ELE 539B: Coding Theory - Lecture 7

Introduction to Spectral Analysis

Bounded Disparity Coding

Spectra of Markov Chains

1.

Introduction of Spectral Nulls

Transformer coupled circuits, magnetic and optical recording channels have very poor transmission characteristics at low frequencies.

Line Codes are used to avoid spending signal power on those parts of the spectrum where channel SNR is poor.

AMI (Alternate Mark Inversion): transmission over twisted pair cables

$0 \to 0$ and $1 \to \pm 1$ using alternate signs

Null at zero frequency in the transmitted spectrum facilitates simple recovery of the data over transformer coupled circuits.

Buried Servo: Low frequency pilot tracking tones for positioning the read head over a selected track. Line codes that produce a spectral null at some arbitrary frequency are used to make space for pilot tones.

N. Hansen, A head-positioning system using buried servos, IEEE Trans. on Magnetics, vol. MAG-17, pp. 2735-2738, 1981.

Introduction to Spectral Analysis

$$X(t) = \sum_{j=-\infty}^{\infty} x_j s(t-jT)$$

Pulse shape $s(t)$ is prescribed by the application and beyond our control. Spectral shaping is provided by controlling the correlation characteristics of the ensemble of codewords (x_j).

$$X(t) = \lim_{M\to\infty} X_M(t) \quad \text{where} \quad X_M(t) = \sum_{j=-M}^{M} x_j s(t-jT)$$

Fourier transform $\hat{X}_M(\omega) = \hat{x}_M(\omega)\hat{s}(\omega)$ converges, where

$$\hat{x}_M(\omega) = \sum_{\ell=-M}^{M} x_\ell e^{-\omega \ell T} \quad \text{and} \quad \hat{s}(\omega) = \int_{-\infty}^{\infty} s(t) e^{-\omega t} dt.$$

Inverse Fourier transform expresses $X_M(t)$ as a superposition of plane waves

$$X_M(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{X}_M(\omega) e^{-i\omega t} dt$$

Spectral Analysis (contd.)

$H_X(\omega)$: expected power spectral density of the modulated signal $X(t)$.

$$H_X(\omega) = \lim_{M\to\infty}\left[\frac{1}{2M+1} E\left[|\hat{X}_M(\omega)|^2\right]\right]$$

$$= \lim_{M\to\infty}\left[\frac{1}{2M+1} E\left[|\hat{x}_M(\omega)|^2\right] |\hat{s}(\omega)|^2\right]$$

$$= |\hat{s}(\omega)|^2 \lim_{M\to\infty}\left[\frac{1}{2M+1} E\left[\left|\sum_{\ell=-M}^{M} x_\ell e^{-i\omega\ell T}\right|^2\right]\right]$$

$H_x(\omega)$: power spectral density of the line code

$$H_x(\omega) = \lim_{M\to\infty}\left[\frac{1}{2M+1} E\left[\left|\sum_{\ell=-M}^{M} x_\ell e^{-\omega\ell T}\right|^2\right]\right]$$

This is determined by the correlation characteristics of the codewords (x_j)

Wide-Sense Stationarity: Codewords (x_j) are generated by a Markov process – the mean $E[x_j]$ and autocorrelation $E[x_j x_{j+m}]$ are time invariant (independent of j).

Spectral Analysis (contd.)

$$\frac{1}{2M+1} E\left[\left|\sum_{\ell=-M}^{M} x_\ell e^{-i\omega \ell T}\right|^2\right] = \frac{1}{2M+1} \sum_{k,\ell=-M}^{M} E[x_\ell x_k] e^{-i\omega(\ell-k)T}$$

$$= \frac{1}{2M+1} \sum_{k,\ell=-M}^{M} R_x(\ell-k) e^{-i\omega(\ell-k)T}$$

Here $E[x_j x_{j+m}] = R_x(m) = E[x_j x_{j-m}]$

Now set $\ell - k = m$, so that $-2M \le m \le 2M$ and observe that there are $2M+1-|m|$ pairs ℓ, k with $\ell - k = m$

$$= \frac{1}{2M+1} \sum_{m=-2M}^{2M} R_x(m) e^{-i\omega m T} (2M+1-|m|)$$

$$= \left(\sum_{m=-2M}^{2M} R_x(m) e^{-i\omega m T}\right)$$

$$- \left(\frac{1}{2M+1} \sum_{m=-2M}^{2M} R_x(m) |m| e^{-i\omega m T}\right)$$

Spectral Analysis (contd.)

