Radar Signal Processing

PTM Pulse Trains
Reed-Müller Pulse Trains
PTM Sequencing Across Frequency

- p-Pulse Train: Transmission of a Golay pair x and y is coordinated according to a unimodular sequence $\mathbf{p} = \{p_n\}$, $n = 0, \ldots, 2^M - 1$ over $N = 2^M$ PRIs.
- At nth PRI: $\frac{1}{2}(1 + (-1)^{p_n})x + \frac{1}{2}(1 - (-1)^{p_n})y$
- Composite ambiguity function:

$$G(k,\theta) = \underbrace{\frac{1}{2}[C_x(k) + C_y(k)] \sum_{n=0}^{2^M-1} e^{jn\theta}}_{\text{Sidelobe free}} + \underbrace{\frac{1}{2}[C_x(k) - C_y(k)] \sum_{n=0}^{2^M-1} (-1)^{p_n} e^{jn\theta}}_{\text{Range sidelobes}}$$

- Key observation: Magnitude of range sidelobes are proportional to the magnitude of the spectrum of the sequence $(-1)^{p_n}$:

$$S_{\mathbf{p}}(\theta) = \sum_{n=0}^{2^M-1} (-1)^{p_n} e^{jn\theta}$$

- Approach: Design $\mathbf{p} = \{p_n\}$ to shape the spectrum $S_{\mathbf{p}}(\theta)$.

PTM Pulse Train

- The PTM pulse train clears out the range sidelobes in a Doppler interval around the zero-Doppler axis.
- Length-2^M PTM sequence zero forces low-order terms of the Taylor expansion of $S_{\mathbf{p}}(\theta)$ around $\theta = 0$:

$$S_{\mathbf{p}}^{(m)}(0) = 0, \quad m = 1, \ldots, M-1$$

- Theorem: The PTM pulse train of length 2^M has the following composite ambiguity function:

$$G(k,\theta) = e^{j(2^{M-1}-1)\theta} \frac{\sin(2^M\theta/2)}{\sin(\theta/2)} L\delta_{k,0} + \frac{1}{2}\left(\prod_{m=0}^{M-1}(1-e^{j2^m\theta})\right)[C_x(k) - C_y(k)]$$

$$= e^{j(2^{M-1}-1)\theta} \frac{\sin(2^M\theta/2)}{\sin(\theta/2)} L\delta_{k,0}$$

$$+ \frac{1}{2}[(-j)^M 2^{M(M-1)/2}\theta^M + \mathcal{O}(\theta^{M+1})][C_x(k) - C_y(k)]$$

PTM Pulse Train

Proof:

$$
\begin{aligned}
S_{\mathbf{p}}(\theta) &= \sum_{n=0}^{2^M-1} (-1)^{p_n} e^{jn\theta} \\
&= \sum_{n_0=0}^{1} \cdots \sum_{n_{M-1}=0}^{1} (-1)^{n_0+\ldots+n_{M-1}} e^{j(n_0 2^0 + n_1 2^1 + \ldots + n_{M-1} 2^{M-1})\theta} \\
&= \prod_{m=0}^{M-1} (1 - e^{j2^m\theta}) = \prod_{m=0}^{M-1} \Lambda_m(\theta)
\end{aligned}
$$

where $\Lambda_m(\theta) = 1 - e^{j2^m\theta}$. Since $\Lambda_m(0) = 0$, we have

$$
S_{\mathbf{p}}^{(\ell)}(0) = 0, \quad \text{if } \ell < M,
$$

and

$$
S_{\mathbf{p}}^{(M)}(0) = M! \prod_{m=0}^{M-1} \Lambda_m^{(1)}(0) = (-j)^M 2^{M(M-1)/2} M!
$$

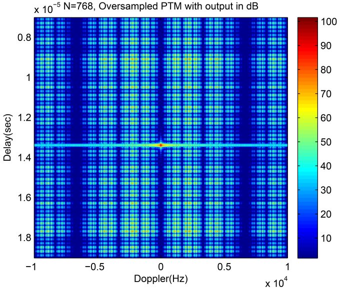
Sidelobe Suppression at Higher Doppler Frequencies

Question: Can we clear out range sidelobes in other Doppler intervals?

