

# Tratamiento Digital de la Información

## Ejercicios 1

1. Dadas la verosimilitudes de ejemplo en clase sobre la culpabilidad o inocencia de nuestro sospechoso, calcule:
  - (a) Las regiones de decisión para la solución de máxima verosimilitud.
  - (b) Las regiones de decisión para la solución de máximo a posteriori.
  - (c) Los costes  $C_{ic}$  y  $C_{ci}$  que empleara un estado policial para considerar culpable a sospechoso con una sola prueba positiva (o bien es reconocido por un testigo o la prueba de ADN es positiva). Asuma que  $C_{ii} = C_{cc} = 0$ .
  - (d) Calcule la curva ROC ( $P_{FA}$  versus  $P_D$ ).  
Nota: Para este problema al ser el espacio muestral de entrada discreto la curva ROC son 3 puntos (número de elementos en  $\mathcal{X}$  menos 1), además del 0,0 y el 1,1. Una los puntos con trazos lineales.
2. A city has two taxi companies distinguished by the color of their taxis 85% of the taxis are Yellow and the rest are Blue. A taxi was involved in a hit and run accident and was witnessed by Mr. Green. Unfortunately, Mr. Green is mildly color blind and can only correctly identify the color 80% of the time. In the trial, Mr. Green testified that the color of the taxi was blue. Should you trust him? Why?
3. For a multiple hypothesis problem with  $y \in \{1, 2, \dots, M\}$  and  $\mathbf{x} \in \mathbb{R}^d$ , define the decision region for each of the  $M$  hypothesis. Assume the likelihood  $p(\mathbf{x}|y)$ , the prior probabilities  $p(y = j) = \pi_j$  and the loss-function matrix  $(\mathbf{L})_{ij} = L_{ij}$  are known. To define the decision regions for each class use thresholds for the likelihood based on the prior probabilities and the loss function.
4. Problem 2(a) y (c).

5. Problem 4(a) y (c). For (a) assume that the prior probabilities are the same.
6. We have the following communication channel in which we transmit a 0 or a 1 and we receive an integer from 0 until  $N$ . The likelihood for  $y = 0$  is given by:

$$P(x|y = 0) = \binom{N}{x} p^x (1-p)^{N-x}$$

and the likelihood of  $y=1$  is given by:

$$P(x|y = 1) = \binom{N}{x} (1-q)^x q^{N-x}$$

- (a) Define the optimal maximum a posteriori rule for  $p(y = 0) = 1/2$  and  $N = 3$  and  $q = 3p < 0.5$ . Is it equal to the maximum likelihood? Why?
- (b) Define the Neyman-Pearson rule for a false alarm rate of 0.01 for  $p = q = 0.1$  and  $N = 3$ .
- (c) Find the minimum value of  $N$  for which we can obtain a false alarm rate of 0.01 and a detection rate of 0.99 for  $p = q = 0.1$
- (d) The likelihoods in this problem define the sufficient statistics for one famous block code, which one is it? (extra)

## size- $\alpha$ tests, $P_D(\tilde{\delta}; \theta)$ is given for $\theta$ near $\theta_0$ by

$$P_D(\tilde{\delta}; \theta) \cong \alpha + (\theta - \theta_0) P'_D(\tilde{\delta}; \theta_0). \quad (\text{II.E.28})$$

Thus for  $\theta$  near  $\theta_0$  we can achieve approximate maximum power with size  $\alpha$  by choosing  $\tilde{\delta}$  to maximize  $P'_D(\tilde{\delta}; \theta_0)$ . A test that maximizes  $P'_D(\tilde{\delta}; \theta_0)$  subject to false-alarm constraint  $P_F(\tilde{\delta}) \leq \alpha$ , is called an  $\alpha$ -level *locally most powerful* (LMP) test, or simply a *locally optimum* test.