Theorem 7.1. If symbols x_j and x_{j+m} become uncorrelated as $m \to \infty$, then for sufficiently large m, the autocorrelation $|R_x(m)| \le C\lambda^{m-1}$ for some constant C and some $0 \le \lambda < 1$.

Corollary 7.2. $H_x(\omega) = \sum_{m=-\infty}^{\infty} R_x(m) e^{-i\omega m T}$

Proof: $\lim_{M\to\infty} \frac{1}{2M+1} \sum_{m=-2M}^{2M} R_x(m) |m| e^{-i\omega m T} = 0$

Running Digital Sum (RDS) of a sequence $x_0, x_1, \ldots$ at time N

$$v_N = \sum_{i=0}^{N} x_i$$

Theorem 7.3 For line codes generated by a finite state machine, the condition that the RDS be bounded is necessary and sufficient for the existence of a spectral null at zero frequency.

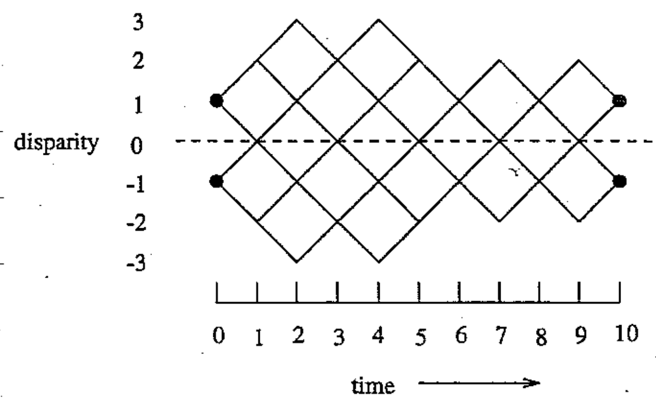
Necessity makes essential use of the finite state machine - see Yoshida and Yajima (1976), Justesen (1982) and Pierobon (1984).

Bounded Disparity Coding.

Sufficiency is straightforward:

$$H_x(0) = \lim_{M\to\infty} \left[\frac{1}{2M+1} E\left[|V_M - V_{-M}|^2\right] \right] = 0.$$

Widmer-Franaszek Code: A rate 8/10 code with a spectral null at dc that is used in magnetic and optical recording.



Q1. Describe this code in terms of a rate 5/6 code of length 6 followed by a rate 3/4 code of length 4.

Q2. Is zero disparity coding possible for this combination of rate and block length?

Spectra of Markov Chains

$S_x(z)$: spectrum of a line code (x_k) generated by a finite state machine

$$S_x(z) = \sum_{m=-\infty}^{\infty} R_x(m)\, z^m, \quad \text{where } z = e^{i\theta}, \text{ and } |\theta| \le \pi$$

Irreducible Markov Chain: Given any pair of encoder states α, β there is a sequence of transitions from α to β.

Theorem 7.4 (Perron-Frobenius) If P is the matrix of state transition probabilities for an irreducible Markov chain, then there is a unique steady state probability distribution π that satisfies $\pi P = \pi$.

Proof: See Horn and Johnson — Matrix Analysis

Remark: The spectral theory of positive matrices was developed by Perron around 1907 and these results were generalized to non-negative matrices by Frobenius.

Non-negative matrix A is irreducible $\Leftrightarrow$ all entries of the matrix $(I+A)^{N-1}$ are positive

Spectra of Markov Chains (contd.)

$R_x(0)$: average transmitted signal power

Let y_{ki} be the symbol associated with the transition from state k to state i – if there is more than one symbol, then y_{ki} is the appropriate average.

$a_i = \sum_k \pi_k P_{ki} y_{ki}$ the weighted sum of symbols that can be transmitted on arriving at state i

$d_j = \sum_\ell P_{j\ell} y_{j\ell}$ the weighted sum of symbols that can be transmitted on departing state j.

