A Converse Bound via the Nussbaum–Szkoła Mapping for Quantum Hypothesis Testing

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***Abstract*—Quantum hypothesis testing concerns the discrimination between quantum states. This paper introduces a novel lower bound for asymmetric quantum hypothesis testing that is based on the Nussbaum–Szkoła mapping. The lower bound provides a unified recovery of converse results across all major asymptotic regimes, including large-, moderate-, and small-deviations. Unlike existing bounds, which either rely on technically involved information-spectrum arguments or suffer from fixed prefactors and limited applicability in the non-asymptotic regime, the proposed bound arises from a single expression and enables, in some cases, the direct use of classical results. It is further demonstrated that the proposed bound provides accurate approximations to the optimal quantum error trade-off function at small blocklengths. Numerical comparisons with existing bounds, including those based on fidelity and information spectrum methods, highlight its improved tightness.**

I. Introduction

Hypothesis testing is a central problem in information theory, concerned with distinguishing between two competing sources based on observed data. In the quantum setting, this task amounts to distinguishing between copies of two quantum states using measurements described by positive operator-valued measures (POVMs). The non-commutativity of quantum states gives rise to phenomena with no classical analogue, making the characterization of optimal error probabilities substantially more challenging.

A binary hypothesis test involves two types of errors: a type-I error, which occurs when the null hypothesis is incorrectly rejected, and a type-II error, which occurs when it is incorrectly accepted. The fundamental trade-off between these errors is typically formalized by optimizing one error probability subject to a constraint on the other. In the quantum setting, this trade-off admits an exact characterization via a semidefinite program (SDP). However, the dimension of the resulting SDP grows rapidly with both the dimension of the quantum states and the number of copies. This rapid scaling renders exact computations infeasible beyond very small systems, thereby motivating the development of analytically tractable and computationally efficient bounds that capture the essential behavior of the error probability.

Two lower bounds are particularly relevant in this context: The first, established by Nussbaum and Szkoła [1], provides a fundamental link between quantum and classical hypothesis testing. This result bounds the optimal average error probability by one-half of the classical optimal error probability associated with the so-called Nussbaum–Szkoła distributions, which are explicitly constructed from the spectral decompositions of the original quantum states ρ and σ, effectively embedding the non-commutative geometry of the Hilbert space into a joint classical distribution. The second, proposed by Pereira *et al.* [2], provides a lower bound in the asymmetric case based on the quantum fidelity $0 \leq F(\rho, \sigma) \leq 1$, which quantifies the overlap between the two quantum states.

For large numbers of copies n, asymptotic methods characterize the type-II error probability β_n under a prescribed scaling of the type-I error α_n. The *small-deviations regime* covers the asymptotic analysis of β_n when α_n is fixed. The quantum Stein's lemma establishes the error exponent in this setting [3], [4], thereby providing the leading term of order n in the expansion of $-\log \beta_n$. A second-order analysis refines this result by characterizing the correction term of order $\sqrt{n}$. In the quantum setting, two pioneering works approach this problem from different perspectives: Li [5] employs elementary algebraic techniques and probability bounds derived from the Nussbaum–Szkoła mapping to characterize the second-order term, while Tomamichel and Hayashi [6] use one-shot entropic quantities—such as the information spectrum relative entropy—to obtain the same result.

When the type-I error decays subexponentially, the problem falls in the *moderate-deviations regime*. The exponent of the type-II error was analyzed asymptotically in [7], [8]. In [7], Chubb *et al.* used a combination of relative entropy inequalities [6] and a Berry–Esseen–type result due to Rozovsky [9]. In [8], Cheng and Hsieh derived the achievability by using a martingale inequality due to Sason [10], originally developed for classical moderate deviations, while their converse follows directly from a sharp converse Hoeffding bound.

