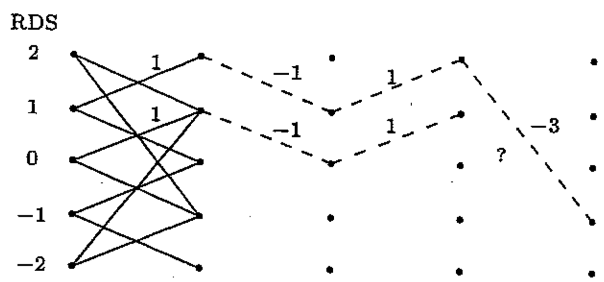
ELE 539 B: Coding Theory - Lecture 8

Viterbi Decoding with Finite Delay

Higher Order Nulls and Vanishing Moments

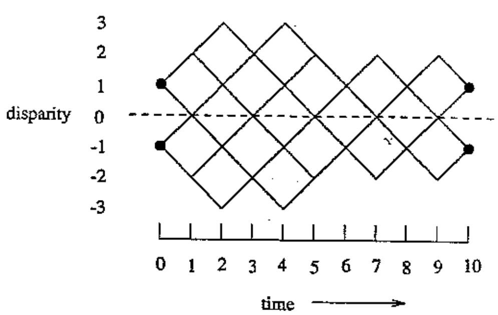
Spectral Constraints and Minimum Distance

Viterbi Decoding with Finite Delay



Viterbi decoding is implemented with a finite decoding delay L. If the codeword (x_k) fails to take on the value ± 3 in a window of length L, then the decoder will not be able to separate two parallel paths, each consistent with (x_k).

Widmer-Franaszek Code



Q. What is the maximum length of a run of identical symbols in the Widmer-Franaszek code?

Higher Order Nulls and Vanishing Moments

As more derivatives vanish the spectrum rises less rapidly from zero, and a wider null is useful in applications requiring greater rejection of low frequency components.

It is also possible to increase the minimum Euclidean distance between codewords by imposing spectral constraints on a line code.

Lemma 8.1 All odd derivatives of the line code spectrum $S_x(\theta)$ vanish at zero frequency.

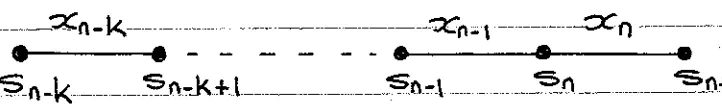
Proof: Since $R_x(m) = R_x(-m)$ the spectrum is an even function of θ

$$S_x(\theta) = \sum_{m=-\infty}^{\infty} R_x(m) e^{im\theta} = S_x(-\theta)$$

Hence $S_x^{(n)}(\theta) = (-1)^n S_x^{(n)}(-\theta)$.

Definition. The line code spectrum $S_x(\theta)$ has a Kth order spectral null at zero frequency if $S_x(0) = 0$ and if $S_x^{(n)}(0) = 0$ for $n = 1, 2, \ldots, 2K-1$.

Coboundary Conditions for Higher Order Spectral Nulls



Definition. A finite state machine satisfies an order K coboundary condition at zero frequency if there exists a function ψ from the state set S to the complex numbers such that

$$x_n = \sum_{l=0}^{K} (-1)^l \binom{K}{l} \psi(s_{n+1-l})$$

K=1: An order 1 coboundary condition restricts RDS to a finite range. We require $x_n = \psi(s_{n+1}) - \psi(s_n)$, and by Theorem 7.3 every state is associated with a particular RDS value, so we set $v_n = \sum_{i \le n} x_i = \psi(s_{n+1})$.

K=2: RDS and RDS sum (RDSS) are restricted to a finite range.

$$x_n = \psi(s_{n+1}) - 2\psi(s_n) + \psi(\cancel{s_{n-1}})$$
$$x_{n-1} = \psi(s_n) - 2\psi(\cancel{s_{n-1}}) + \psi(s_{n-2})$$
$$x_{n-2} = \psi(\cancel{s_{n-1}}) - 2\psi(s_{n-2}) + \psi(s_{n-3})$$

Hence

RDS $v_n = \psi(s_{n+1}) - \psi(s_n)$

and

$$\psi(s_{n+1}) = \sum_{i \le n} v_i$$

Coboundary Conditions (contd.)

