbf{G}'_1 = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right], \quad \mathbf{G}'_2 = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$$

III) Parity check matrices for both cases are

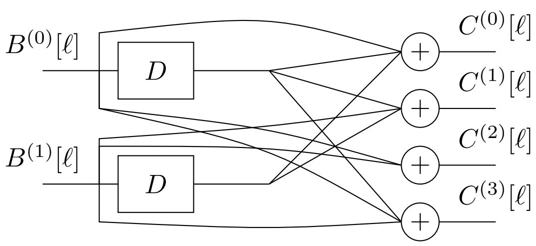
$$\mathbf{H}_1 = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right], \quad \mathbf{H}_2 = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

IV) Syndrome tables

$\boldsymbol{e}$	$\mathbf{s}$	$\boldsymbol{e}$	$\mathbf{s}$
0 0 0 0	0 0	0 0 0 0	0 0
1 0 0 0	1 0	1 0 0 0	1 0
0 1 0 0	1 1	0 1 0 0	0 1
0 0 0 1	0 1	0 0 0 1	1 1

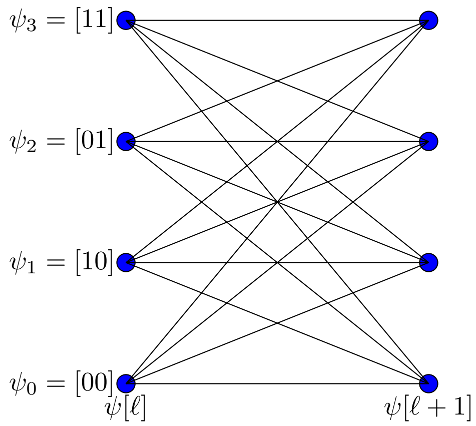
b) For the convolutional encoder

I) Schematic representation



II) Trellis diagram is shown below. Labels can be easily obtained from the following table:

$B^{(0)}[\ell]$	$B^{(1)}[\ell]$	$B^{(0)}[\ell-1]$	$B^{(1)}[\ell-1]$	$C^{(0)}[\ell]$	$C^{(1)}[\ell]$	$C^{(2)}[\ell]$	$C^{(3)}[\ell]$
1	1	1	1	1	1	0	1
0	1	1	1	0	1	1	0
1	0	1	1	1	0	1	0
0	0	1	1	0	0	0	1
1	1	0	1	0	0	0	0
0	1	0	1	1	0	1	1
1	0	0	1	0	1	1	1
0	0	0	1	1	1	0	0
1	1	1	0	0	0	0	1
0	1	1	0	1	0	1	0
1	0	1	0	0	1	1	0
0	0	1	0	1	1	0	1
1	1	0	0	1	1	0	0
0	1	0	0	0	1	1	1
1	0	0	0	1	0	1	1
0	0	0	0	0	0	0	0



$\psi[\ell]$	$\psi[\ell+1]$	Etiquetas
ψ_3	ψ_3	11|1101
	ψ_2	01|0110
	ψ_1	10|1010
	ψ_0	00|0001
ψ_2	ψ_3	11|0000
	ψ_2	01|1011
	ψ_1	10|0111
	ψ_0	00|1100
ψ_1	ψ_3	11|0001
	ψ_2	01|1010
	ψ_1	10|0110
	ψ_0	00|1101
ψ_0	ψ_3	11|1100
	ψ_2	01|0111
	ψ_1	10|1011
	ψ_0	00|0000

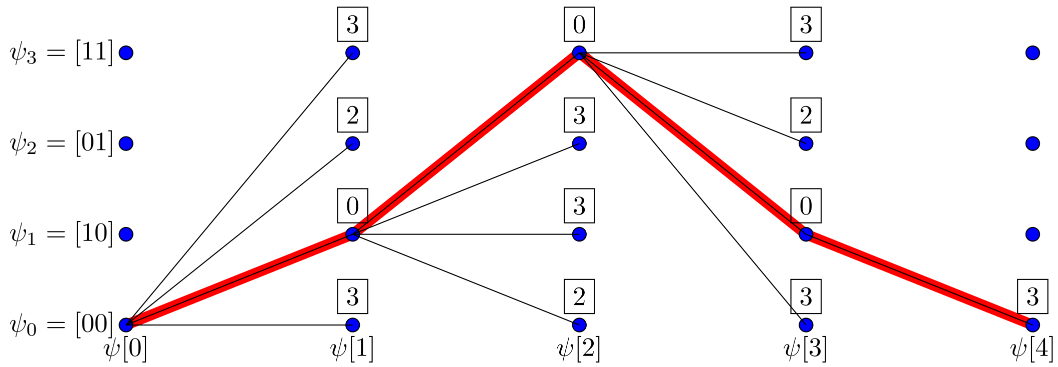
III) Minimum Hamming distance $D^H_{min} = 3$

IV) Decoded sequence

ℓ	0	1	2	3	4	5
$B[\ell]$	1	0	1	1	1	0

Acummulated metrics (survival path highlighted in boldface) in each state after applying Viterbi's algorithm

	$\psi[1]$	$\psi[2]$	$\psi[3]$	$\psi[4]$
$\psi_3 = [1,1]$	3	5|3|**0**|6	**3**|5|6|4	–
$\psi_2 = [0,1]$	2	6|4|**3**|5	**2**|4|3|5	–
$\psi_1 = [1,0]$	0	6|4|**3**|5	**0**|6|5|3	–
$\psi_0 = [0,0]$	3	3|5|**2**|4	**3**|5|6|4	6|4|**3**|5



c) Probability of error for the block code

$$P_e = \varepsilon\,(1-\varepsilon)^3 + \sum_{e=2}^{4} \binom{4}{e}\,\varepsilon^e\,(1-\varepsilon)^{4-e}.$$

And for the convolutional code

$$P_e \approx c \sum_{e=2}^{8} \binom{8}{e}\,\varepsilon^e\,(1-\varepsilon)^{8-e},$$

Exercise 5.5 (Solution) a) For the linear block code

I) Values providing best performance ($d_{min} = 3$) are

$$a = 1, \quad b = 1, \quad c = 1, \quad d = 0$$

II) Syndrome table

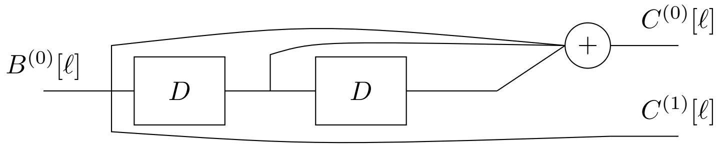
$\boldsymbol{e}$	$\mathbf{s}$
0 0 0 0 0	0 0 0
1 0 0 0 0	1 1 1
0 1 0 0 0	1 0 1
0 0 1 0 0	1 0 0
0 0 0 1 0	0 1 0
0 0 0 0 1	0 0 1
1 0 0 0 1	1 1 0
0 0 0 1 1	0 1 1

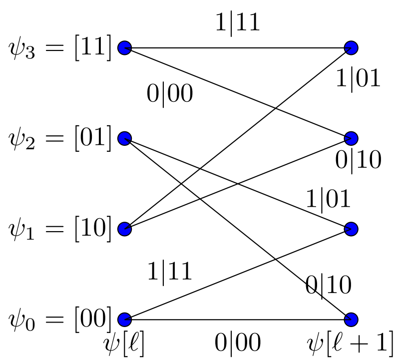
III) Decoded words

$$\mathbf{b}_0 = 00, \quad \mathbf{b}_1 = 01, \quad \mathbf{b}_2 = 11$$

b) For the convolutional encoder

I) Schematic representation and trellis diagram





II) Encoded sequence

m	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$C[m]$	1	1	1	0	0	1	0	1	0	0	1	0	0	0	0	0

III) Performance working from hard output

$$P_e \approx c \sum_{e=2}^{6} \binom{6}{e} \varepsilon^e (1-\varepsilon)^{6-e},$$

where ε is the bit error rate (BER) associated to a 4-QAM modulation

$$\varepsilon = Q\left(\frac{1}{\sqrt{N_0/2}}\right) - \frac{1}{2} Q^2\left(\frac{1}{\sqrt{N_0/2}}\right)$$