When both errors decay exponentially, the problem falls in the *large-deviations regime*, with performance described by the error exponents $A = -\lim_{n\to\infty} \frac{1}{n} \log \alpha_n$ and $B = -\lim_{n\to\infty} \frac{1}{n} \log \beta_n$. In the quantum setting, the converse part of this analysis was established by Nussbaum and Szkoła [1], while the achievability was proven by Audenaert *et al.* [11], together yielding the quantum generalization of the classical Chernoff exponent. The Hoeffding bound in the asymmetric setting was established by Hayashi and Nagaoka [12], [13].

The authors have received funding from the Spanish Ministerio de Ciencia, Innovación y Universidades under Grant PID2024-159557OB-C21 (MICIU/AEI/10.13039/501100011033 and ERDF/UE) and from the Comunidad de Madrid under Grant IDEA-CM (TEC-2024/COM-89).

In this work, we propose a novel converse bound for asymmetric quantum hypothesis testing based on the Nussbaum–Szkoła mapping. The bound is parametrized by a scalar $0 \le s < 1$ and provides a unified and conceptually simple framework that recovers converse results across the main asymptotic regimes: large-, moderate-, and small-deviations. This unification follows from a single expression by selecting appropriate sequences of the parameter $s \equiv s_n$. In certain regimes, it further enables a direct transfer of classical hypothesis testing results to the quantum setting. We further show that our bound is asymptotically tight for pure, asymptotically orthogonal states, implying that the associated multiplicative constants cannot be improved in general.

II. Preliminaries

We study the problem of discriminating between two quantum states. Specifically, let us consider the density operators[1] ρ and σ, acting on some finite-dimensional complex Hilbert space $\mathcal{H}$ with dimension d, and define the hypotheses

$$\mathrm{H}_0\colon \rho, \qquad \mathrm{H}_1\colon \sigma. \tag{1}$$

In this binary setting, we distinguish between two error types:

- The type-I error is the error of accepting H_1 when the true state is the null hypothesis $\mathrm{H}_0\colon \rho$.
- The type-II error is the error of accepting H_0 when the true system state is the alternative hypothesis $\mathrm{H}_1\colon \sigma$.

A binary test is defined by a positive self-adjoint operator Π acting on $\mathcal{H}$ such that $0 \preceq \Pi \preceq I$, where I denotes the identity matrix and the notation $A \preceq B$ means that $B - A$ is positive semidefinite. For a test Π associated with H_1, let $\bar{\Pi} \triangleq I - \Pi$. The type-I and type-II error probabilities are, respectively,

$$\alpha(\Pi) = \mathrm{Tr}[\Pi\rho], \qquad \beta(\Pi) = \mathrm{Tr}[\bar{\Pi}\sigma] = 1 - \mathrm{Tr}[\Pi\sigma]. \tag{2}$$

The two error probabilities cannot be made arbitrarily small at the same time. The best achievable trade-off between these probabilities corresponds to the Pareto-optimal boundary

$$\beta_\alpha(\rho,\sigma) = \inf_{\Pi:\alpha(\Pi)\le\alpha} \beta(\Pi). \tag{3}$$

For classical distributions P and Q, $\beta_\alpha(P,Q)$ is defined analogously. The error trade-off admits the following formulation:

Lemma 1 ([14, Lemma 2]): Let ρ and σ be two density operators. Then,

$$\beta_\alpha(\rho,\,\sigma) = \sup_{t\ge 0} \left\{ \mathrm{Tr}[\sigma\Pi_t] + t\left(\mathrm{Tr}[\rho\bar{\Pi}_t] - \alpha\right) \right\}, \tag{4}$$

where $\Pi_t \triangleq \{t\rho - \sigma \succeq 0\}$ is the projection onto the non-negative eigenspace of $t\rho - \sigma$ and $\bar{\Pi}_t = I - \Pi_t$.

When the quantum states are n-fold tensor products, i.e., $\rho \equiv \rho^{\otimes n}$ and $\sigma \equiv \sigma^{\otimes n}$, we consider the asymptotic behaviors of the type-I and type-II error probabilities as $n \to \infty$.