Theorem 8.2. For line codes generated by a finite state machine, the condition that the finite state machine satisfy an order K coboundary condition at zero frequency is (necessary and) sufficient for the existence of a Kth order spectral null at zero frequency.

Proof: The coboundary condition is a convolution

$$\hat{x}(\omega) = \sum_{n,\ell} e^{-i(n+1-\ell)\omega - i(\ell-1)\omega} \psi(s_{n+1-\ell}) \left((-1)^{\ell}\binom{K}{\ell}\right)$$

$$= e^{i\omega}\left(\sum_m \psi(s_m) e^{-im\omega}\right)\left(\sum_{\ell=0}^{K} (-1)^{\ell}\binom{K}{\ell} e^{-i\ell\omega}\right), \quad \text{where } m = n+1-\ell$$

$$= e^{i\omega}\, \hat{\psi}(\omega) \left(1 - e^{-i\omega}\right)^K$$

Writing $z = e^{i\omega}$, the power spectrum $S_x(z)$ is given by

$$S_x(z) = S_\psi(z)\, |z|^2\, |(1 - 1/z)|^{2K} = S_\psi(z) \left[\frac{-(1-z)^2}{z}\right]^K$$

and the Kth order spectral null is revealed.

Spectral Constraints and Minimum Distance

Definition A sequence $e = e_0, \ldots, e_n$ has a sign change at position u if $e_u \neq 0$ and $\text{sign}(e_u) = -\text{sign}(e_t)$, where $t = \max\{i < u \mid e_i \neq 0\}$.

Descartes' Rule of Signs: Let $e(D)$ be a real polynomial with K positive real roots, not necessarily distinct. Then there are at least K sign changes in the sequence e of coefficients of $e(D)$

Proof: See A.S. Householder, Principles of Numerical Analysis.

Proposition 8.3. Let C be a line code generated by a finite state machine with an order K spectral null at $\omega = 0$. If the entries of codewords (x_k) are odd integers, then the minimum squared distance between distinct codewords is at least $8K$.

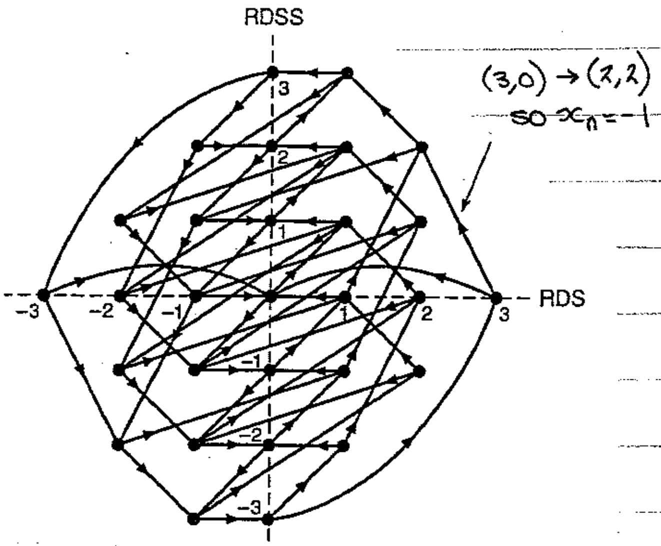
Proof: Write $e_n = x_n - y_n$

$$e_n = \sum_{\ell=0}^{K} (-1)^{\ell}\binom{K}{\ell} \left[\psi(s_{n+1-\ell}) - \psi(s'_{n+1-\ell})\right]$$

so that $e(z) = (1-z)^K \left[\phi_x(z) - \phi_y(z)\right]$.

Introducing a Double Spectral Null at Zero Frequency

Example: A 29-state line code with symbols $\pm 1, \pm 3$ and a double spectral null at zero frequency.