Working from soft output

$$P_e \approx c\, Q\left(\frac{2}{\sqrt{N_0/2}}\right)$$

IV) Decoded sequence

m	0	1	2	3
$B[m]$	1	0	1	1

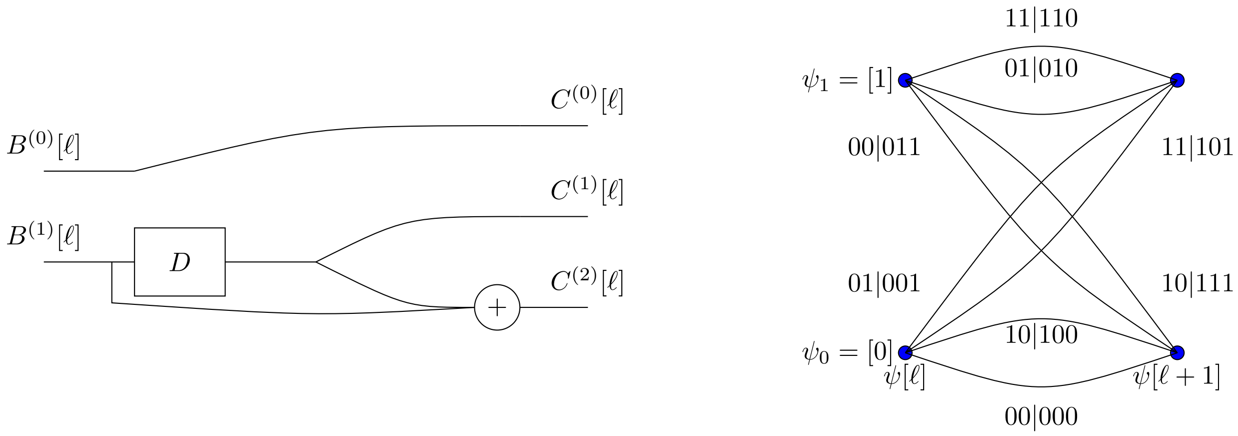
Exercise 5.6 (Solution) a) Both codes are systematic, one by the beginning and the second one by the end.

b) Minimum distance of code 1 is $d_{min} = 3$, and therefore the code is able to detect up to $d = 2$ errors and is able to correct all patterns of up to $t = 1$ error. For code 2, minimum distance is $d_{min} = 2$, allowing to detect $d = 1$ error, and to correct $t = 0$ errors.

c) Decoded words are

$$\hat{\mathbf{b}}_a = 01,\ \hat{\mathbf{b}}_b = 11.$$

Exercise 5.7 (Solution) a) Coding rate is $R = \frac{2}{3}$, and schematic representation



b) Trellis diagram is also shown in the previous figure (above). Minimum distance is

$$D^H_{min} = 3.$$

c) Encoded sequence is

$$C[m'] = 101\ 010\ 111\ 000 \mid 000\ 000$$

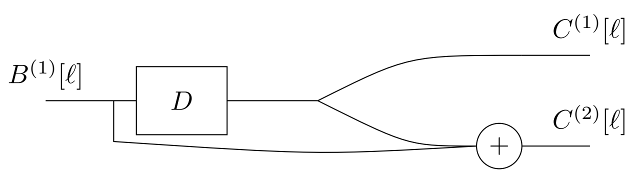
The last 6 bits correspond to the encoding of the cyclic header of zeros transmitted after the information bits.

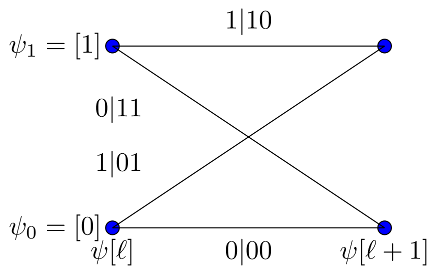
d) Yes, it is a possible fragment for this encoder, corresponding to the following information piece

$$B[m] = \cdots 11\ 01\ 10\ 00\ \cdots$$

as long as it is transmitted after $B[m] = \cdots 01$ or $B[m] = \cdots 11$ (to start from state $\psi_1 = 1$).

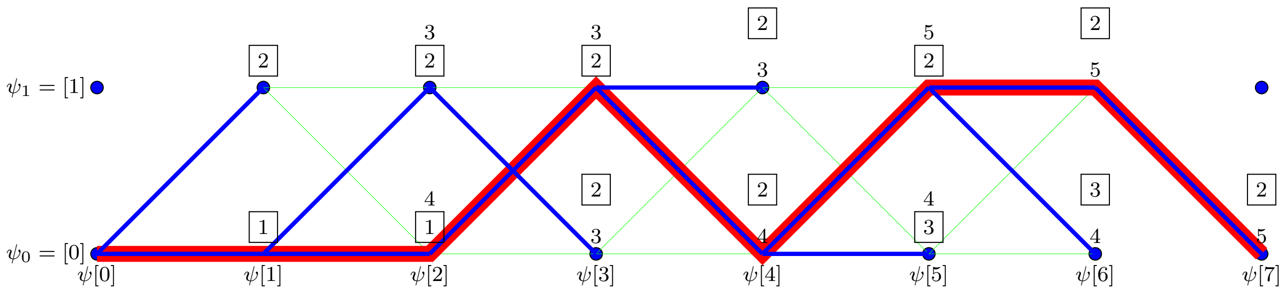
e) Now the encoder is





Decoded bit sequence is

ℓ	0	1	2	3	4	5
$B[\ell]$	0	0	1	0	1	1



Exercise 5.8 (Solution) a) Solution is provided for every code

1) For the first code, coding rate is $R = \frac{1}{2}$, and minimum distances is two.
2) For the second code, coding rate is $R = \frac{1}{5}$, and minimum distances is also two.
3) For the third code, coding rate is $R = \frac{2}{5}$, and minimum distances is again two.

b) First code is non linear. Second code is linear, and its generating matrix is

$$G_2 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

Third code is linear, and its generating matrix is for instance (this is not the only option)

$$G_3 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

c) The only two codes that can be systematic are $\mathcal{C}_1$ and $\mathcal{C}_2$.

d) To augment the minimum distance, it is possible to replace the second coded word by 11111, thus having a minimum distance of five.

e) For code $\mathcal{C}_2$ the maximum likelihood word is the second one, 01010. For code $\mathcal{C}_3$ thre are two coded word with the same minimum likelihood, 01111 and 11011, which means that is possible to decide any of them.

Exercise 5.9 (Solution) a) The dictionary of the code is

i	$\mathbf{b}_i$	$\mathbf{c}_i$
0	0 0 0	0 0 0 0 0 0
1	0 0 1	0 0 1 0 1 1
2	0 1 0	0 1 0 1 0 1
3	0 1 1	0 1 1 1 1 0
4	1 0 0	1 0 0 1 1 0
5	1 0 1	1 0 1 1 0 1
6	1 1 0	1 1 0 0 1 1
7	1 1 1	1 1 1 0 0 0

Parity check matrix is

$$\mathbf{H} = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

and minimum distance $d_{min} = 3$.

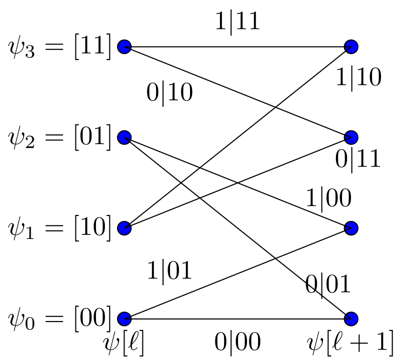
b) Probability of error is

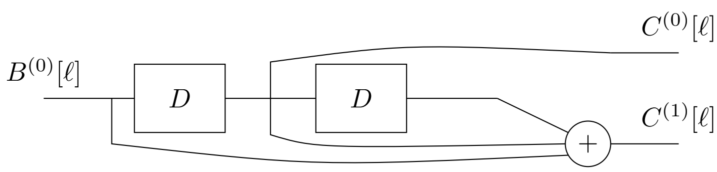
$$P_e = 14\,\varepsilon^2\,(1-\varepsilon)^4 + \sum_{e=3}^{6} \binom{6}{e}\,\varepsilon^e\,(1-\varepsilon)^{6-e},$$

where

$$\varepsilon = Q\left(\frac{d_{min}^{BPSK}}{2\sqrt{N_0/2}}\right).$$

c) Trellis diagram (and schematic representation)





d) Probability of error is

$$P_e \approx c \sum_{e=2}^{6} \binom{6}{e}\,\varepsilon^e\,(1-\varepsilon)^{6-e},$$

where ε is the bit error rate (BER) associated to the BPSK (or 2-PAM) constellation, i.e.

$$\varepsilon = Q\left(\frac{1}{2\sqrt{N_0/2}}\right)$$

This result takes into account that for this convolutional encoder $D^H_{min} = 4$, and this distance is achieved through 3 transitions in the trellis diagram.

