Golay, Heisenberg and Weyl

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Princeton University

Abstract

Sixty years ago, efforts by Marcel Golay to improve the sensitivity of far infrared spectrometry led to his discovery of pairs of complementary sequences. These sequences are finding new application in active sensing, where the challenge is how to see faster, to see more finely where necessary, and to see with greater sensitivity, by being more discriminating about how we look.

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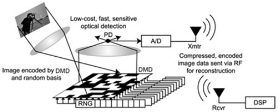


Rice single pixel camera: compressive sensing and the challenge of dimension reduction

1950 Hamming Codes

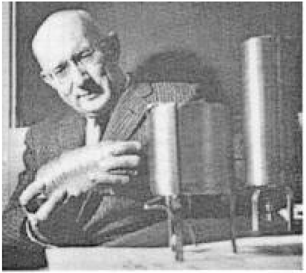
1949 Multi-slit Golay spectrometer

1942 Discovery of the first order Reed-Muller codes by R. A. Fisher

1935 Development of weighing designs by F. Yates

Given a measurement device with mean zero and variance σ^2 how many measurements are needed to measure each of N objects with variance σ^2/N?

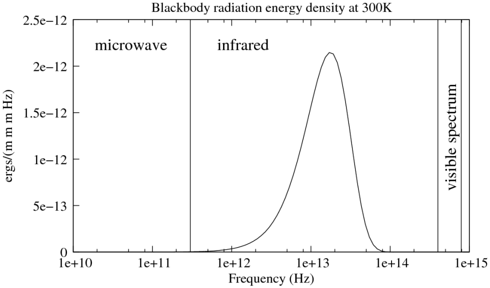
Far Infrared Spectrometry identifies molecules by detecting the characteristic absorption frequencies of specific chemical bounds.



- **1949** — Spectrometer with spinning disks and slits encoding Walsh functions
- **1951** — Spectrometer with fixed slits encoding Golay complementary pairs
- **1952** — Bridges across the infrared radio gap – Proc. IRE.

Obstacles to Infrared Spectrometry

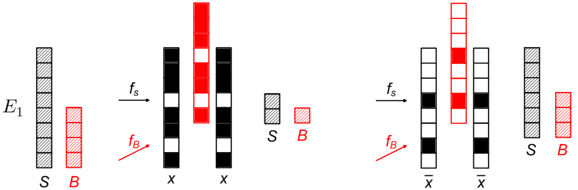
- Sources of interest are typically small thus emit and absorb weakly.
- Blackbody radiation from the environment and the equipment itself at room temperature is strongly concentrated in the infrared spectrum and overlaps the signal of interest.



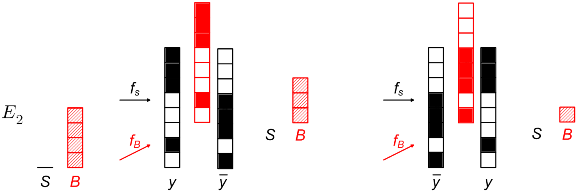
- Detectors were temperature sensors that could not by themselves distinguish between different frequencies of infrared radiation but merely integrated total thermal energy received.

The Origin of Golay Complementary Pairs

PATH 1: x $= + + + - + + - +$



PATH 2: y $= + + + - - - + -$



$$E_S = E_1 - E_2 \Longleftrightarrow R_x(k) + R_y(k) = 2L\delta_{k,0}$$

Golay Complementary Sequences (Golay Pairs)

Definition: Two length L unimodular sequences $x(\ell)$ and $y(\ell)$ are Golay complementary if the sum of their autocorrelation functions satisfies

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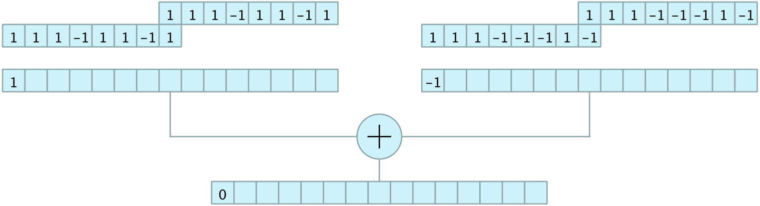
for all $-(L-1) \leq k \leq L-1$.



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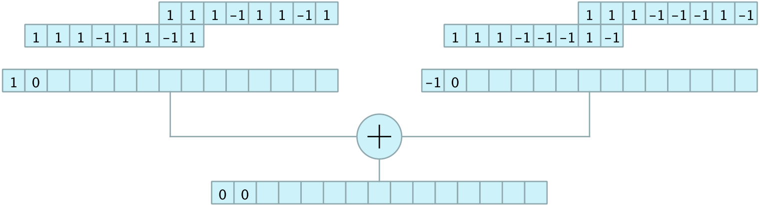

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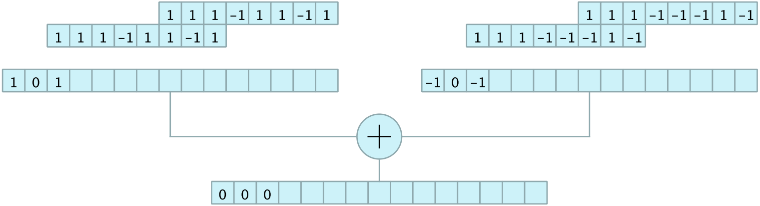

