Another interpretation of the optimum decision rule based on the ML criterion is obtained by expanding the distance metrics in Equation (7.5.41) as

$$\begin{aligned} D(\mathbf{r}, \mathbf{s}_m) &= \sum_{n=1}^{N} r_n^2 - 2\sum_{n=1}^{N} r_n s_{mn} + \sum_{n=1}^{N} s_{mn}^2 \\ &= \|\mathbf{r}\|^2 - 2\mathbf{r}\cdot\mathbf{s}_m + \|\mathbf{s}_m\|^2, \quad m = 1, 2, \ldots, M \end{aligned} \tag{7.5.42}$$

The term $|\mathbf{r}|^2$ is common to all decision metrics and, hence, it may be ignored in the computations of the metrics. The result is a set of modified distance metrics

$$D'(\mathbf{r}, \mathbf{s}_m) = -2\mathbf{r}\cdot\mathbf{s}_m + \|\mathbf{s}_m\|^2 \tag{7.5.43}$$

Note the selecting the signal $\mathbf{s}_m$ that minimizes $D'(\mathbf{r}, \mathbf{s}_m)$ is equivalent to selecting the signal that maximizes the metric $C(\mathbf{r}, \mathbf{s}_m) = -D'(\mathbf{r}, \mathbf{s}_m)$; i.e.,

$$C(\mathbf{r}, \mathbf{s}_m) = 2\mathbf{r}\cdot\mathbf{s}_m - \|\mathbf{s}_m\|^2 \tag{7.5.44}$$

The term $\mathbf{r}\cdot\mathbf{s}_m$ represents the projection of the received signal vector onto each of the M possible transmitted signal vectors. The value of each of these projections is a measure of the correlation between the received vector and the mth signal. For this reason, we call $C(\mathbf{r}, \mathbf{s}_m)$, $m = 1, 2, \ldots, M$, the *correlation metrics* for deciding which of the M signals was transmitted. Finally, the terms $\|\mathbf{s}_m\|^2 = \mathcal{E}_m, m = 1, 2, \ldots, M$, may be viewed as bias terms that serve as compensation for signal sets that have unequal energies, such as PAM. If all signals have the same energy, $\|\mathbf{s}_m\|^2$ may also be ignored in the computation of the correlation metrics $C(\mathbf{r}, \mathbf{s}_m)$ and the distance metrics $D(\mathbf{r}, \mathbf{s}_m)$ or $D'(\mathbf{r}, \mathbf{s}_m)$.

In summary, we have demonstrated that the optimum ML detector computes a set of M distances $D(\mathbf{r}, \mathbf{s}_m)$ or $D'(\mathbf{r}, \mathbf{s}_m)$ and selects the signal corresponding to the smallest (distance) metric. Equivalently, the optimum ML detector computes a set of M correlation metrics $C(\mathbf{r}, \mathbf{s}_m)$ and selects the signal corresponding to the largest correlation metric.

The above development for the optimum detector treated the important case in which all signals are equally probable. In this case, the MAP criterion is equivalent to the ML criterion. However, when the signals are not equally probable, the optimum MAP detector bases its decision on the probabilities $P(\mathbf{s}_m \mid \mathbf{r})$, $m = 1, 2, \ldots, M$, given by Equation (7.5.38) or, equivalently, on the posterior probability *metrics,*

$$\mathrm{PM}(\mathbf{r}, \mathbf{s}_m) = f(\mathbf{r} \mid \mathbf{s}_m)P(\mathbf{s}_m) \tag{7.5.45}$$

The following example illustrates this computation for binary PAM signals.

Example 7.5.3

Consider the case of binary PAM signals in which the two possible signal points are $s_1 = -s_2 = \sqrt{\mathcal{E}_b}$, where $\mathcal{E}_b$ is the energy per bit. The prior probabilities are $P(s_1) = p$ and $P(s_2) = 1 - p$. Determine the metrics for the optimum MAP detector when the transmitted signal is corrupted with AWGN.