e) The sequence of encoded bits is

$$C[m] = 100110\ 110011$$

The decoded sequence can be any of the following ones:

$$B[\ell] = 100\ 110, \quad B[\ell] = 001\ 110, \text{ or } B[\ell] = 010\ 110$$

f) Encoded sequence is

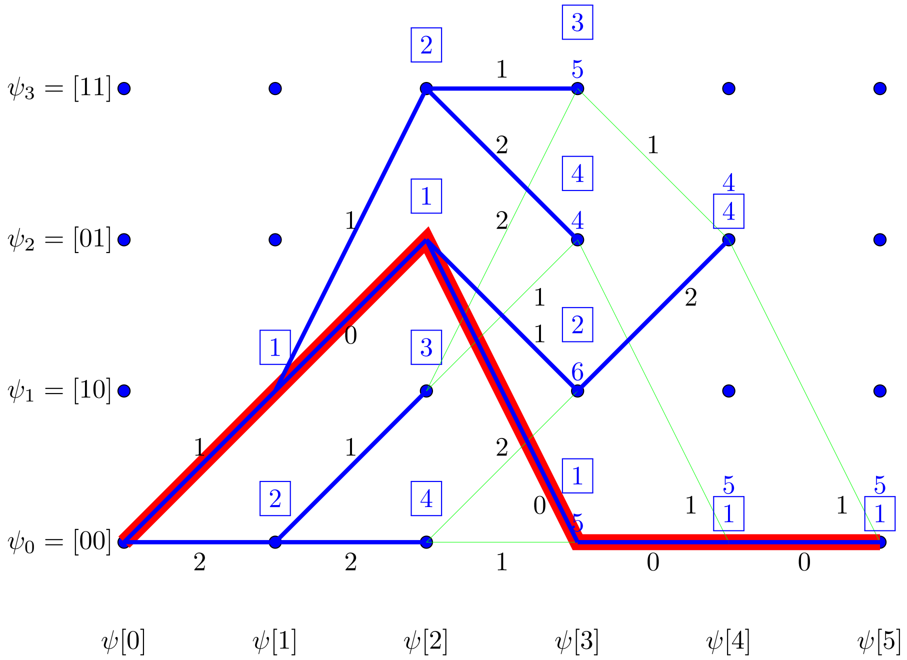
$$C[m] = 01\ 11\ 01\ 00\ 00$$

wich means that the received sequence is

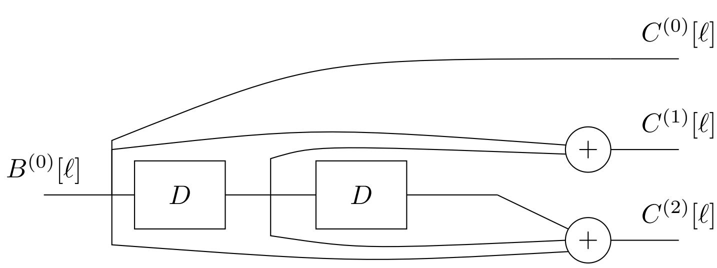
$$R[m] = 11\ 11\ 01\ 00\ 00$$

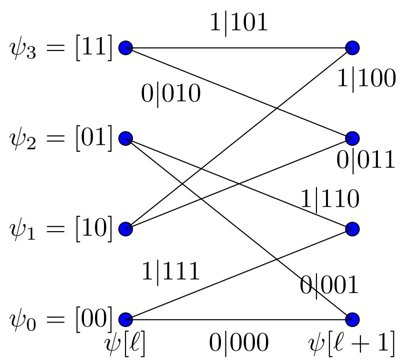
Decoded information sequence is

$$B[\ell] = 100$$



Exercise 5.10 (Solution) a) Schematic representation and trellis diagram





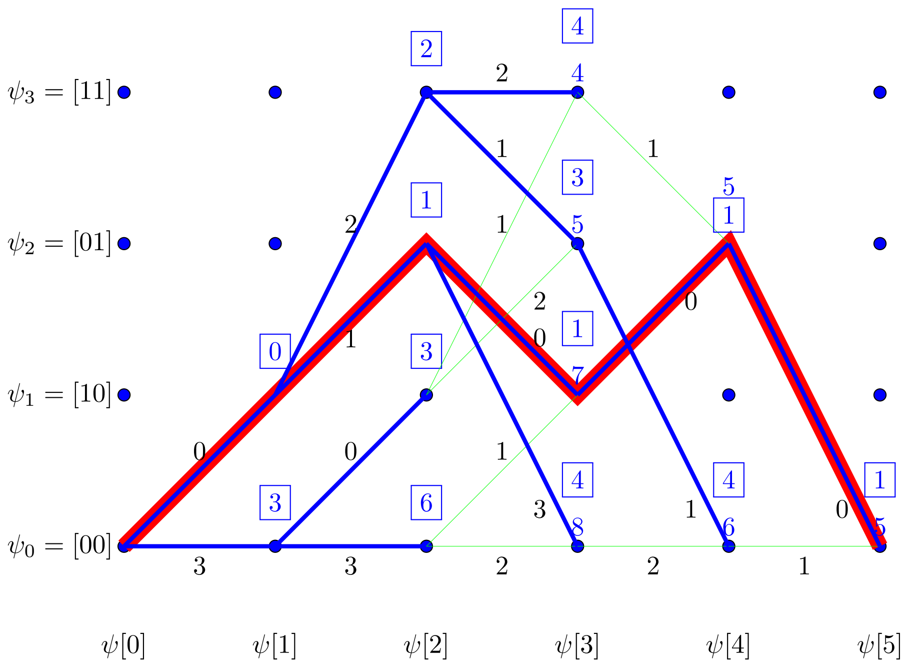
b) Yes, because the first output is just a replica of the input.

c) The maximum likelihood sequence is

m'	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$C[m']$	1	1	1	0	1	1	1	1	0	0	1	1	0	0	1

And the decoded information sequence is

ℓ	0	1	2
$B[\ell]$	1	0	1

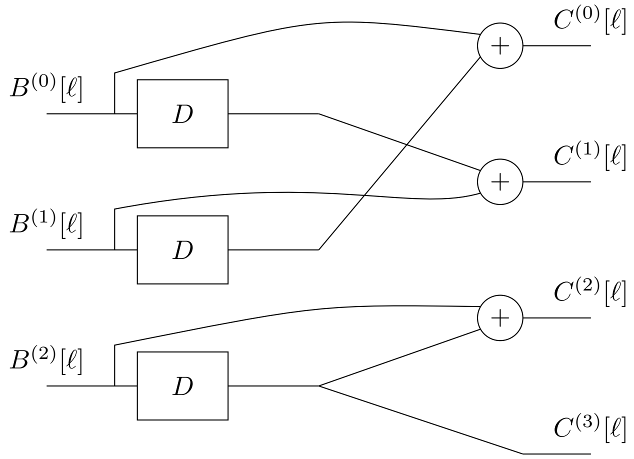


Exercise 5.11 (Solution) a) The dictionnary of the code is

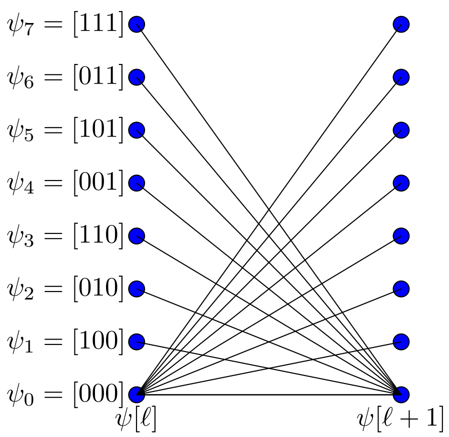
i	$\mathbf{b}_i$	$\mathbf{c}_i$
0	0	0 0 0
1	0	1 1 1

and minimum distance is $d_{min} = 3$.

b) Schematic representation



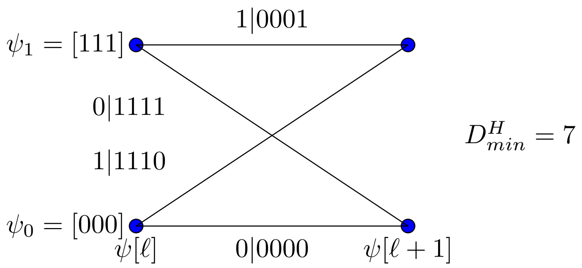
Branches moving from and to the all zeros state are plotted in the figure



$\psi[\ell]$	$\psi[\ell+1]$	Etiquetas
ψ_0	ψ_7	111|1110
	ψ_6	011|0110
	ψ_5	101|1010
	ψ_4	001|0010
	ψ_3	110|1100
	ψ_2	010|0100
	ψ_1	100|1000
	ψ_0	000|0000
ψ_7	ψ_0	000|1111
ψ_6		000|1011
ψ_5		000|0111
ψ_4		000|0011
ψ_3		000|1100
ψ_2		000|1000
ψ_1		000|0100
ψ_0		000|0000

Minimum Hamming distance of the code is $D^H_{min} = 2$.