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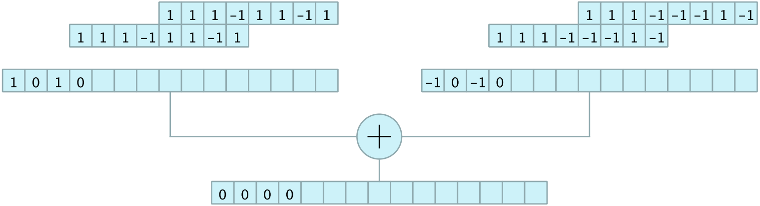

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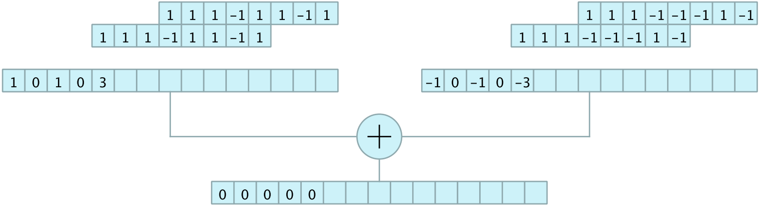

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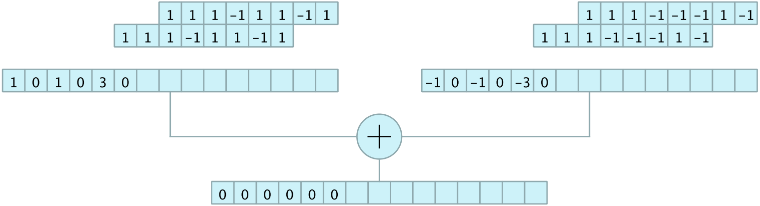

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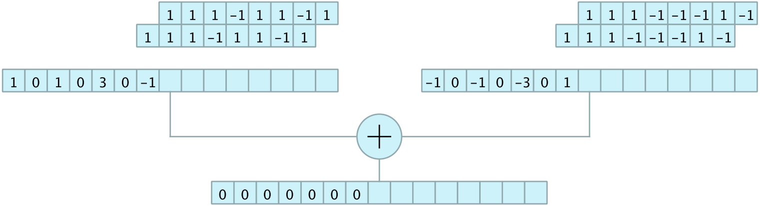

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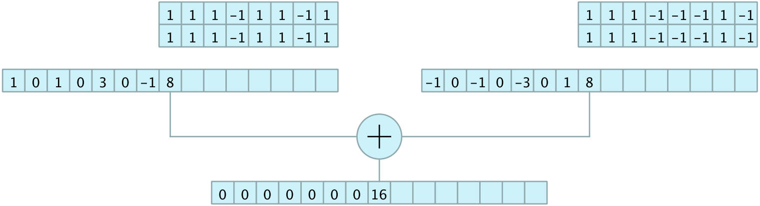

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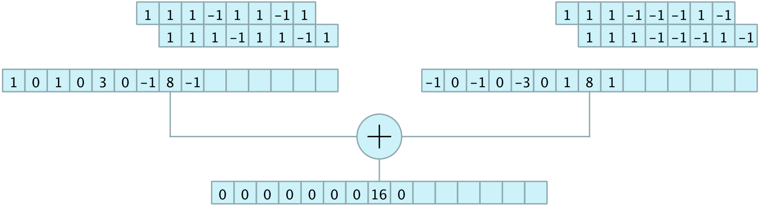

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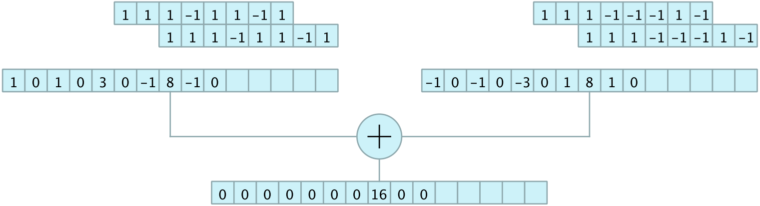

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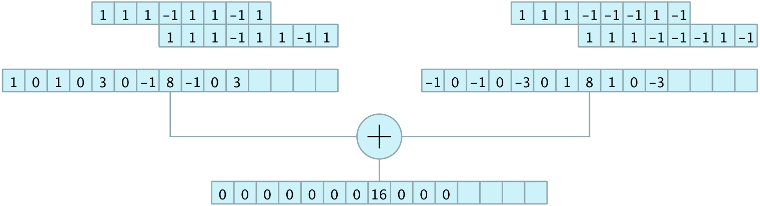

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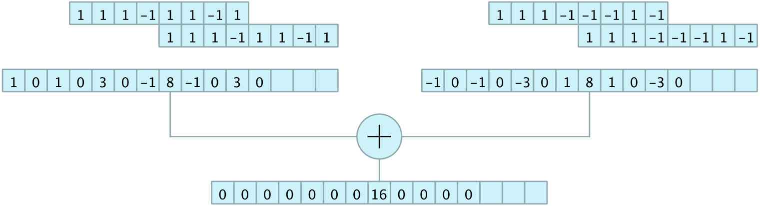