Solution The received signal vector (one dimensional) for binary PAM is

$$r = \pm\sqrt{\mathcal{E}_b} + y_n(T) \tag{7.5.46}$$

where $y_n(T)$ is a zero-mean Gaussian random variable with variance $\sigma_n^2 = N_0/2$. Consequently, the conditional PDFs $f(r \mid s_m)$ for the two signals are

$$f(r \mid s_1) = \frac{1}{\sqrt{2\pi}\sigma_n} e^{-(r-\sqrt{\mathcal{E}_b})^2/2\sigma_n^2} \tag{7.5.47}$$

$$f(r \mid s_2) = \frac{1}{\sqrt{2\pi}\sigma_n} e^{-(r+\sqrt{\mathcal{E}_b})^2/2\sigma_n^2} \tag{7.5.48}$$

Then the metrics $\mathrm{PM}(\mathbf{r}, \mathbf{s}_1)$ and $\mathrm{PM}(\mathbf{r}, \mathbf{s}_2)$ defined by Equation (7.5.45) are

$$\begin{aligned} \mathrm{PM}(r, s_1) &= p f(r \mid s_1) \\ &= \frac{p}{\sqrt{2\pi}\sigma_n} e^{-(r-\sqrt{\mathcal{E}_b})^2/2\sigma_n^2} \end{aligned} \tag{7.5.49}$$

$$\mathrm{PM}(\mathbf{r}, \mathbf{s}_2) = \frac{1-p}{\sqrt{2\pi}\sigma_n} e^{-(r+\sqrt{\mathcal{E}_b})^2/2\sigma_n^2} \tag{7.5.50}$$

If $\mathrm{PM}(\mathbf{r}, \mathbf{s}_1) > \mathrm{PM}(\mathbf{r}, s_2)$, we select s_1 as the transmitted signal; otherwise, we select s_2. This decision rule may be expressed as

$$\frac{\mathrm{PM}(\mathbf{r}, \mathbf{s}_1)}{\mathrm{PM}(\mathbf{r}, \mathbf{s}_2)} \underset{s_2}{\overset{s_1}{\gtrless}} 1 \tag{7.5.51}$$

But

$$\frac{\mathrm{PM}(\mathbf{r}, \mathbf{s}_1)}{\mathrm{PM}(\mathbf{r}, \mathbf{s}_2)} = \frac{p}{1-p} e^{[(r+\sqrt{\mathcal{E}_b})^2 - (r-\sqrt{\mathcal{E}_b})^2]/2\sigma_n^2} \tag{7.5.52}$$

so that Equation (7.5.51) may be expressed as

$$\frac{(r+\sqrt{\mathcal{E}_b})^2 - (r-\sqrt{\mathcal{E}_b})^2}{2\sigma_n^2} \underset{s_2}{\overset{s_1}{\gtrless}} \ln\frac{1-p}{p} \tag{7.5.53}$$

or, equivalently,

$$\sqrt{\mathcal{E}_b}\, r \underset{s_2}{\overset{s_1}{\gtrless}} \frac{\sigma_n^2}{2}\ln\frac{1-p}{p} = \frac{N_0}{4}\ln\frac{1-p}{p} \tag{7.5.54}$$

This is the final form for the optimum detector. It computes the correlation metric $C(\mathbf{r}, \mathbf{s}_1) = r\sqrt{\mathcal{E}_b}$ and compares it with the threshold $(N_0/4)\ln(1-p)/p$.

It is interesting to note that in the case of unequal prior probabilities, it is necessary to know not only the values of the prior probabilities but also the value of the power-spectral density N_0, in order to compute the threshold. When $p = 1/2$, the threshold is zero, and knowledge of N_0 is not required by the detector.

We conclude this section with the proof that the decision rule based on the maximum-likelihood criterion minimizes the probability of error when the M signals are equally probable a priori. Let us denote by R_m the region in the N-dimensional space for