$$R_x(m) = a P^{m-1} d$$

Write

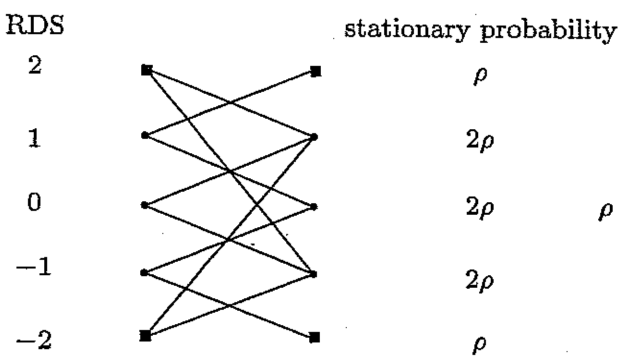
$$a = \frac{(a,\pi)}{(\pi,\pi)}\pi + a_1 \quad \text{with } a_1 \perp \pi, \quad d = \frac{(d,\underline{1})}{(\underline{1},\underline{1})}\underline{1} + d_1 \quad \text{with } d_1 \perp \underline{1}$$

Then
$$aP^{m-1}d = \frac{(a,\pi)(d,\underline{1})}{(\pi,\pi)(\underline{1},\underline{1})}\pi.\underline{1} + \epsilon$$

Spectra of Markov Chains (contd.)

Note that $|\epsilon| \leq C\lambda^{m-1}$ where C is some constant and $0 \leq \lambda < 1$ is the size of the second largest eigenvalue of P. If symbols x_j and x_{j+m} become uncorrelated as $m \to \infty$, then the first term must vanish and $R_x(m)$ decays as asserted in Theorem 7.1.

Example: A 5-state line code with symbols $\pm1, \pm3$ and a spectral null at dc



$$\rho = \frac{1}{8}$$

Calculate stationary probabilities by setting $\pi_{\pm2} = \rho$ and working towards the center:

$$\frac{1}{2}\pi_{\pm1} = \pi_{+2} = \rho$$

so that $\pi_{\pm1} = 2\rho$, and

$$\frac{1}{2}\pi_1 + \frac{1}{2}\pi_{-1} = \pi_0$$

so that $\pi_0 = 2\rho$

Hence $\rho = 1/8$

$$a = \frac{1}{16}(2,4,0,-4,-2)$$

$$d = (-2,0,0,0,2)$$

Spectral Calculations

$$P = \frac{1}{2}\;\begin{array}{c} \\ 2 \\ 1 \\ 0 \\ \bar{1} \\ \bar{2} \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} 2 & 1 & 0 & \bar{1} & \bar{2} \end{array} \\ \left[\begin{array}{ccccc} & 1 & & 1 & \\ 1 & & 1 & & \\ & 1 & & 1 & \\ & & 1 & & 1 \\ & 1 & & 1 & \end{array}\right] \end{array}$$

$R_x(0) = \frac{1}{8}\left(5 + 3\times2\times1 + 5\right) = 2$

this is the average transmitted signal power.

$R_x(1) = ad = -1/2$

$R_x(2) = aPd = a\,(0,-1,0,1,0)^T = -1/2$

$a = \frac{1}{16}(2,4,0,-4,-2)$

$R_x(3) = aP^2d = a\,(0,0,0,0,0)^T = 0$

Hence $R_x(m) = 0$ for $m \geqslant 3$

$d = (-2,0,0,0,2)$

$S_x(z) = 2 - (z+z^{-1})/2 - (z^2+z^{-2})/2, \quad z = e^{i\theta}, \ |\theta| \leqslant \pi$

$S_x(\theta) = 2 - \cos\theta - \cos 2\theta.$

Block Coded Symbols

In many line codes, the output is a word of length N rather than a single symbol.

$a \rightarrow$ matrix with N rows

$d \rightarrow$ matrix with N columns

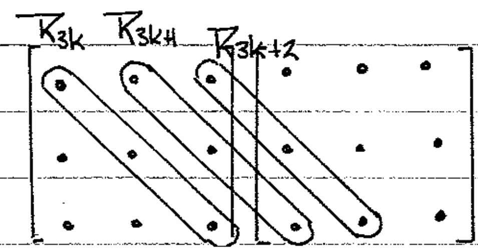
$aP^{t-1}d \rightarrow N\times N$ matrix

$$[aP^{t-1}d]_{ij} = E\left[x_{lN+i}\,x_{(l+t)N+j}\right]$$

The autocorrelation $R_x(m)$ is obtained by averaging over the possible positions of the symbol x_j within a word of length N.

Write $m = kN + l$ where $0 \leqslant l \leqslant N-1$

$$N R_x(m) = \sum_{i=1}^{N-l}\left(aP^k d\right)_{i,i+l} + \sum_{i=N-l+1}^{N}\left(aP^{k+1}d\right)_{i,i+l-N}$$



Deriving the autocorrelation by averaging when N=3