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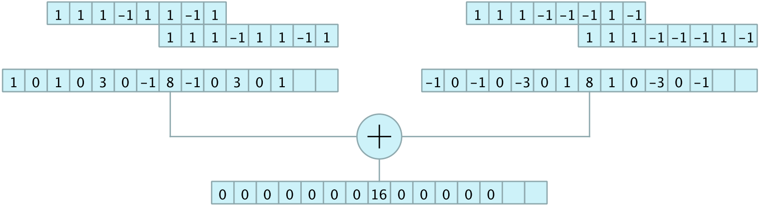

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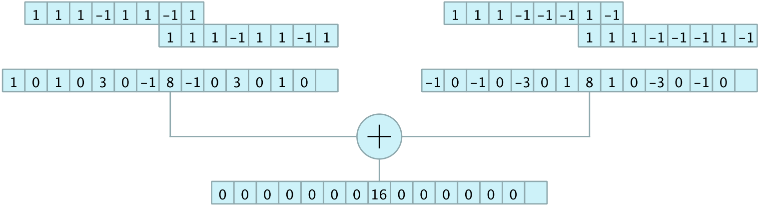

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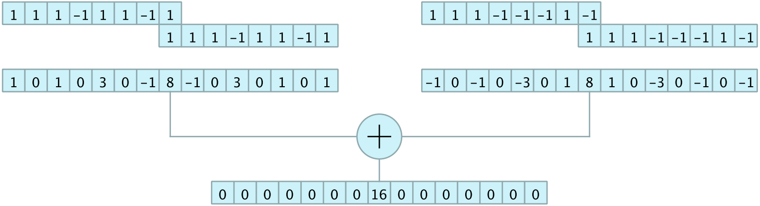
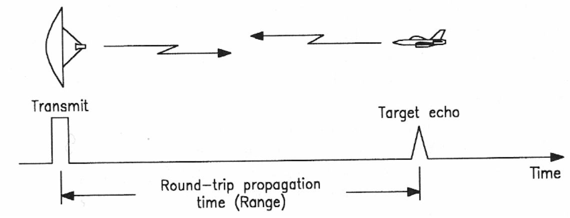
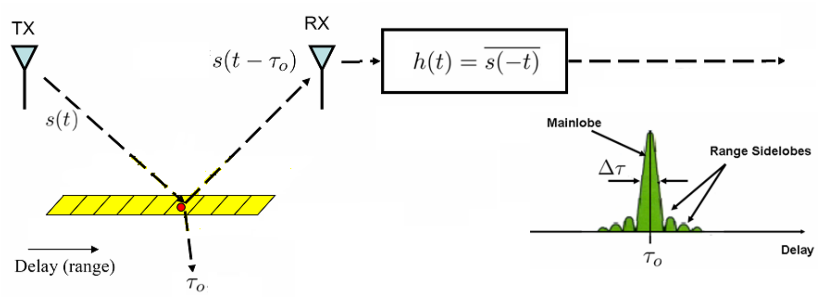


Radar Fundamentals

Illuminate a scene with a waveform and analyze the return to

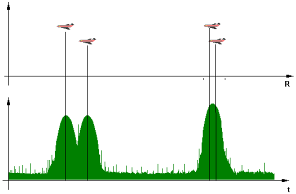


- Detect the presence of a target
- Estimate target range from round trip delay
- Estimate target velocity from Doppler effect



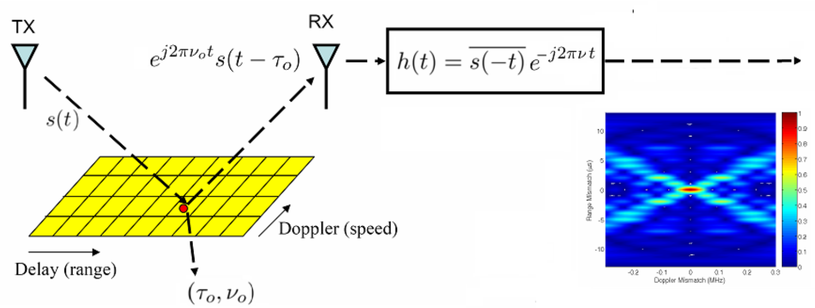
Autocorrelation Function:

$$R_s(\tau) = \int_{-\infty}^{\infty} s(t)\overline{s(t-\tau)}dt$$



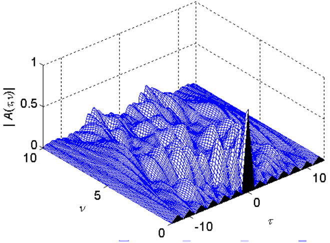
Ideal: Impulse-like

Radar Imaging



Ambiguity Function:

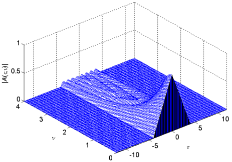
$$A_s(\tau,\nu)=\int_{-\infty}^{\infty} s(t)\overline{s(t-\tau)}e^{-j2\pi\nu t}dt$$

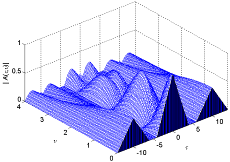


Ideal: Thumbtack

Pulse Train: Sequence of waveforms separated in time

$$\mathcal{S}(t) = \sum_{n=0}^{N-1} s(t-nT)$$





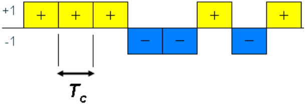
Ambiguity function of pulse train:

$$A_{\mathcal{S}}(\tau,\nu) = \underbrace{\left(\sum_{n=0}^{N-1} e^{jn2\pi\nu T}\right)}_{\text{Doppler shifts over PRIs}} A_s(\tau,\nu) + \text{terms at } mT$$