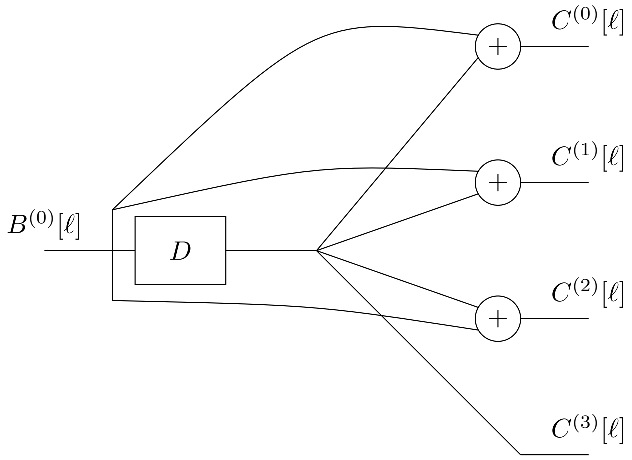
c) Codign rate of the concatenated code is $R = \frac{1}{4}$, and its trellis diagram



The concatenated code has better performance than both individual codes, because its minimum distances is much higher, allowing to correct up to 3 error in every block of 8 encoded bits.

With respect to required bandwidth, to transmit the same number of information bits using the concatenated code, it is necessary to transmit at a rate that is 4/3 higher, which means that the required bandwidth is 4/3 times higher also (with the block code the rate for encoded data is 3 times the information rate, while using the concatenated code the rate of encoded data is 4 times the information rate).

d) The equivalent convolutional encoder is



$$\mathbf{G}(D) = [1+D, 1+D, 1+D, D]$$

Exercise 5.12 (Solution) a) Code A is a linear code because every linear combination of several coded words is another coded word. Code B is not linear, because for instance

$$\mathbf{b}_1 + \mathbf{b}_2 = 01111 \text{ is not a coded word.}$$

Code A is not systematic, because neither the first two bits or the last two bits of the encoded words $\mathbf{c}_i$ do not correspond with the uncoded words $\mathbf{b}_i$. However, Code B is systematic, because the last two bits of every encoded word correspond with its associated information bits (uncoded word).

The number of errors that each code is able to correct:

Using Code A: $t = 1$ error.

Using Code B: $t = 0$ errors.

b) Generating and parity check matrices for Code A are

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{H} = \left[\begin{array}{cc|ccc} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{array}\right].$$

c) Syndrome table

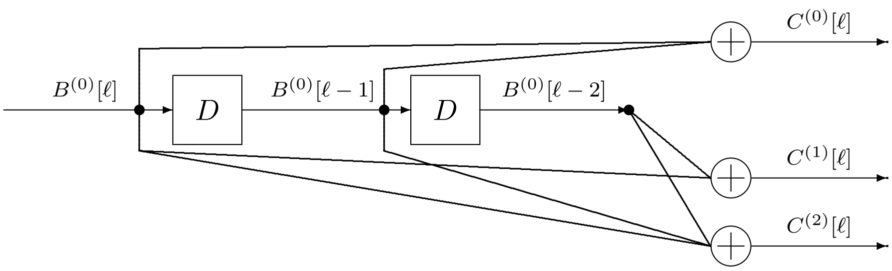
$\boldsymbol{e}$	$\mathbf{s}$
0 0 0 0 0	0 0 0
1 0 0 0 0	1 1 1
0 1 0 0 0	1 0 1
0 0 1 0 0	1 0 0
0 0 0 1 0	0 1 0
0 0 0 0 1	0 0 1
0 0 1 1 0	1 1 0
0 0 0 1 1	0 1 1

Decoded works are

$$\hat{\mathbf{b}}_a = \mathbf{b}_3 = 1\ 1, \ \hat{\mathbf{b}}_b = \mathbf{b}_2 = 1\ 0.$$

Exercise 5.13 (Solution) a) Generating matrix with D polynomials and schematic representation are

$$\mathbf{G}(D) = \left[\ 1+D, \quad 1+D^2, \quad 1+D+D^2\ \right].$$



b) Encoded sequence

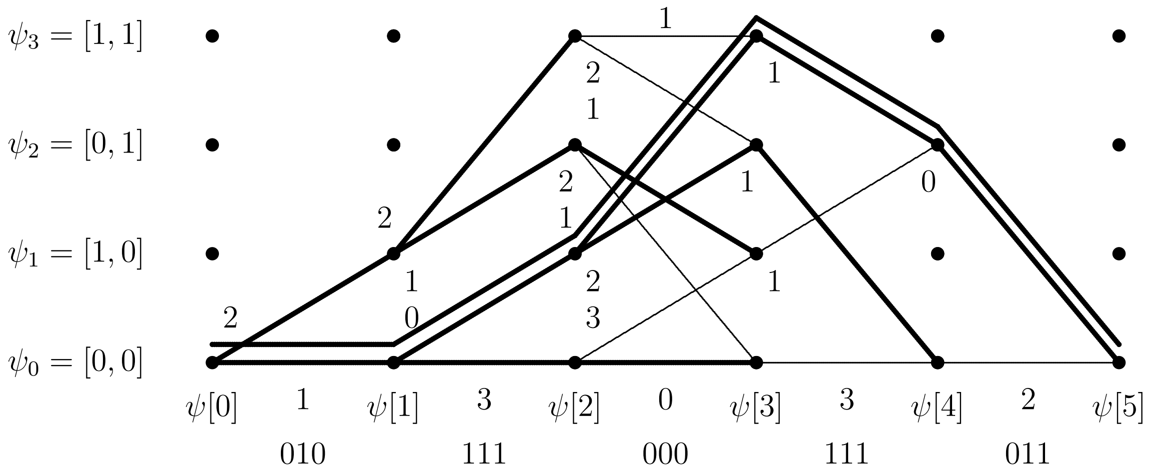
$$C[m] = 000\ 111\ 010\ 110\ 100\ 101\ 011$$

Approximation of the probability of error

$$Pe \approx c \sum_{e=4}^{9} \binom{9}{e} \varepsilon^e \, (1-\varepsilon)^{9-e}$$

c) Viterbi's algorithm to decode the received sequence:

The figure shows the branch metric and highlight the survival paths (thick line) and the final solution (double line)



and table includes the accummulated metric for each state, with boldface highlighting the metric of the survival path. Finally, decoded bits are provided.

	$\ell=1$	$\ell=2$	$\ell=3$	$\ell=4$	$\ell=5$	
ψ_3		4	5/**2**	-	-	$\hat{B}^{(0)}[0]=0$
ψ_2		3	6/**3**	**3**/5	-	$\hat{B}^{(0)}[1]=1$
ψ_1	2	1	**4**/7	-	-	$\hat{B}^{(0)}[2]=1$
ψ_0	1	4	5/**4**	**4**/7	**3**/6	

Exercise 5.14 (Solution) a) Generating matrices

$$\mathbf{G}_1 = \left[\begin{array}{c|cc} 1 & 1 & 1 \end{array}\right],\ \mathbf{G}_2 = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right],\ \mathbf{G}_3 = \left[\begin{array}{ccc|cc} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{array}\right]$$

b) Capabilities of detection and correction:

- Código 1: $d_{min}=3$, detecta $d=2$ errores, corrige $t=1$ error
- Código 2: $d_{min}=2$, detecta $d=1$ errores, corrige $t=0$ errores
- Código 3: $d_{min}=3$, detecta $d=2$ errores, corrige $t=1$ error

Therefore, code 3 is better than code 2.

c) If codes 1 and 2 are concatenated, the resulting code has size $k=1$, $n=4$, and the coded words are

k	C1	C1+C2
0	000	0000
1	111	1111

Minimim distance is now $d_{min} = 4$, which allows to increase the detection capability of code 1, and its better than the one of code 2.

d) If now codes C2 and C3 are concatenated, the size of the code is $k = 3$ and $n = 4$. Generating matrix is

$$\mathbf{G}_{2+3} = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Minimum distance is again $d_{min} = 4$, and again the concatenation improves the performance of the individual codes.

e) Syndrome table

$\boldsymbol{e}$	$\mathbf{s}$
00000	000
10000	100
01000	010
00100	001
00010	011
00001	101
11000	110
01001	111

Decoding consists in the following steps:

1.- Computation of syndrome: $\mathbf{s} = \mathbf{r}\mathbf{H}^T = 001$

2.- Identification of the error patter (from syndrome table): $\mathbf{s} = 001 \rightarrow \boldsymbol{e} = 00100$

3.- Correction: $\hat{\mathbf{c}} = \mathbf{r} + \boldsymbol{e} = 11011 = \mathbf{c}_3$

4.- Decoding (from dictionary): $\hat{\mathbf{c}} = \mathbf{c}_3 \rightarrow \hat{\mathbf{b}} = \mathbf{b}_3 = 11$

Exercise 5.15 (Solution) a) Encoded words are

$$\mathbf{c}_0 = 0\,0\,0\,0\,0\,0$$

$$\mathbf{c}_2 = 0\,1\,1\,1\,1\,0$$

$$\mathbf{c}_6 = 1\,1\,0\,0\,1\,1$$

The code is not systematic, because the 3 bits of the uncoded word is not includen in the first/last 3 bits of the encoded words.