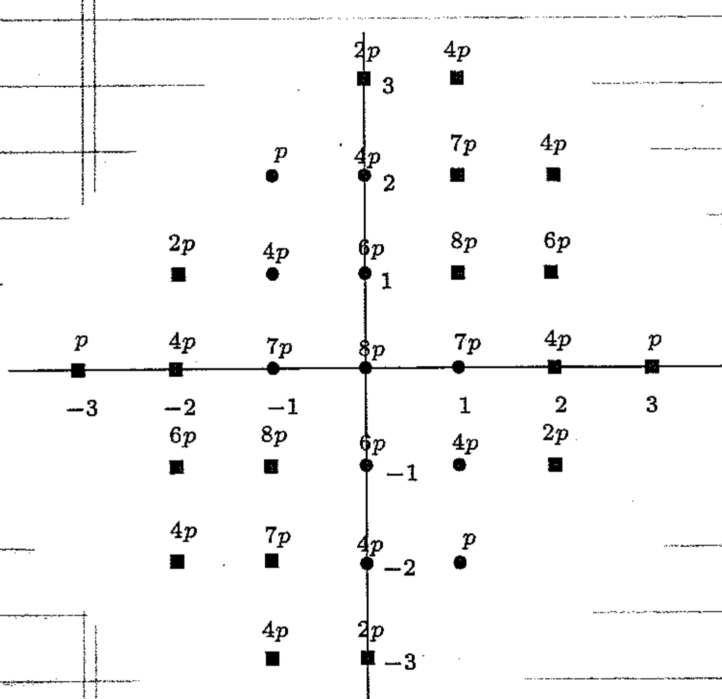
Encoder states are pairs (v_n, w_n) where v_n is the RDS and w_n is the RDSS. Transmission of x_n causes the state transition

$$(v_n, w_n) \to (v_n + x_n, w_n + v_n + x_n)$$

The coboundary operator is the RDSS state label.

Possible to employ a symbol by symbol decoder that ignores encoder state and simply decides on congruence of symbols modulo 4.

Solving for the Stationary Probabilities



Set $\pi_{(-3,0)} = p$ and solve for the remaining stationary probabilities one at a time. We find $p = 1/128$

States indicated by circles use the symbol alphabet ± 1 and states indicated by squares use the alphabet $\{1,3\}$ or the alphabet $\{-1,-3\}$.

Deriving the Spectrum

$a = \frac{1}{256}$

RDSS \ RDS	−3	−2	−1	0	1	2	3
3				$\overline{4}$	0		
2			$\overline{2}$	$\overline{6}$	6	6	
1		$\overline{12}$	$\overline{16}$	$\overline{4}$	16	28	
4	$\overline{6}$	$\overline{22}$	$\overline{22}$	0	22	22	6
−1		$\overline{28}$	$\overline{16}$	4	16	12	
−2		$\overline{6}$	$\overline{6}$	6	2		
−3			0	4			

, $d =$

RDSS \ RDS	−3	−2	−1	0	1	2	3
3				−2	−2		
2					−2	−2	
1		2			−2	−2	
4	2	2				−2	−2
−1		2	2			−2	
−2		2	2				
−3			2	2			

The spectrum $S_x(\theta)$ is remarkably simple since $aP^t d = 0$ for $t \geq 6$.

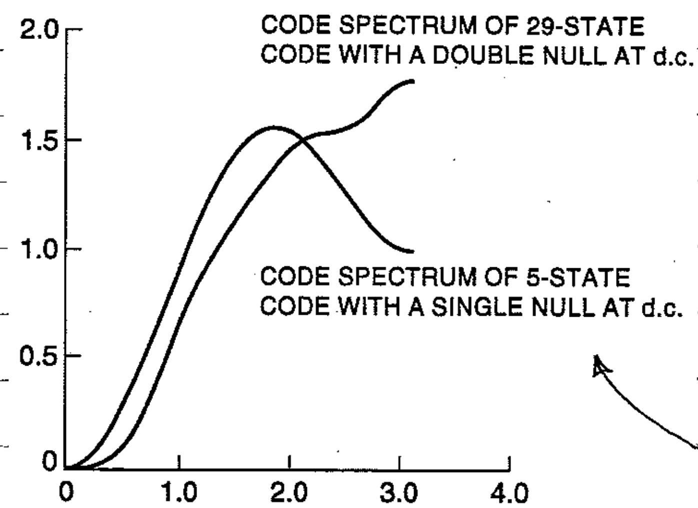
$$S_x(\theta) = \frac{1}{27}\left(27 - 23\cos\theta - 5\cos 2\theta - \cos 3\theta + \cos 4\theta + \cos 6\theta\right)$$