Radar Waveforms

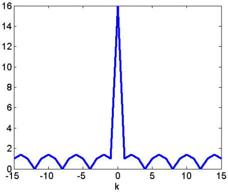
Phase Coded Waveforms:

$$s(t) = \sum_{\ell=0}^{L-1} x(\ell)\mathrm{rect}\left(\frac{t-\ell T_c}{T_c}\right)$$

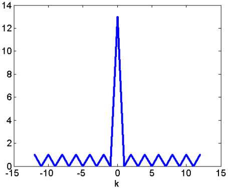


$\{x(\ell)\}_{\ell=0}^{L-1}$: length-$L$ unimodular sequence (typically 1 and -1)

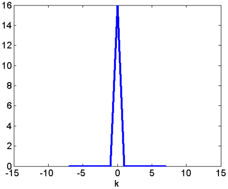
Autocorrelation Functions:



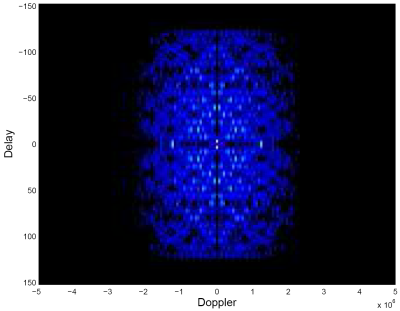
Frank Code



Barker Code



Golay Complementary Codes

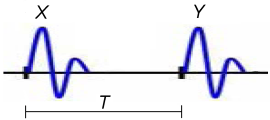


$$A_{s_x}(\tau,\nu) + e^{j2\pi\nu T} A_{s_y}(\tau,\nu)$$

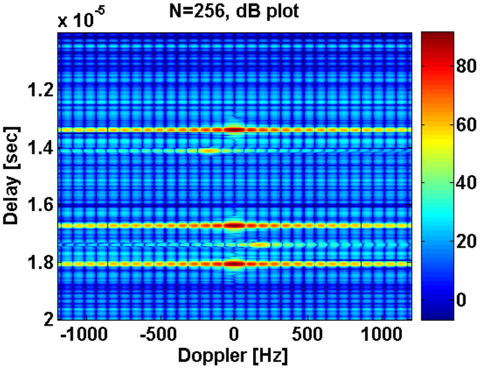
"Although the autocorrelation sidelobe level is zero, the ambiguity function exhibits relatively high sidelobes for nonzero Doppler." [Levanon, Radar Signals, 2004, p. 264]

Why? Roughly speaking

$$R_x(k) + R_y(k)e^{j\theta} \neq \alpha(\theta)\delta_{k,0}$$



Range Sidelobe Problem: A weak target located near a strong target can be masked by the range sidelobes of the ambiguity function centered around the strong target.



Range-Doppler image obtained with conventional pulse train

$x \quad y \quad \cdots \quad x \quad y$

Coordinating Waveforms in Time:

Question: Is it possible to design a *Doppler resilient sequence* of Golay pairs $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$ to have

$$\sum_{n=0}^{N-1} e^{jn\theta} R_{x_n}(k) \approx \beta(\theta)\delta_{k,0}; \quad \text{for all} \quad \theta \in \Theta$$

in a given Doppler interval Θ?

Doppler Resilient Golay Pairs

- Two Golay pairs (x_0, x_1) and (x_2, x_3) over 4 PRIs:

$$R_{x_0}(k)+e^{j\theta}R_{x_1}(k)+e^{j2\theta}R_{x_2}(k)+e^{j3\theta}R_{x_3}(k) \approx \beta(\theta)\delta_{k,0}, \quad \forall\theta \in \Theta$$

- How about around zero Doppler? Taylor Expansion
- First order approximation:

$$\underbrace{\underbrace{0R_{x_0}(k) + R_{x_1}(k)}_{1R_{x_1}(k)} + \underbrace{2R_{x_2}(k) + 3R_{x_3}(k)}_{2 \times 2L\delta_{k,0} + 1R_{x_3}(k)}}_{3 \times 2L\delta_{k,0}}$$

- Condition: (x_1, x_3) also Golay pair.
- Example:

x_0	x_1	x_2	x_3
x	y	y	x

Doppler Resilient Pulse Trains

p-Pulse Train: Transmission of a Golay pair x and y is coordinated according to a binary sequence $\mathbf{p} = \{p_n\}$, $n = 0, \ldots, 2^M - 1$ over $N = 2^M$ PRIs:

$$\underbrace{\frac{1}{2}[R_x(k) + R_y(k)] \sum_{n=0}^{2^M-1} e^{jn\theta}}_{\text{Sidelobe free}} + \underbrace{\frac{1}{2}[R_x(k) - R_y(k)] \sum_{n=0}^{2^M-1} (-1)^{p_n} e^{jn\theta}}_{\text{Range sidelobes}}$$

Key observation: Magnitudes of range sidelobes are proportional to the magnitude of the spectrum of the sequence $(-1)^{p_n}$:

$$S_{\mathbf{p}}(\theta) = \sum_{n=0}^{2^M-1} (-1)^{p_n} e^{jn\theta}$$

Approach: Design $\mathbf{p} = \{p_n\}$ to shape the spectrum $S_{\mathbf{p}}(\theta)$.