Capacity of detection

$$d = 2 \text{ errors}.$$

Capacity of correction

$$t = 1 \text{ error}.$$

b) Matrices

$$\mathbf{G} == \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{H} = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

c) Syndrome table

$\boldsymbol{e}$	$\mathbf{s}$
0 0 0 0 0 0	0 0 0
1 0 0 0 0 0	1 1 0
0 1 0 0 0 0	1 0 1
0 0 1 0 0 0	0 1 1
0 0 0 1 0 0	1 0 0
0 0 0 0 1 0	0 1 0
0 0 0 0 0 1	0 0 1
0 0 1 1 0 0	1 1 1

To decode the received word, the required steps are:

1.- Computation of the syndrome

$$\mathbf{s} = \mathbf{r}\mathbf{H}^T = 0\ 1\ 1.$$

2.- Identification of the error pattern

$$\boldsymbol{e} = 0\ 0\ 1\ 0\ 0\ 0.$$

3.- Correction

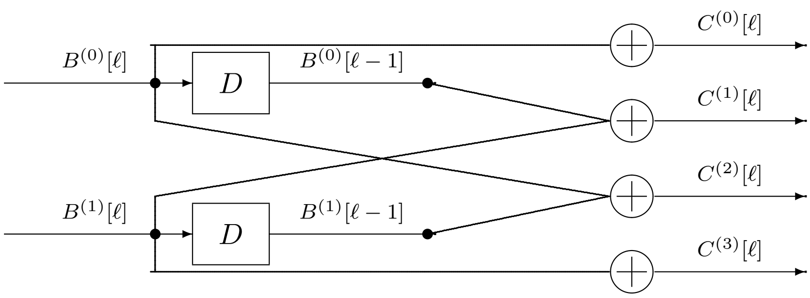
$$\hat{\mathbf{c}} = \mathbf{r} + \varepsilon = 1\ 1\ 0\ 0\ 1\ 1 = \mathbf{c}_6.$$

4.- Decoding

$$\hat{\mathbf{b}} = \mathbf{b}_6 = 1\ 1\ 0.$$

Exercise 5.16 (Solution) a) Generating matrix and schematic representation are

$$\mathbf{G}(D) = \begin{bmatrix} 1 & D & 1 & 0 \\ 0 & 1 & D & 1 \end{bmatrix}.$$



b) Encoded sequence is

$$C[m] = 1010\ 1011\ 0011\ 0010\ 0000$$

Probability of error

$$Pe \approx c \sum_{e=2}^{8} \binom{8}{i} \varepsilon^e\ (1-\varepsilon)^{8-e}$$

c) To obtain the decoded word it is necessary to apply Viterbi's algorithm. The figure shows the branch metric used in this algorthm, highlighting the survival paths (wider lines) and the final solution (double line).

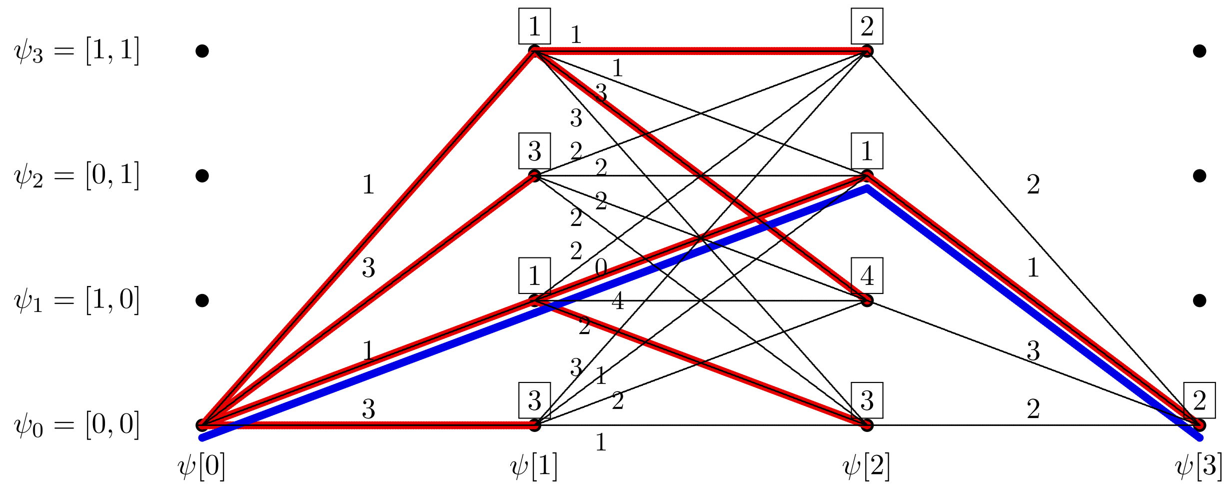


Table with accumulated metrics for each state, highlighting the one corresponding to the survival path, and presenting the decoded bits.

	$\ell=1$	$\ell=2$	$\ell=3$
ψ_3	[1]	[2]/5/3/6	
ψ_2	[3]	2/5/[1]/4	
ψ_1	[1]	[4]/5/5/5	
ψ_0	[3]	4/5/[3]/4	4/[2]/7/5

$\hat{B}^{(0)}[0]=1$
$\hat{B}^{(1)}[0]=0$
$\hat{B}^{(0)}[1]=0$
$\hat{B}^{(1)}[1]=1$

La solución es

$$\hat{B}[m] = 1\ 0\ 0\ 1$$

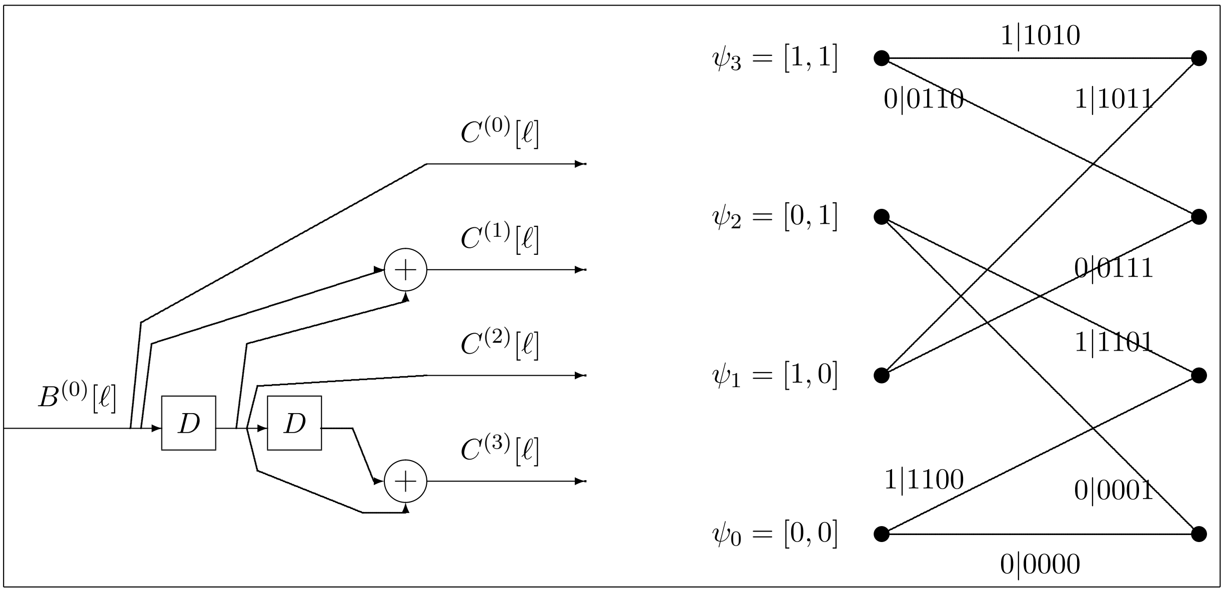
Exercise 5.17 (Solution) a) For the first code

$$P_e \approx c \sum_{e=1}^{4} \binom{4}{e} \varepsilon^e\, (1-\varepsilon)^{4-e}.$$

For the second code

$$P_e \approx c \sum_{e=2}^{8} \binom{8}{e} \varepsilon^e\, (1-\varepsilon)^{8-e}.$$

b) Schematic representation and trellis diagram for the concatenated code



Exercise 5.18 (Solution) a) Generating matrix is

$$\mathbf{G}_2 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

In this case, the number of errors that can be corrected is

$$t = \left\lfloor \frac{d_{min}}{2} \right\rfloor = 0$$

for both codes, $\mathcal{C}_1$ and $\mathcal{C}_2$.