Theorem: The k-oversampled PTM sequence of length $2^M k$ produces an Mth order null at $\theta = 2\pi\ell/k$ for all co-prime ℓ and k.

Corollary: Oversampled PTM produces an $(M-1)$th order null at $\theta = 0$ and $(M-h-1)$th order nulls at all $\theta = 2\pi\ell/2^h k$.

Example: $M = 3, k = 3 \longrightarrow \{p_n\} = 000111111000\cdots$



Reed-Müller Pulse Trains

- First order Reed-Müller code $RM(1, M)$ consists of 2^M code words of the form

$$r_b(n) = \sum_{m=0}^{2^M-1} b_m n_m \quad \text{for } n = 0, \cdots, 2^M - 1$$

  where n_m denotes the mth binary digit of n.

- Walsh functions are the exponentiated Reed-Müller codes

$$w_b(n) = (-1)^{r_b(n)}, \quad \text{for } n = 0, \cdots, 2^M - 1$$

- The length-2^M PTM sequence is equal to $r_b(n)$ with $b = (1, 1, \ldots, 1)$, and $(-1)^{r_b(n)}$ is its corresponding Walsh function.

Reed-Müller Pulse Trains

- Theorem: For a Reed-Müller code $p_n = r_b(n)$ of length 2^M the magnitude spectrum $|S_{\mathbf{p}}(\theta)|$ is given by

$$\begin{aligned} |S_{\mathbf{p}}(\theta)| &= \left| \sum_{n=0}^{2^M-1} (-1)^{r_b(n)} e^{jn\theta} \right| \\ &= 2^M \prod_{\substack{m=0 \\ b_m=0}}^{M-1} |\cos(2^{m-1}\theta)| \prod_{\substack{m=0 \\ b_m=1}}^{M-1} |\sin(2^{m-1}\theta)| \end{aligned}$$

  where b_m, $m = 0, \ldots, M-1$ is the mth entry in the binary M-tuple b.

  Proof: Homework

- Question: Given a Doppler interval, which first-order RM (or Walsh function) minimizes $|S_{\mathbf{p}}(\theta)|$?

Reed-Müller Pulse Trains

- $|\mathbf{S}_p(\theta)|$ is $2\pi-$periodic. Only need to look at $0 \leq \theta \leq 2\pi$.

$$|S_{\mathbf{p}}(\theta)| = 2^M \prod_{\substack{m=0 \\ b_m=0}}^{M-1} |\cos(2^{m-1}\theta)| \prod_{\substack{m=0 \\ b_m=1}}^{M-1} |\sin(2^{m-1}\theta)|$$

- Theorem: Among first order Reed-Müller codes of length 2^M there is a single code which minimizes $|S_{\mathbf{p}}(\theta)|$ across the entire Doppler interval $[\pi k/2^M, \pi(k+1)/2^M]$, where k is an integer.

- This allows us to clear out the range sidelobes along a particular Doppler bin.

Proof:

$$|S_{\mathbf{p}}(\theta)| = 2^M \prod_{\substack{m=0\\b_m=0}}^{M-1} |\cos(2^{m-1}\theta)| \prod_{\substack{m=0\\b_m=1}}^{M-1} |\sin(2^{m-1}\theta)|$$

$$= 2^M \prod_{m=0}^{M-1} |\cos(2^{m-1}\theta + \frac{\pi}{2} b_m)|$$

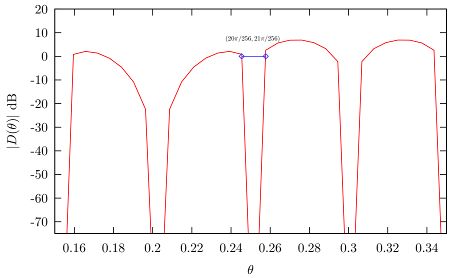
Minimize one by one to find the optimal RM code:

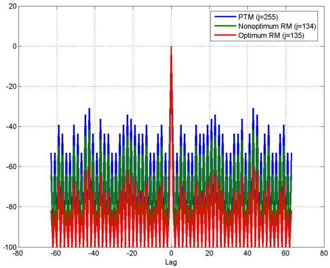
$$b_m = \begin{cases} 1, & 2^m\theta \in [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi] \mod(2\pi) \\ 0, & \text{otherwise} \end{cases}$$

For all θ inside a given Doppler interval $[\pi k/2^M, \pi(k+1)/2^M]$ the minimizers b_m, $m = 0, 1, \ldots, M-1$ stay unchanged.