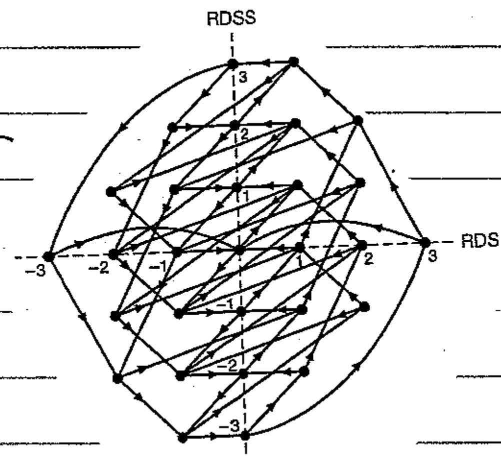
(normalized to unit power)

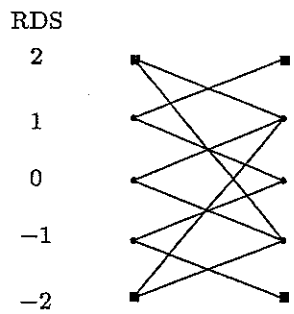
9.

Tale of Two Spectra

Spectrum $S_x(\theta)$







Higher Order Running Digital Sums

$RDS_f^k(x_n)$ enables analysis of spectral nulls at $f = pf_s/q$ where f_s is the symbol frequency

$$RDS_f^1(x_n) = RDS_f(x_n) = \sum_{i=0}^{n} \omega^i x_i \qquad \text{where } \omega = e^{-2\pi i p/q}$$

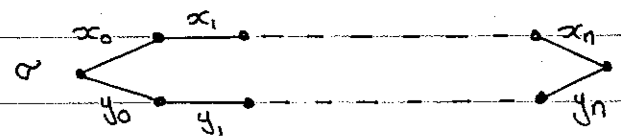
$$RDS_f^k(x_n) = \sum_{i=0}^{n} RDS_f^{k-1}(x_n)$$

Lemma 8.4 $\quad RDS_f^k = \sum_{i=0}^{n} \binom{n+k-1-i}{k-1} \omega^i x_i$

The proof uses induction and standard binomial identities

Example: $RDS_0^2(x_n) = \sum_{i=0}^{n} (n+1-i) x_i$

Difference Event $(e_n) = (x_n) - (y_n)$



Vanishing Moments

$RDS_0^1(e_n) = 0$ for a line code with a first order spectral null at $f=0$

$RDS_f^k(e_n) = 0$ for a line code with a kth order spectral null at f.

Order k Moment at $f = pf_s/q$: $\quad M_f^k(x_n) = \sum_{i=0}^{n} i^k \omega^i x_i$

Theorem 8.5 The following are equivalent

(a) $RDS_f^k(x_n) = 0$ for all $1 \le k \le M$

(b) $M_f^k(x_n) = 0$ for all $0 \le k \le M-1$.

Proof: Start with (a) $\Rightarrow$ (b)

$$RDS_f^k(x_n) = \sum_{i=0}^{n} \binom{n+k-1-i}{k-1} \omega^i x_i$$

$$= \frac{1}{(k-1)!} \sum_{i=0}^{n} (n-i+1) \cdots (n-i+k-1) \omega^i x_i$$

Define functions $g_{j-1}(n) \neq 0$ for $1 \le j \le k$ by

$$\binom{n+k-1-i}{k-1} = \sum_{j=1}^{k} g_{j-1}(n)\, i^{j-1}$$

Vanishing Moments and Spectral Nulls

$$RDS_f^k(x_n) = \sum_{j=1}^{k} M_f^{j-1}(x_n)\, g_{j-1}(n) = 0 \quad \text{for all } 1 \leq k \leq M \qquad (*)$$

k=1: $RDS_f^1(x_n) = 0 \Rightarrow M_f^0(x_n) = 0$

k=2: $RDS_f^2(x_n) = 0 \Rightarrow M_f^0(x_n)\, g_0(n) + M_f^1(x_n)\, g_1(n) = 0$

$\Rightarrow M_f^1(x_n) = 0$

Proceeding step by step we arrive at (b).

(b) $\Rightarrow$ (a) just follows from (*)

Theorem 8.6 A finite state machine generates a line code with a Kth order spectral null at f if and only if for every difference event e the order k moments $M_f^k(e)$ vanish for $k = 0, 1, \ldots, K-1$