Radar Signal Processing

Sensitivity of Golay Pairs to Doppler
Doppler Resilient Golay Pairs
Prouhet-Thue-Morse Pulse Train

Sensitivity of Golay Pairs to Doppler

- Pulse train of Golay pairs $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$:

- Correlator output in the presence of Doppler shift:

$$G(k, \theta) = \sum_{n=0}^{N-1} e^{jn\theta} C_{x_n}(k)$$

where $\theta = -\omega T$ is the relative Doppler shift over a PRI, and Doppler shift at the chip rate is ignored.

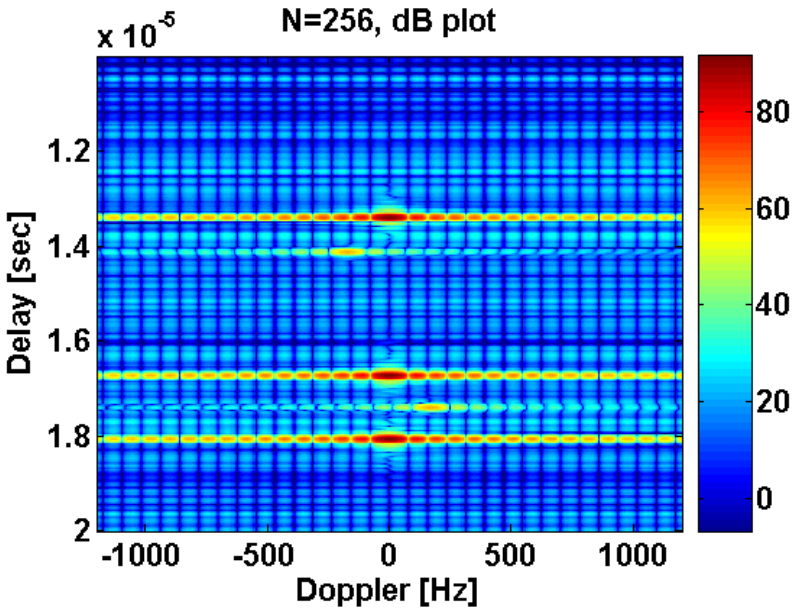
- We call $G(k, \theta)$ the "composite ambiguity function".
- Doppler shift perturbs the perfect auto-correlation property and creates range sidelobes:

$$\sum_{n=0}^{N-1} e^{jn\theta} C_{x_n}(k) \neq NL\delta_{k,0}$$

Sensitivity to Doppler

"Although the autocorrelation sidelobe level is zero, the ambiguity function exhibits relatively high sidelobes for nonzero Doppler." [Levanon, Radar Signals, 2004, p. 264]

Range Sidelobes Problem: A weak target located near a strong target can be masked by the range sidelobes of the ambiguity function centered around the strong target.



Range-Doppler image obtained with conventional pulse train

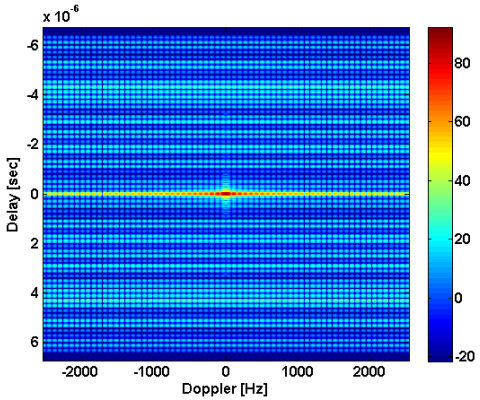
$x \quad y \quad \cdots \quad x \quad y$

Doppler Resilient Golay Pairs

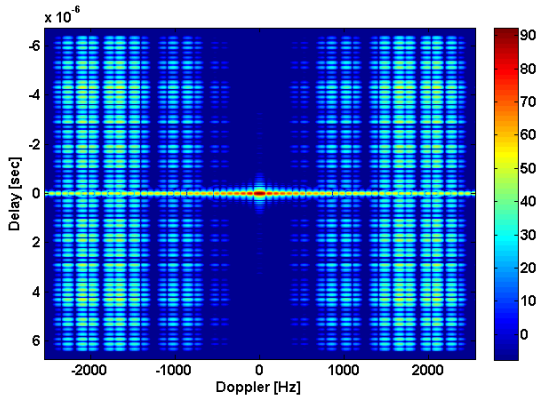
- Question: Is it possible to design a *Doppler resilient sequence* of Golay pairs $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$ so that for a reasonable range of Doppler shifts

$$G(k, \theta) = \sum_{n=0}^{N-1} e^{jn\theta} C_{x_n}(k) \approx NL\delta_{k,0}$$

- Design the Golay pairs so that the composite ambiguity function has a high-order null along $\theta = 0$.