PTM Pulse Train: Zero-forcing Taylor Moments

Theorem: To zero-force up to M Taylor moments of the spectrum $S_{\mathbf{p}}(\theta)$ around $\theta = 0$, coordinate the transmission of a Golay pair (x, y) according to the length $N = 2^{M+1}$ PTM sequence, with 0 locations corresponding to x and 1 locations corresponding to y.

Prouhet-Thue-Morse Sequence: The nth term in the PTM sequence p_n is the sum of the binary digits of n mod 2:

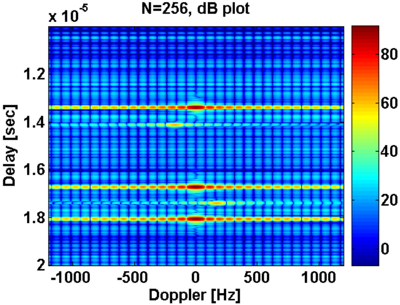
n	(0)=0000	(1)=0001	(2)=0010	(3)=0011
p_n	0	1	1	0

Example: Length-8 PTM Pulse Train

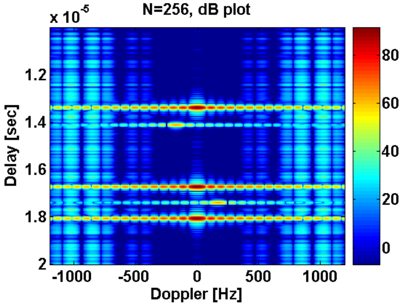
$$\begin{array}{cccccccc} x & y & y & x & y & x & x & y \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{array}$$

PTM Pulse Train in Action

Alternating Pulse Train



PTM Pulse Train



By transmitting a Golay pair according to the PTM sequence we can clear out the range sidelobes along modest Doppler frequencies.

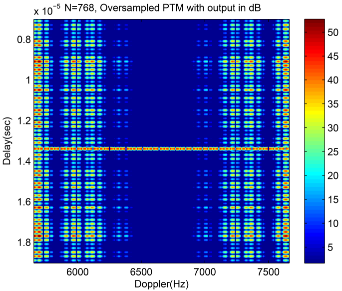
Range Sidelobe Suppression at Higher Doppler Frequencies

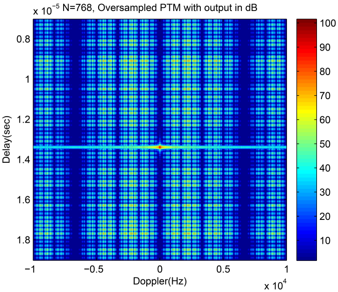
Theorem: There exists a unique first-order RM codeword that minimizes the range sidelobes in the Doppler interval $[\frac{\pi k}{2^M}, \frac{\pi(k+1)}{2^M}]$.

Theorem: The k-oversampled PTM sequence of length $2^M k$ produces an Mth order null at $\theta = 2\pi\ell/k$ for all co-prime ℓ and k.

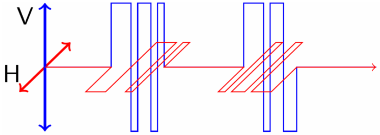
Corollary: Oversampled PTM sequence produces an $(M-1)$th order null at $\theta = 0$ and $(M-2)$th order nulls at all $\theta = \pi\ell/k$.

Example: $M = 3, k = 3 \longrightarrow \{p_n\} = 000111111000\cdots$





Polarization: Alamouti space-time block code is used to coordinate transmission on V and H channels



Multiple Dimensions: Paraunitary filter banks introduced by Tseng and Liu to study acoustic surface waves

Polamouti Transmission:

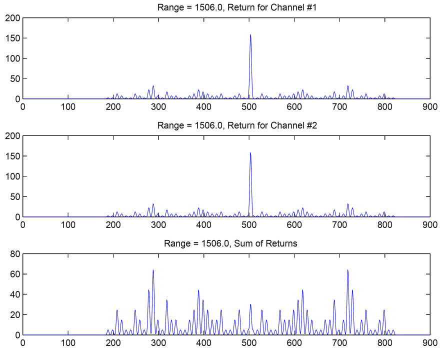
$$R = \begin{pmatrix} h_{VV} & h_{VH} \\ h_{HV} & h_{HH} \end{pmatrix} \begin{pmatrix} x & -\widetilde{y} \\ y & \widetilde{x} \end{pmatrix} + \text{Noise}$$

Unitary property: Interplay between Alamouti signal processing and perfect autocorrelation property of Golay pairs

$$\begin{pmatrix} x & -\widetilde{y} \\ y & \widetilde{x} \end{pmatrix} \begin{pmatrix} \widetilde{x} & \widetilde{y} \\ -y & x \end{pmatrix} = \begin{pmatrix} 2L & 0 \\ 0 & 2L \end{pmatrix}$$

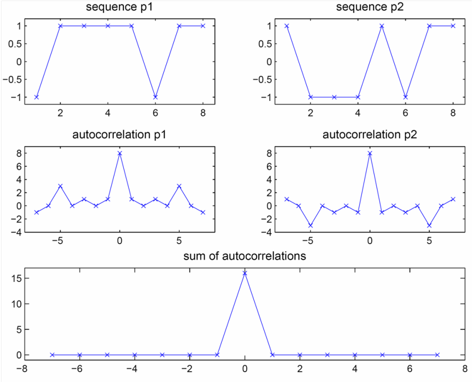
Instantaneous Radar Polarimetry eliminates range sidelobes and improves detection performance, without adding to signal processing complexity

Degrees of Freedom–Frequency



Roadblock to OFDM radar: A pair of complementary waveforms cannot be multiplexed in frequency because of an unknown range-dependent phase term, thereby preventing coherent combining; this limits the applicability of any set of orthogonal waveforms.