Both codes are not systematic, because neither the first or last k bits of the encoded workd do not correspond to the uncoded information bits (being $k = 2$ for code $\mathcal{C}_1$ and $k = 3$ for code $\mathcal{C}_2$). This is equivalent to not having an identity matrix in the first or last k colums of the generating matrix.

b) Dictionary of the concatenated code is

$\mathbf{b}_i(\mathcal{C}_1)$	$\mathbf{c}_i(\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2)$
00	00000
01	01111
10	10110
11	11001

And generating matrix

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

The concatenated code is systematic: now the first two bits of every coded word correspond with the uncoded information bits. Equivalently, the first two columns of the generating matrix contain an identity matrix. The code is able to correct all pattern of

$$t = 1 \text{ error.}$$

c) Parity check matrix

$$\mathbf{H} = \left[\begin{array}{cc|ccc} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

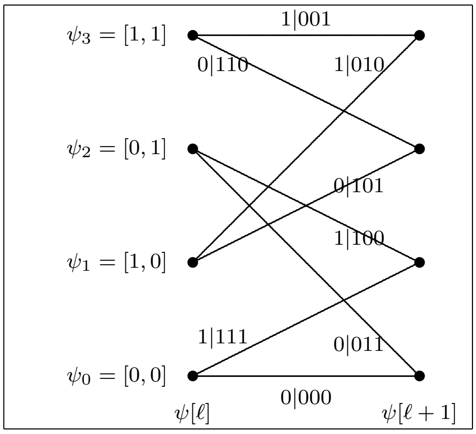
Syndrome table

$\boldsymbol{e}$	$\mathbf{s}$
00000	000
10000	110
01000	111
00100	100
00010	010
00001	001
00011	011
00101	101

Exercise 5.19 (Solution) a) Generating matrix

$$\mathbf{G}(D) = [1 + D,\ 1 + D^2,\ 1 + D + D^2]_{1\times 3}.$$

Trellis diagram

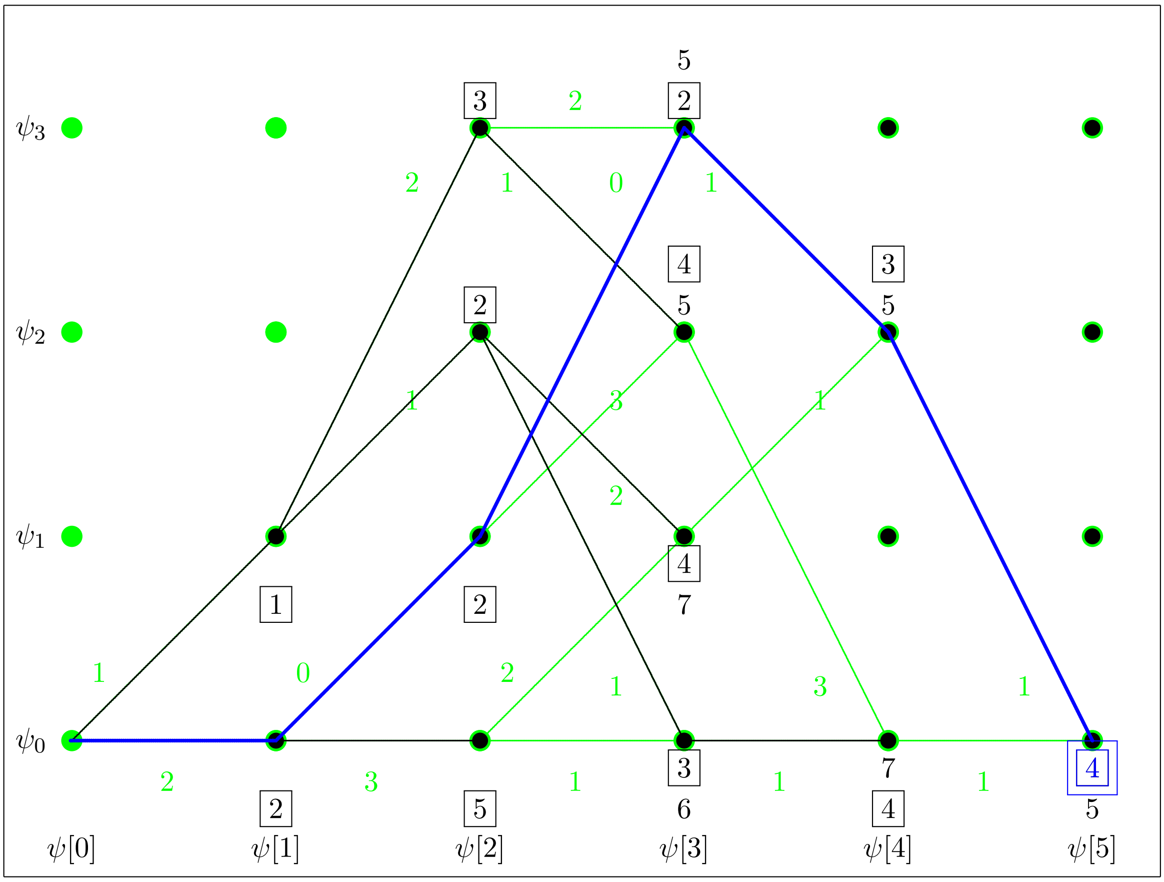


Probability of error is

$$P_e \approx c \sum_{e=4}^{9} \binom{9}{e} \varepsilon^e \, (1-\varepsilon)^{9-e}$$

where $\varepsilon = 10^{-4}$ in this case.

b) To decode the received sequence, Viterbi's algorithm is used



The result is the following decoded information sequence

$$\hat{B}^{(0)}[0] = 0,\ \hat{B}^{(0)}[1] = 1,\ \hat{B}^{(0)}[2] = 1.$$

Exercise 5.20 (Solution) a) Coding rate

$$R = \frac{4}{7}$$

Generating matrix

$$\mathbf{G} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

$$d_{min} = 3$$

and the code can detect patterns of up to

$$d = 2 \text{ errors}$$

and is able to correct all patterns of up to

$$t = 1 \text{ error}$$

It is a perfect code because it has the minimum required redundancy to correct up to 1 error, which meand that the following condition is satisfied

$$\sum_{j=0}^{t} \binom{n}{j} = 2^{n-k} \text{ in this case } \Rightarrow \binom{7}{0} + \binom{7}{1} = 1 + 7 = 8$$

b) Parity check matrix is

$$\mathbf{H} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Syndrome table

$\boldsymbol{e}$	$\mathbf{s}$
0000000	000
1000000	011
0100000	101
0010000	110
0001000	111
0000100	100
0000010	010
0000001	001

c) Syndrome based decoding consists of the following steps:

- Computation of the syndrome

$$\mathbf{s} = \mathbf{r} \times \mathbf{H} = 111$$

- Identification of the error pattern (from syndrome table)

$$\mathbf{s} = 111 \rightarrow \boldsymbol{e} = 0001000$$

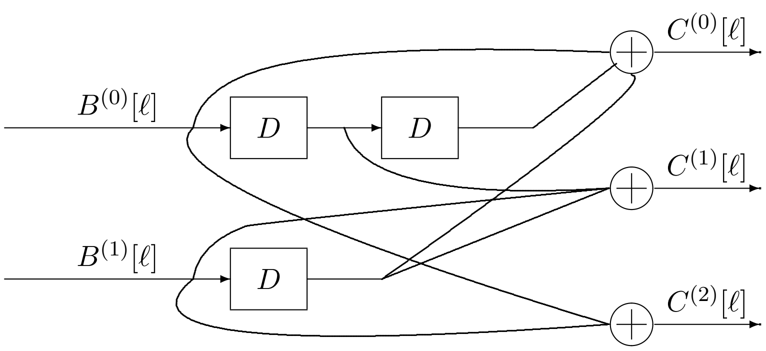
- Correction

$$\hat{\mathbf{c}} = \mathbf{r} + \boldsymbol{e} = 0110011$$

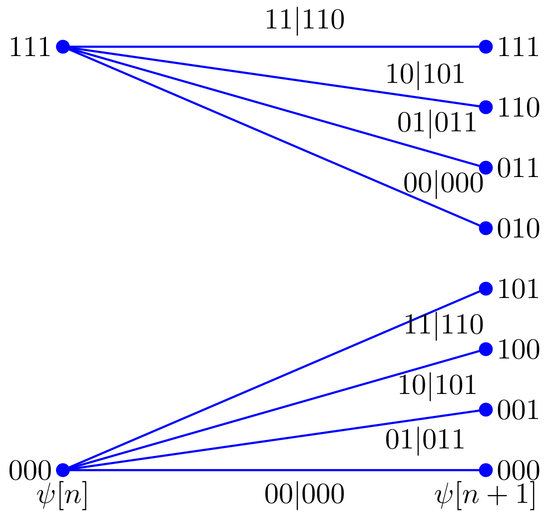
- Decoding (through the dictionary)

$$\hat{\mathbf{c}} \rightarrow \hat{\mathbf{b}} = 1101$$

Exercise 5.21 (Solution) a) Schematic representation



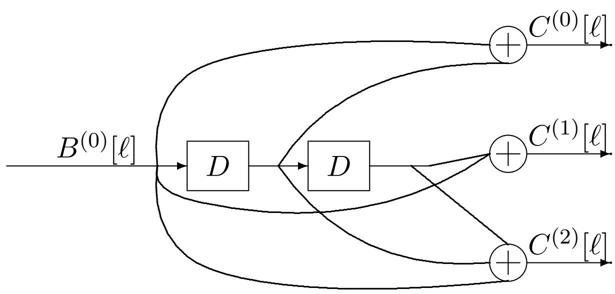
Trellis diagram



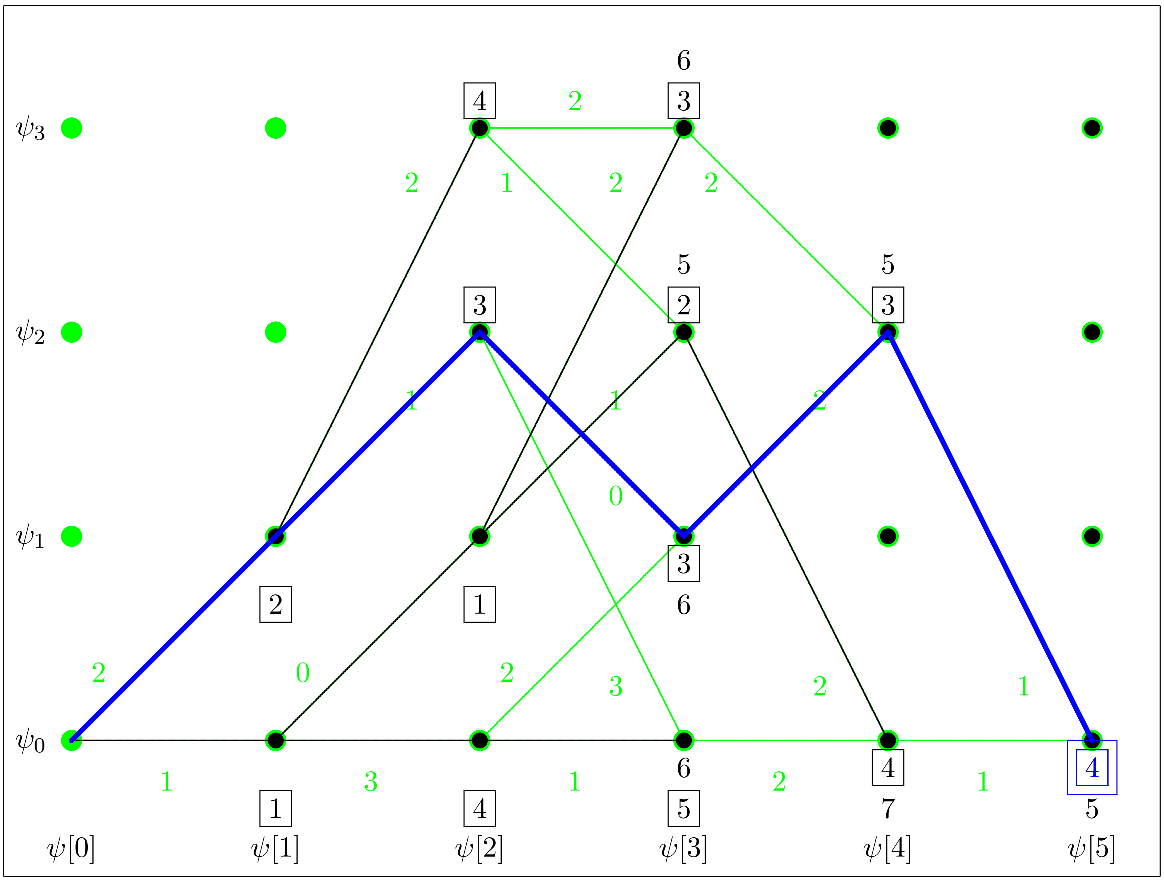
b) Generating matrix

$$\mathbf{G} = \begin{bmatrix} 1+D & 1+D^2 & 1+D+D^2 \end{bmatrix}$$

Schematic representation



c) To decode the received sequence, Viterbi's algorithm is used. The result of this algorithm is shown in ghe figure



The decoded information sequence is

$$\hat{B}^{(0)}[0] = 1,\ \hat{B}^{(0)}[1] = 0,\ \hat{B}^{(0)}[2] = 1.$$

Exercise 5.22 (Solution) a) Parity check matriz and syndrome table

$$\mathbf{H} = \left[\begin{array}{cc|ccc} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$\boldsymbol{e}$	$\mathbf{s}$
00000	000
10000	011
01000	110
00100	100
00010	010
00001	001
11000	101
10100	111

b) Coding rate

$$R = \frac{2}{5}$$

Generator matrix

$$\mathbf{G} = \left[\begin{array}{ccccc} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

The number of errors that a linear block code can correct is related with the minimum Hamming distance of the code. This distance can be obtained by the lower number of ones (weight) of a coded word different than the all zeros coded word. In this case

$$d_{min} = 3$$

and the number of errors that can be detected and corrected, respectively

Detect up to $d = 2$ errors

Correct all patterns of up to $t = 1$ error

It is a non perfect code because it does not have the minimum necessary redundancy to corect one error, which means

$$\sum_{j=0}^{t} \binom{n}{j} \neq 2^{n-k}, \text{ in this case } \Rightarrow \binom{5}{0} + \binom{5}{1} = 1 + 5 < 8$$

This can be also seen through the syndrom table: after introducing all the patterns with up to $t = 1$ errors, there are still 2 unassigned syndromes, that later are associated to error patterns of two error.

The probability of error is

$$P_e = \left[\binom{5}{2} - 2\right] \varepsilon^2 \, (1-\varepsilon)^3 + \sum_{e=3}^{5} \binom{5}{e} \varepsilon^e \, (1-\varepsilon)^{5-e},$$

where $\varepsilon = Q\left(\frac{1}{\sqrt{N_0/2}}\right)$ is the bit error rate with a 2-PAM constellation.

c) Syndrome-based decoding:

- Syndrome

$$\mathbf{s} = \mathbf{r} \times \mathbf{H}^T = 010$$

- Error pattern

$$\mathbf{s} = 010 \rightarrow \boldsymbol{e} = 00010$$

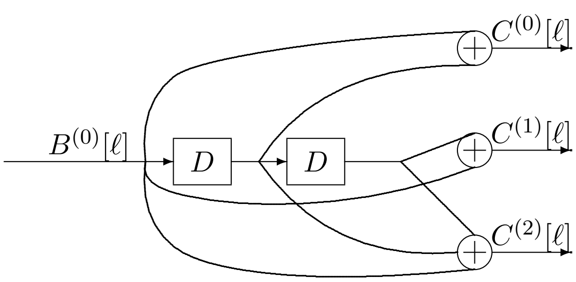
- Correction

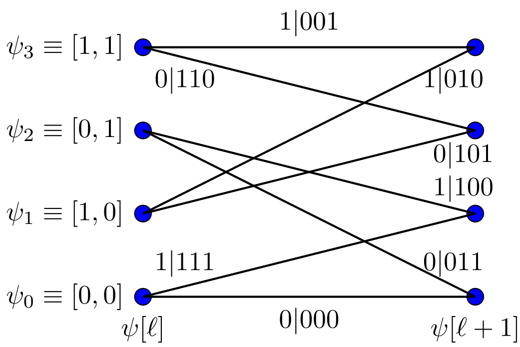
$$\hat{\mathbf{c}} = \mathbf{r} + \boldsymbol{e} = 11101$$