To see the general structure of LMP tests we note that, assuming that  $P_\theta$  has density  $p_\theta$  for each  $\theta \in \Lambda_1$ , we can write

$$\begin{aligned} P_D(\tilde{\delta}; \theta) &= E_\theta\{\tilde{\delta}(Y)\} \\ &= \int_{\Gamma} \tilde{\delta}(y)p_\theta(y)\mu(dy). \end{aligned} \quad (\text{II.E.29})$$

Assuming sufficient regularity on  $\{p_\theta(y); \theta \in \Lambda_1\}$  that we can interchange order of integration and differentiation in (II.E.29), we have

$$P'_D(\tilde{\delta}; \theta_0) = \int_{\Gamma} \tilde{\delta}(y) \frac{\partial}{\partial \theta} p_\theta(y) |_{\theta=\theta_0} \mu(dy). \quad (\text{II.E.30})$$

Comparison of (II.E.30) with (II.D.4) indicates that the  $\alpha$ -level LMP design problem is the same as the  $\alpha$ -level Neyman-Pearson design problem, if we replace  $p_1(y)$  with  $\partial p_\theta(y)/\partial \theta|_{\theta=\theta_0}$ . Using this analogy, it is straightforward to show that, within regularity, an  $\alpha$ -level LMP test for (II.E.26) is given by

$$\tilde{\delta}_{lo}(y) = \begin{cases} 1 & \text{if } \frac{\partial}{\partial \theta} p_\theta(y) |_{\theta=\theta_0} > \eta p_{\theta_0}(y), \\ \gamma & \text{if } \frac{\partial}{\partial \theta} p_\theta(y) |_{\theta=\theta_0} < \eta p_{\theta_0}(y), \\ 0 & \text{otherwise,} \end{cases} \quad (\text{II.E.31})$$

where  $\eta$  and  $\gamma$  are chosen so that  $P_F(\tilde{\delta}_{lo}) = \alpha$ . Details of this development can be found in the book by Ferguson (1968). LMP tests are discussed further in Chapter III.

In the absence of applicability of any of the above-mentioned optimality criteria, a test that is often used for composite problems in which  $\theta$  is the union of disjoint  $\Lambda_0$  and  $\Lambda_1$  is that based on comparing the quantity

$$\frac{\max_{\theta \in \Lambda_1} p_\theta(y)}{\max_{\theta \in \Lambda_0} p_\theta(y)} \quad (\text{II.E.32})$$

to a threshold. This test is sometimes known as the *generalized likelihood-ratio test* or a *maximum-likelihood test*, and further motivation for tests of this type is found in Chapter IV.

## II.F Exercises

- Find the minimum Bayes risk for the binary channel of Example II.B.1.

- Suppose  $Y$  is a random variable that, under hypothesis  $H_0$ , has pdf

$$p_0(y) = \begin{cases} (2/3)(y+1), & 0 \leq y \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

and, under hypothesis  $H_1$ , has pdf

$$p_1(y) = \begin{cases} 1, & 0 \leq y \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

- Find the Bayes rule and minimum Bayes risk for testing  $H_0$  versus  $H_1$  with uniform costs and equal priors.

- Find the minimax rule and minimax risk for uniform costs.
- Find the Neyman-Pearson rule and the corresponding detection probability for false-alarm probability  $\alpha \in (0, 1)$ .

- Repeat Exercise 2 for the situation in which  $p_j$  is given instead by

$$p_j(y) = \frac{(j+1)}{2} e^{-(j+1)|y|}, \quad y \in \mathbb{R}, j = 0, 1.$$

For parts (a) and (b) assume costs

$$C_{ij} = \begin{cases} 0, & \text{if } i = j \\ 1, & \text{if } i = 1 \text{ and } j = 0 \\ 3/4, & \text{if } i = 0 \text{ and } j = 1, \end{cases}$$

and for part (a) assume priors  $\pi_0 = 1/4$  and  $\pi_1 = 3/4$ .

- Repeat Exercise 2 for the situation in which  $p_0$  and  $p_1$  are given instead by

$$p_0(y) = \begin{cases} e^{-y}, & y \geq 0 \\ 0, & y < 0 \end{cases}$$

and

$$p_1(y) = \begin{cases} \sqrt{2/\pi} e^{-y^2/2}, & y \geq 0 \\ 0, & y < 0. \end{cases}$$

For part (a) consider arbitrary priors.