Golay Pairs: Autocorrelation Properties



$$R_{p_1}(k) = -R_{p_2}(k), \quad \text{for } k \neq 0$$

$$R_{p_1}^2(k) = R_{p_2}^2(k), \quad \text{for } k \neq 0$$

$$R_{p_1}(2k) = R_{p_2}(2k) = 0, \quad \text{for } k \neq 0$$

$$R_{p_1}(k) + R_{p_2}(k) = 2L\delta(k)$$

Modified Golay Pairs

- Design a pair of sequences such that

$$R_p^2(k) + R_q^2(k) = C\delta(k)$$

- At least one of the squared autocorrelations must be negative at some values of k.
- Possible only if the sequence has imaginary components.
- Let $p_1(n)$ and $p_2(n)$ be a Golay pair. Define

$$q_2(n) = p_2(n)e^{j\frac{\pi}{2}n} \longrightarrow R_{q_2}(k) = R_{p_2}(k)e^{j\frac{\pi}{2}k}$$

Then

$$R_{q_2}^2(k) = R_{p_2}^2(k)e^{j\pi k} = \begin{cases} -R_{p_1}^2(k) & k \text{ odd} \\ 0 & k \neq 0 \text{ even} \\ R_{p_1}^2(k) & k = 0 \end{cases}$$

$$\longrightarrow R_{q_2}^2(k) + R_{p_1}^2(k) = 2L^2\delta(k)$$

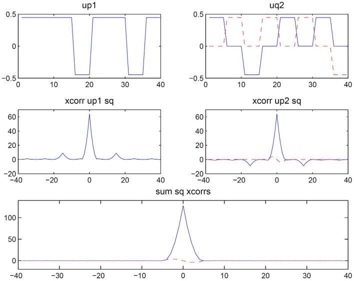
- Modified Golay pair p_1 and q_2 is used to phase code a pulse.
- First code is transmitted at carrier frequency.
- Second code is transmitted twice, offset equally above and below the carrier.

Received signal:

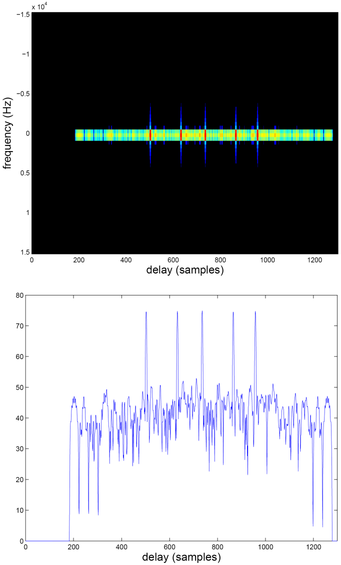
$$\begin{aligned} y_1(t) &= ae^{-j\omega_c d} s_1(t-\tau) \\ y_{2a}(t) &= ae^{-j(\omega_c+\omega_b)d} s_2(t-\tau) \\ y_{2b}(t) &= ae^{-j(\omega_c-\omega_b)d} s_2(t-\tau) \end{aligned}$$

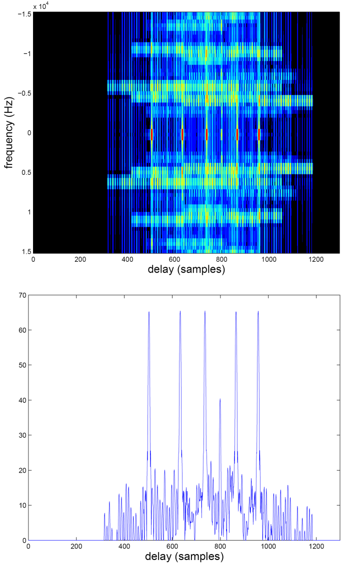
Receiver signal processing:

$$\Gamma(\tau) = R^2_{s_1 y_1}(\tau) + R_{s_2 y_{2a}}(\tau) \times R_{s_2 y_{2b}}(\tau)$$



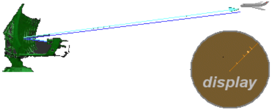
Optimizable Waveforms





SISO Radar:

- Transmits a fixed waveform over multiple pulse repetition intervals (PRIs) for range-Doppler imaging.