- Decoding

$$\hat{\mathbf{c}} \rightarrow \hat{\mathbf{b}} = 11$$

Exercise 5.23 (Solution) a) Schematic representation and trellis diagram



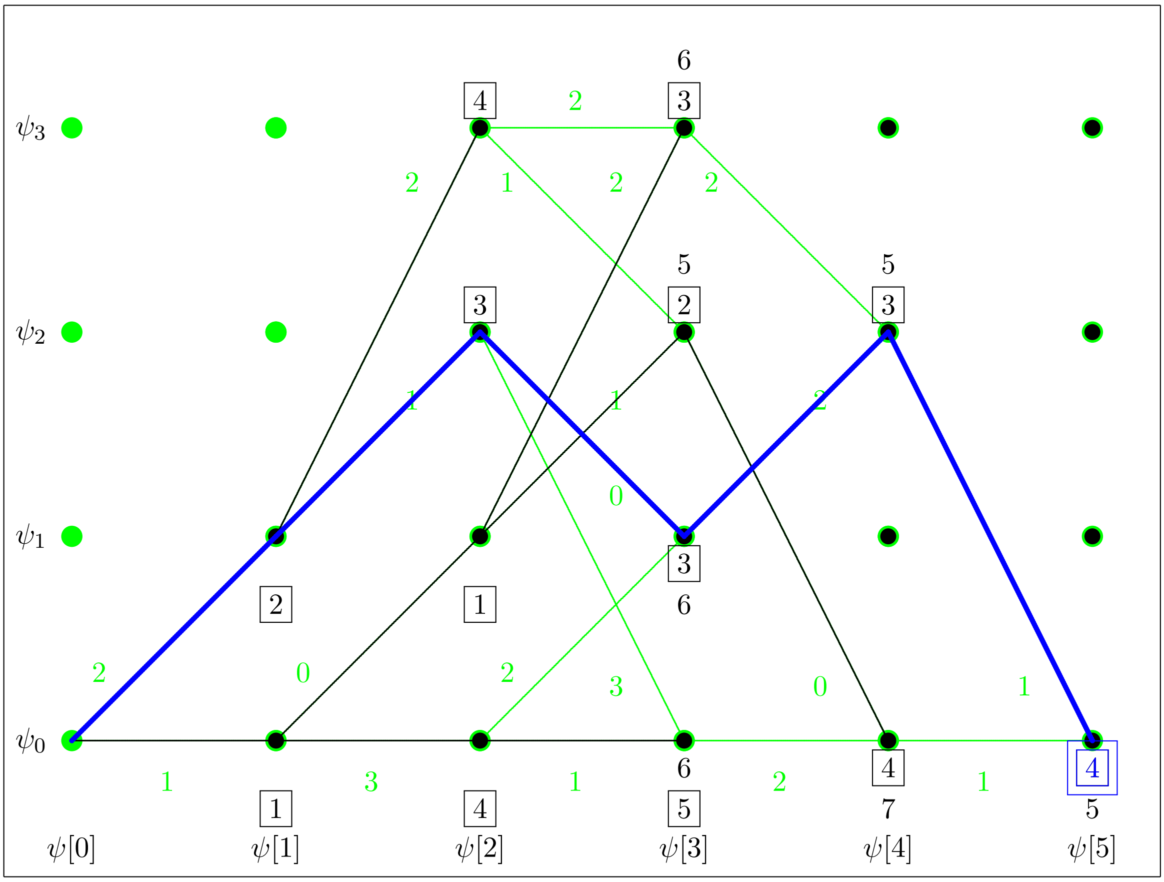


b) Probability of error

$$P_e \approx c \sum_{e=4}^{9} \binom{9}{e} \varepsilon^e \, (1-\varepsilon)^{9-e}$$

where $\varepsilon = Q\left(\frac{1}{\sqrt{N_0/2}}\right)$ is the bit error rate using a 2-PAM constellation.

c) Decoding: Viterbi's algorithm



The decoded bit sequence is

$$\hat{B}^{(0)}[0] = 1,\ \hat{B}^{(0)}[1] = 0,\ \hat{B}^{(0)}[2] = 1.$$

Exercise 5.24 (Solution) a) Coding rate

$$R = \frac{1}{2}.$$

Minimum Hamming distance is given by the lowest number of ones (weight) in all coded word different than the "all zeros" coded word: $d_{min} = 3$. Then

$$d = 2 \text{ (errors can be detected)},\ t = 1 \text{ (error can be corrected)}$$

b) Parity check matrix

$$\mathbf{H} = \left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Syndrome table:

$\boldsymbol{e}$	s
000000	000
100000	011
010000	110
001000	101

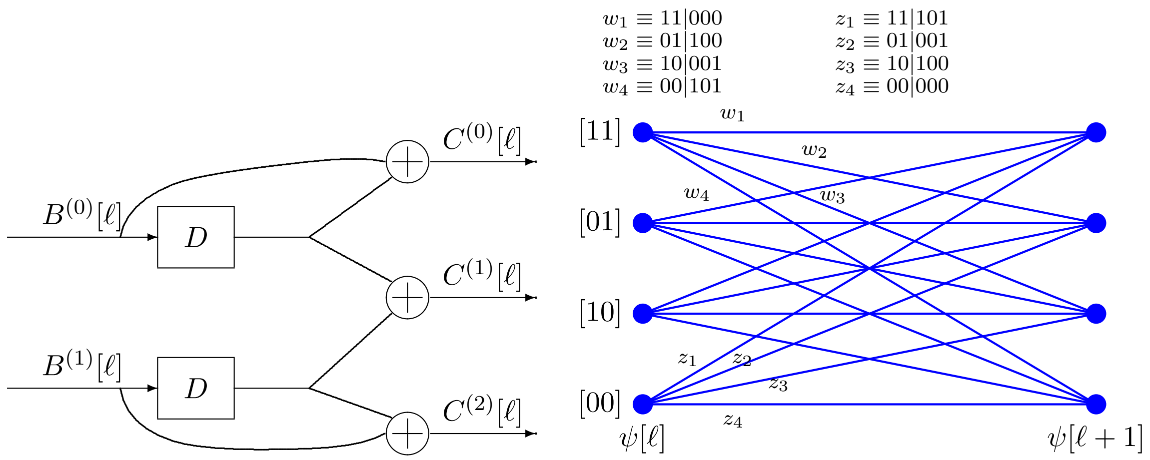
$\boldsymbol{e}$	s
000100	100
000010	010
000001	001
100100	111

The error pattern for the last syndrome is not unique (it is one of the possible patterns).

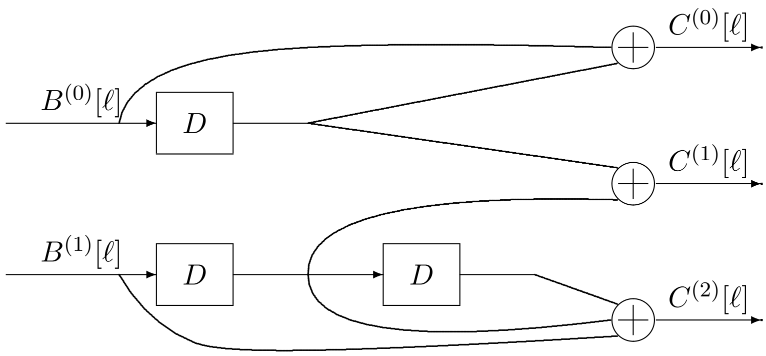
c) Syndrome-based decoding:

- Syndrome
$$\mathbf{s} = \mathbf{r} \times \mathbf{H}^T = 110$$
- Error pattern
$$\mathbf{s} = 110 \rightarrow \boldsymbol{e} = 010000$$
- Correction
$$\hat{\mathbf{c}} = \mathbf{r} + \boldsymbol{e} = 101110 = \mathbf{c}_2$$
- Decoding
$$\hat{\mathbf{c}} = \mathbf{c}_2 \rightarrow \hat{\mathbf{b}} = \mathbf{b}_2 = 010$$

Exercise 5.25 (Solution) a) Schematic representation and trellis diagram



b) Generator matrix and trellis diagram for the second encoder



$$\mathbf{G} = \begin{bmatrix} 1+D & D & 0 \\ 0 & D & 1+D+D^2 \end{bmatrix}$$

c) Encoded bit sequence
$$C[m'] = 100011101101111\cdots$$