Reed-Müller Pulse Train in Action

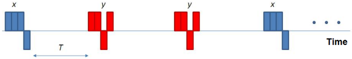
- Suppose we want to minimize sidelobes in the region of $\theta = 0.25$ using an RM pulse train of length 256.
- This means minimizing $|S_{\mathbf{p}}(\theta)|$ in the interval $[20\pi/256, 21\pi/256]$.
- The right sequence is $r_b(n)$, with b being the binary representation of 135.





Clearing Doppler Sidelobes

Time-Domain PAM:

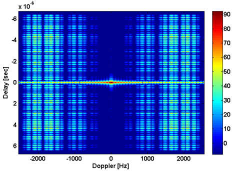


$$z(t) = \sum_{m=0}^{M-1} a_m s(t - mT_c)$$

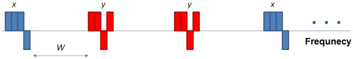
Impulse in Delay:

$$\chi_z(\tau,\nu) = \int_{-\infty}^{\infty} z(t)\overline{z(t-\tau)}e^{-j\nu t}$$

$$\approx \delta(\tau)\alpha(\nu), \quad \forall \nu \in \Delta\nu$$



Frequency-Domain PAM:

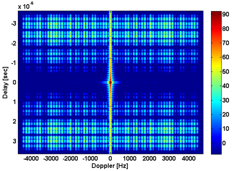


$$z(\omega) = \sum_{m=0}^{M-1} a_m s(\omega - mW_c)$$

Impulse in Doppler:

$$\chi_z(\nu,\tau) = \int_{-\infty}^{\infty} z(\omega)\overline{z(\omega-\nu)}e^{-j\tau\omega}$$

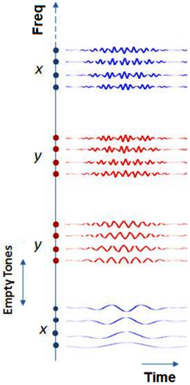
$$\approx \delta(\nu)\alpha(\tau), \quad \forall \tau \in \Delta\tau$$



PTM Sequencing Across Frequency

Time-domain OFDM $\iff$ Frequency-domain PAM

$$Z(t) = \left(\sum_{m=0}^{M-1} a_m e^{-jm\omega_c t} \right) S(t) \iff z(\omega) = \sum_{m=0}^{M-1} a_m s(\omega - mW_c)$$



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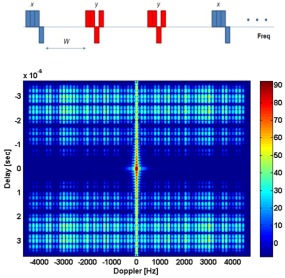
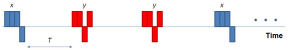


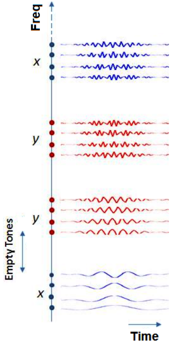
$$\int_{-\infty}^{\infty} Z(t)\overline{Z(t-\tau)}e^{j\nu t}dt$$
$$\approx \alpha(\tau)\delta(\nu), \quad \forall \tau \in \Delta\tau$$

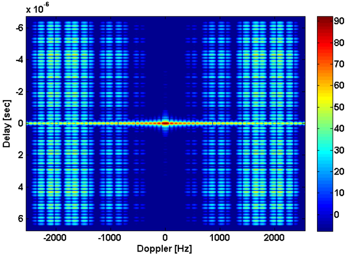
Clearing in a Desired Range/Doppler Interval

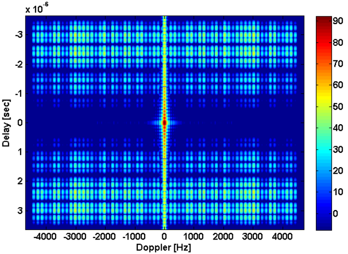
Sequencing in Time:



Stacking in Frequency:







Game: **Shaping the spectrum of the coordinating sequence**

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