Without resilience



With resilience

Doppler Resilient Golay Pairs

- Approach: Select the Golay pairs $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$ so that in the Taylor expansion of $G(k, \theta)$ around $\theta = 0$ all terms up to a certain order, say M, vanish at all nonzero delays (become impulses).
- Taylor expansion of $G(k, \theta)$ around $\theta = 0$:

$$G(k, \theta) = \sum_{m=0}^{\infty} D_m(k)(j\theta)^m,$$

$$D_m(k) = \sum_{n=0}^{N-1} n^m C_{x_n}(k), \qquad \text{for} \;\; m = 0, 1, 2, 3, \ldots$$

- Objective: Design $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$ so that $D_m(k)$, $m = 1, \ldots, M$ vanish at all nonzero delays.

Doppler Resilient Golay Pairs: 1st Order

- Transmit 2 Golay pairs (x_0, x_1) and (x_2, x_3) over 4 PRIs.
- Making $D_1(k)$ vanish:

$$D_1(k) = \underbrace{\underbrace{0C_{x_0}(k) + C_{x_1}(k)}_{1C_{x_1}(k)} + \underbrace{2C_{x_2}(k) + 3C_{x_3}(k)}_{2 \times 2L\delta_{k,0} + 1C_{x_3}(k)}}_{3 \times 2L\delta_{k,0}}$$

- Condition: Golay pairs (x_0, x_1) and (x_2, x_3) must be selected such that (x_1, x_3) is also a Golay pair.
- Example:

x_0	x_1	x_2	x_3
x	y	y	x

where (x, y) is an arbitrary Golay pair.

Doppler Resilient Golay Pairs: 1st & 2nd Order

- Transmit 4 Golay pairs $(x_0, x_1), \ldots, (x_6, x_7)$ over 8 PRIs.
- Making $D_1(k)$ vanish:

$$\underbrace{0C_{x_0}(k) + 1C_{x_1}(k)}_{\underbrace{[(1-0)=1]C_{x_1}(k)}_{3 \times 2L\delta_{k,0}}} + \underbrace{2C_{x_2}(k) + 3C_{x_3}(k)}_{\underbrace{2 \times 2L\delta_{k,0} + [(3-2)=1]C_{x_3}(k)}_{3 \times 2L\delta_{k,0}}} + \underbrace{4C_{x_4}(k) + 5C_{x_5}(k)}_{\underbrace{4 \times 2L\delta_{k,0} + [(5-4)=1]C_{x_5}(k)}_{11 \times 2L\delta_{k,0}}} + \underbrace{6C_{x_6}(k) + 7C_{x_7}(k)}_{\underbrace{6 \times 2L\delta_{k,0} + [(7-6)=1]C_{x_7}(k)}_{11 \times 2L\delta_{k,0}}}$$

Condition: Golay pairs must be selected such that (x_1, x_3) and (x_5, x_7) are also Golay pairs.

Doppler Resilient Golay Pairs: 1st & 2nd Order

- Making $D_2(k)$ vanish:

$$\underbrace{\underbrace{\underbrace{0^2 C_{x_0}(k) + 1^2 C_{x_1}(k)}_{[(1^2-0^2)=1]C_{x_1}(k)} + \underbrace{2^2 C_{x_2}(k) + 3^2 C_{x_3}(k)}_{4\times 2L\delta_{k,0}+\ [(3^2-2^2)=5]C_{x_3}(k)}}_{5\times 2L\delta_{k,0}+\ [(3^2-2^2-1^2+0^2)=4]C_{x_3}(k)} + \underbrace{\underbrace{4^2 C_{x_4}(k) + 5^2 C_{x_5}(k)}_{16\times 2L\delta_{k,0}+\ [(5^2-4^2)=9]C_{x_5}(k)} + \underbrace{6^2 C_{x_6}(k) + 7^2 C_{x_7}(k)}_{36\times 2L\delta_{k,0}+\ [(7^2-6^2)=13]C_{x_7}(k)}}_{61\times 2L\delta_{k,0}+\ [(7^2-6^2-5^2+4^2)=4]C_{x_7}(k)}}_{[(0^2+1^2+2^2+\ldots+7^2)=70]\times 2L\delta_{k,0}}$$

- Condition: Golay pairs must be selected such that (x_1, x_3), (x_5, x_7), and (x_3, x_7) are also Golay pairs.
- Example:

x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x	y	y	x	y	x	x	y

Prouhet-Thue-Morse (PTM) Sequence

- Is there a Pattern? Yes, it's the Prouhet-Thue-Morse sequence!