MIMO Radar (Waveform Agile):

- Capable of simultaneous transmission of multiple waveforms across frequency, polarization, and space

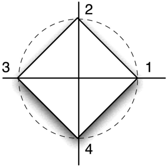
Chesapeake Bay Radar

Radar Networks:

- MIMO radar capabilities plus multiple views

National weather radar network

D_4: The Symmetry Group of the Square



Generated by matrices $\mathrm{x} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\mathrm{z} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$

$\mathrm{xz} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ anticlockwise rotation by $\dfrac{\pi}{2}$

$\mathrm{z} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ reflection in the horizontal axis

D_4 is the set of eight 2×2 matrices $\varepsilon\, D(a,b)$ given by

$$\varepsilon\, D(a,b) = \varepsilon \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)^{a} \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)^{b} \text{ where } \varepsilon = \pm 1 \text{ and } a, b = 0 \text{ or } 1.$$

$$\mathrm{x}^2 = \mathrm{z}^2 = I_2$$

$$\left.\begin{array}{l} \mathrm{zx} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \\[2ex] \mathrm{xz} = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \end{array}\right] \mathrm{xz} = -\mathrm{zx}$$

The Heisenberg-Weyl Group $\mathcal{W}\left(\mathbb{Z}_2^m\right)$

$\mathcal{W}\left(\mathbb{Z}_2^m\right)$ is the m-fold Kronecker product of D_4 extended by iI_{2^m}.

$i^\lambda p_{m-1} \otimes \ldots \otimes p_0$ where $p_j = I_2, \mathrm{x}, \mathrm{z},$ or xz for $j = 0, 1, \ldots, m-1$

There are 2^{2m+2} elements, each represented by a pair of binary m-tuples

$$\mathrm{xz} \otimes \mathrm{x} \otimes \mathrm{z} \otimes \mathrm{xz} \otimes I_2 \leftrightarrow D(\overset{a}{11010}, \overset{b}{10110})$$

Theorem: $D(a,b)D(a',b') = (-1)^{a'.b+b'.a} D(a',b')D(a,b)$

$$D(a,b)^2 = (-1)^{a.b} I_{2^m}$$

$$D(01,11) = \left(\begin{array}{cc|cc} & - & & \\ + & & & \\ \hline & & & + \\ & & - & \end{array}\right), \quad D(10,10) = \left(\begin{array}{cc|cc} & & - & \\ & & & - \\ \hline + & & & \\ & + & & \end{array}\right)$$

Fourier Analysis in the Binary World

$$\begin{tikzcd}
 & \mathcal{HW}(\mathbb{Z}_2^m) \arrow[dl, dash] \arrow[dr, dash] & \\
D(a,0) \arrow[rr, leftrightarrow, "\mathcal{H}"] & & D(0,b)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{HW}(\mathbb{Z}_N),\ N = 2^m \arrow[dl, dash] \arrow[dr, dash] & \\
\Delta(k,0) \arrow[rr, leftrightarrow, "\mathcal{F}"] & & \Delta(0,j)
\end{tikzcd}$$

- The operators $D(a,0)$ are the <u>time shifts</u> of the binary world.
- The operators $D(0,b)$ are the <u>frequency shifts</u> of the binary world.
- Walsh functions are the <u>sinusoids</u> of the binary world– eigenfunctions of the time shift operator.

Chirps in the Binary World

Second order Reed-Muller codewords are the chirps of the binary world.

$$
\begin{array}{ccc}
\text{Maximal Commutative Subgroup} & X & \longrightarrow & X_P = d_P^{-1} X d_P \\
 & | & & | \\
\text{Orthonormal Basis} & H_{2^m} & \longrightarrow & H_{2^m} d_P
\end{array}
\qquad d_P = \text{diag}[i^{vPv^T}]
$$

Example: $m = 3, P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$$
H_8 = \frac{1}{2\sqrt{2}} \left[\begin{array}{cccc|cccc}
+ & + & + & + & + & + & + & + \\
+ & - & + & - & + & - & + & - \\
+ & + & - & - & + & + & - & - \\
+ & - & - & + & + & - & - & + \\
\hline
+ & + & + & + & - & - & - & - \\
+ & - & + & - & - & + & - & + \\
+ & + & - & - & - & - & + & + \\
+ & - & - & + & - & + & + & -
\end{array} \right]
\quad
d_P = \begin{array}{cc}
\left[\begin{array}{cccc|cccc}
1 & & & & & & & \\
 & 1 & & & & & & \\
 & & 1 & & & & & \\
 & & & -1 & & & & \\
\hline
 & & & & i & & & \\
 & & & & & i & & \\
 & & & & & & -i & \\
 & & & & & & & i
\end{array} \right]
&
\begin{array}{c}
000 \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \\ 111
\end{array}
\end{array}
$$

Representation of Operators

Inner Products: $(R, S) = Tr(R^\dagger S)$

Hilbert-Schmidt or Frobenius Norm: $\|S\| = Tr(S^\dagger S)^{\frac{1}{2}}$

Orthonormal Basis: $\frac{1}{\sqrt{N}} D(a,b), a, b \in \mathbb{Z}_2^m$ where $N = 2^m$

$$Tr(D(a,b)^\dagger D(a', b')) = N\delta_{a,a'}\delta_{b,b'}$$

Weyl Transform: Given an operator S write

$$S = \frac{1}{N} \sum_{a,b \in \mathbb{Z}_2^m} Tr(D(a,b)^\dagger S) D(a,b)$$

$$= \sum_{a,b \in \mathbb{Z}_2^m} S(a,b) \left[\frac{1}{\sqrt{N}} D(a,b)\right]$$