1st order: PTM sequence of length $4 = 2^{1+1}$

x_0	x_1	x_2	x_3
x	y	y	x
0	1	1	0

2nd order: PTM sequence of length $8 = 2^{2+1}$

x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x	y	y	x	y	x	x	y
0	1	1	0	1	0	0	1

PTM Pulse Train: Up to Order M

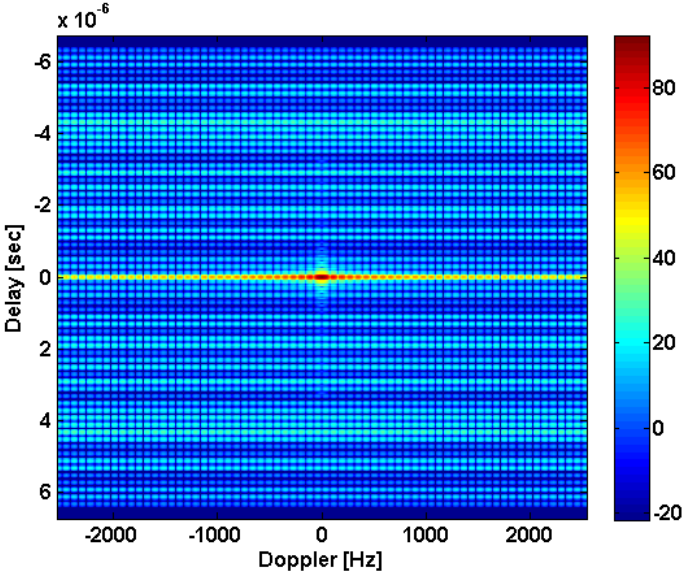
- Prouhet-Thue-Morse Sequence: The nth term in the PTM sequence p_n is the sum of the binary digits of n mod 2:

n	(0)=0000	(1)=0001	(2)=0010	(3)=0011
p_n	0	1	1	0

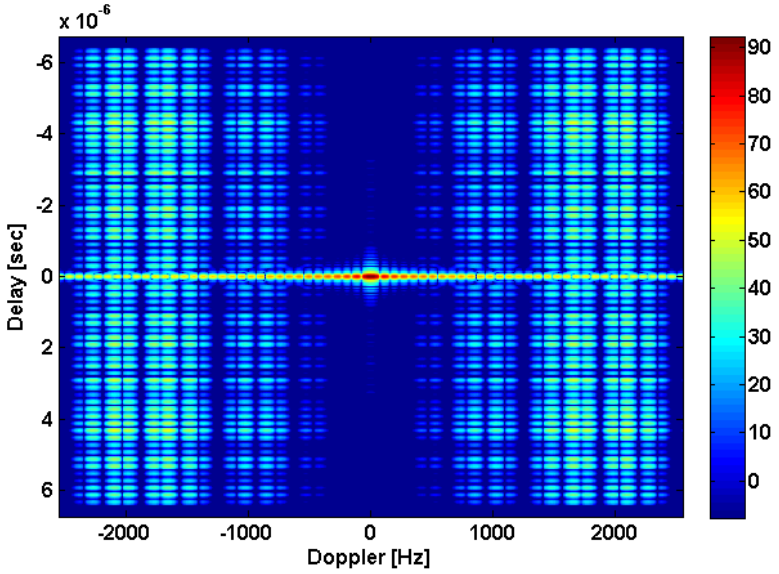
- Theorem: To zero-force up to M Taylor moments, coordinate the transmission of a Golay pair (x, y) according to the length $N = 2^{M+1}$ PTM sequence, with 0 locations corresponding to x and 1 locations corresponding to y.

- The result is the PTM pulse train, which is resilient to modest Dopplers.

Length-256 PTM Pulse Train: Zero-Forcing 7 Moments



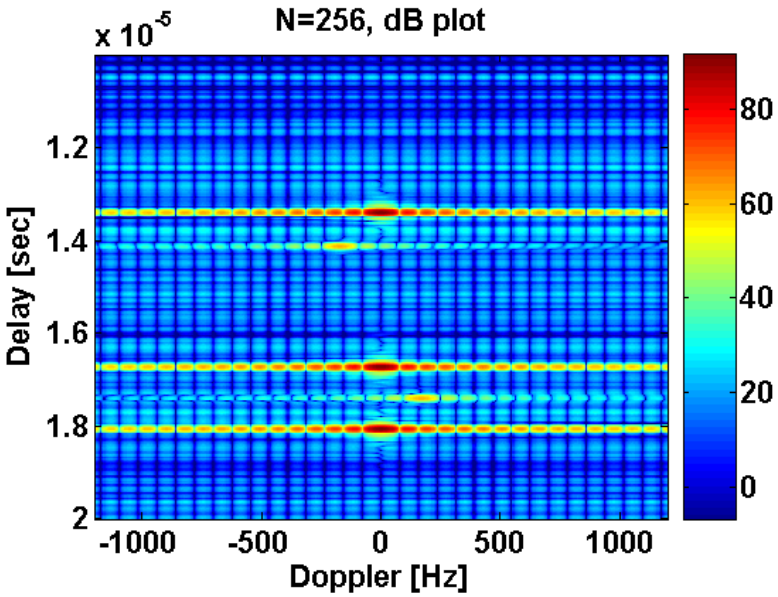
Conventional



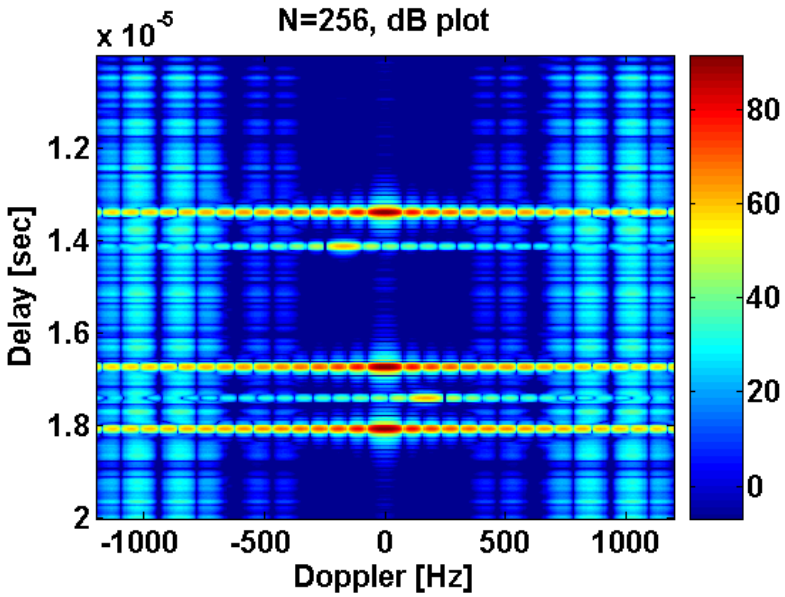
Doppler Resilient

Parameters: $f_0 = 17$ GHz and $T = 0.5$ μsec, $T_c = 100$ nsec.

PTM Pulse Train in Action



Conventional (Alternating)



Doppler Resilient (PTM)

By transmitting a Golay pair according to the PTM sequence we can clear out the range sidelobes along modest Doppler shifts.

Why PTM Sequence?

- Look at the calculations for zero-forcing the 1st and 2nd order moments.
- Key is partitioning of $\mathbb{S} = \{0, 1, \ldots, 7\}$ into disjoint subsets $\mathbb{S}_0 = \{0, 3, 5, 6\}$ and $\mathbb{S}_1 = \{1, 2, 4, 7\}$ that satisfy

$$(0^m + 3^m + 5^m + 6^m) - (1^m + 2^m + 4^m + 7^m) = 0, \quad \text{for } m = 1, 2.$$

- Prouhet's Problem: Let $\mathbb{S} = \{0, 1, \ldots, N-1\}$. Given M, is it possible to partition $\mathbb{S}$ into two disjoint subsets $\mathbb{S}_0$ and $\mathbb{S}_1$ such that

$$\sum_{r \in \mathbb{S}_0} r^m = \sum_{q \in \mathbb{S}_1} q^m$$

for all $0 \leq m \leq M$?

Solution: Possible when $N = 2^{M+1}$. The partitions are identified by the PTM sequence.

References

1. Y. Chi, A. Pezeshki, and A. R. Calderbank, "Complementary waveforms for sidelobe suppression and radar polarimetry," *in Applications and Methods of Waveform Diversity,* V. Amuso, S. Blunt, E. Mokole, R. Schneible, and M. Wicks, Eds., SciTech Publishing, Inc., to appear 2009.
2. Y. Chi, A. Pezeshki, and A. R. Calderbank, "Range sidelobe suppression in a desired Doppler interval," *Proc. Waveform Diversity and Design Conference,* Orlando, FL, Feb. 8-13, 2009.
3. R. Calderbank, S. D. Hoawrd, and W. Moran, "Waveform diversity in radar signal processing: A focus on the use and control of degrees of freedom," *IEEE Signal Processing Magazine,* vol. 26, no. 1, pp. 32-41, Jan. 2009.
4. A. Pezeshki, A. R. Calderbank, W. Moran, and S. D. Howard, "Doppler resilient Golay complementary waveforms," *IEEE Trans. Information Theory,* vol. 54, no. 9, pp. 4254-4266, Sep. 2008.
5. A. Pezeshki, A. R. Calderbank, S. D. Howard, and W. Moran, "Doppler resilient Golay complementary pairs for radar," *Proc. IEEE Workshop on Statistical Signal Processing,* Madison, WI, Aug. 26-29, 2007.