The Weyl Tranform is the isometry

$$S \longleftrightarrow (S(a,b)) = \left(\frac{1}{\sqrt{N}} Tr(D(a,b)^\dagger S)\right)$$

From Sequences to Rank One Projection Operators

Walsh Sequence: $\theta^\dagger = \frac{1}{2}(+-+-) = \frac{1}{2}\mathbf{1}D(00,01)$

Rank One Projection Operator: $\theta\theta^\dagger = \frac{1}{4}\left[\begin{array}{cc|cc} + & - & + & - \\ - & + & - & + \\ \hline + & - & + & - \\ - & + & - & + \end{array}\right]$

$$\theta\theta^\dagger = \frac{1}{4}\left[I_4 - \left[\begin{array}{cc|cc} & 1 & & \\ 1 & & & \\ \hline & & & 1 \\ & & 1 & \end{array}\right] + \left[\begin{array}{cc|cc} & & 1 & \\ & & & 1 \\ \hline 1 & & & \\ & 1 & & \end{array}\right] - \left[\begin{array}{cc|cc} & & & 1 \\ & & 1 & \\ \hline & 1 & & \\ 1 & & & \end{array}\right]\right]$$

$$= \frac{1}{4}\sum_{a\in\mathbb{Z}_2^2}(-1)^{a.(01)}D(a,0)$$

Dirac Sequence: $\varphi^\dagger = \theta^\dagger H_4 = (0100)$

Rank One Projection Operator: $\varphi\varphi^\dagger = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$$\varphi\varphi^\dagger = \frac{1}{4}\sum_{b\in\mathbb{Z}_2^2}(-1)^{(01).b}D(0,b)$$

Weyl Transforms of Operators

The more symmetries of a sequence θ the smaller is the support of Weyl transform of $\theta\theta^\dagger$.

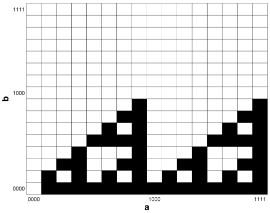
Isotropy Subgroup: $H_\theta = \{g \in \mathcal{W}(\mathbb{Z}_2^m) \,|\, g\theta = c_g\theta\}$

Theorem: H_θ is commutative and $S_\theta(a,b) = 0$ unless $D(a,b)$ commutes with every $D(a',b')$ in H_θ.

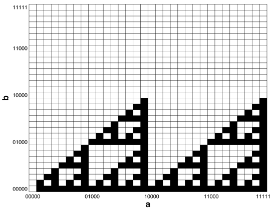
$\mathbf{S_{\Delta,0}}$: Union of supports of cyclic shift operators $\Delta(k,0)$

Theorem: $(a,b) \in S_{\Delta,0} \iff a \neq 0, b_{m-1} = 0$ and a covers b. The support takes the form of a pair of Sierpinski triangles.

$m = 4$:



$m = 5$:



Connecting Periodic and Aperiodic Correlation

$$\theta = \sum_{v, v_{m-1}=0} \theta_v e_v \text{ and } \varphi = \sum_{v, v_{m-1}=0} \varphi_v e_v$$

We may view θ, φ as sequences $\overline{\theta}, \overline{\varphi}$ of length 2^{m-1} or as sequences of length 2^m obtained by padding with zeros.

Proposition: $\overline{\theta}, \overline{\varphi}$ are $\mathbb{Z}$-Golay complementary if θ, φ are $\mathbb{Z}_N$-Golay complementary.

$\mathbb{Z}_N$-Golay Complementary Pairs:

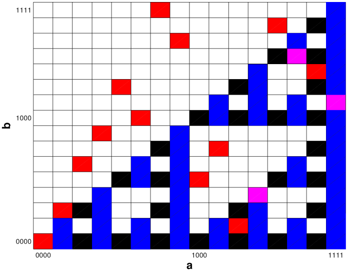
$$\theta^\dagger \Delta(k,0)\theta + \varphi^\dagger \Delta(k,0)\varphi = 0 \quad \text{for } k \neq 0$$

$$Tr\left((P_\theta + P_\varphi)\Delta(k,0)\right) = 0 \quad \text{for } k \neq 0$$

Note: The orthonormal basis $D(a,b)$ from $\mathcal{W}(\mathbb{Z}_2^m)$ provides a sparse representation of P_φ and P_ψ for many widely used sequences φ, ψ.

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\varphi = D(0\dots0, 10\dots0)\theta$$



P minimizes overlap (magenta) between the support of P_θ, P_φ (the subgroup X_P shown in red) and the support of $S_{\Delta,0}$ (black and blue).

$D(0\dots0, 10\dots0)$ removes overlap between the support of $P_\theta + P_\varphi$ and the support of $S_{\Delta,0}$:

$$(S_\varphi + S_\theta)(v, vP) = ((-1)+1)S_\theta(v, vP) = 0$$

Information Theory and Sensing

P. M. Woodward (1953): introduced the narrowband radar ambiguity function to describe the effect of the transmit waveform at matched filter output.

"The reader may feel some disappointment, not unshared by the writer, that the basic question of what to transmit remains substantially unanswered."

Specific Questions:

- How to design measurements?
- How to utilize various modes of diversity with minimal complexity?
- What are the scaling laws? rate-reliability tradeoff?
- How to compress and fuse information?
- How to manage sensor operations and allocate resources?

A Final Thought

Classical Coding Theory

Fourier Analysis