[1] Density operators are self-adjoint, positive semidefinite, and have unit trace.

III. Error Probability Lower Bound

A. The Lower Bound

The Nussbaum-Szkoła mapping relates two quantum states ρ and σ to two classical distributions P and Q. Specifically, for the spectral decompositions of the quantum states ρ and σ

$$\rho = \sum_{i=1}^{d} \lambda_i \left|x_i\right\rangle \left\langle x_i\right|, \quad \sigma = \sum_{j=1}^{d} \mu_j \left|y_j\right\rangle \left\langle y_j\right| \tag{5}$$

the Nussbaum-Szkoła-mapping distributions are given by

$$P_{ij} \triangleq \lambda_i |\langle x_i|y_j\rangle|^2, \quad Q_{ij} \triangleq \mu_j |\langle x_i|y_j\rangle|^2. \tag{6}$$

The next theorem relates the quantum and classical hypothesis testing trade-off functions of (5) and (6), respectively. Specifically, the Pareto-optimal boundary of the quantum test (5) is lower bounded by the Pareto-optimal boundary of the test (6), with the type-I and type-II error probabilities weighted by the respective factors $1-s$ and s, for an arbitrary $0 \le s < 1$.

Theorem 1: For a binary quantum hypothesis test between states ρ and σ (5) with Nussbaum-Szkoła mapping distributions P and Q (6), we have that

$$\beta_\alpha(\rho,\sigma) \ge s\beta_{\frac{1}{1-s}\alpha}(P,Q), \tag{7}$$

for every $0 \le s < 1$ and $0 \le \alpha \le 1 - s$.

Proof: Using that $\sum_{i=1}^{d} |x_i\rangle \langle x_i|$ and $\sum_{j=1}^{d} |y_j\rangle \langle y_j|$ are equal to the identity matrix, together with the idempotency of Π_t and $\bar{\Pi}_t$ alongside the cyclic property of the trace, we rewrite the trace terms in Lemma 1 as

$$\mathrm{Tr}[\sigma\Pi_t] = \sum_{ij} \mu_j |\langle x_i|\Pi_t|y_j\rangle|^2, \tag{8}$$

$$\mathrm{Tr}[\rho\bar{\Pi}_t] = \sum_{ij} \lambda_i |\langle x_i|\bar{\Pi}_t|y_j\rangle|^2. \tag{9}$$

Substituting the identities (8) and (9) into the variational expression of Lemma 1, we obtain

$$\beta_\alpha(\rho,\,\sigma) \ge \sum_{ij} \mu_j |\langle x_i|\Pi_t|y_j\rangle|^2 + \sum_{ij} t\lambda_i |\langle x_i|\bar{\Pi}_t|y_j\rangle|^2 - t\alpha \tag{10}$$

for any fixed $t \ge 0$. Let $t = \delta t'$ for $\delta \ge 0$. After joining both sums, and lower-bounding $t'\lambda_i$ and μ_j by $\min(t'\lambda_i, \mu_j)$, this can be further lower-bounded by

$$\beta_\alpha(\rho,\,\sigma) \ge \sum_{ij} \min(t'\lambda_i, \mu_j)\Big(|\langle x_i|\Pi_t|y_j\rangle|^2 + \delta|\langle x_i|\bar{\Pi}_t|y_j\rangle|^2 \Big) - \delta t'\alpha. \tag{11}$$

Next, for an arbitrary $0 \le s < 1$, define the vectors

$$u \triangleq \left[\sqrt{s},\, \sqrt{1-s}\right], \tag{12}$$

$$v \triangleq \left[|\langle x_i|\Pi_t|y_j\rangle|,\, \sqrt{\delta}|\langle x_i|\bar{\Pi}_t|y_j\rangle| \right]. \tag{13}$$

The Cauchy-Schwarz inequality $|\langle u, v\rangle|^2 \le \langle u, u\rangle \langle v, v\rangle$ yields

$$\Big(\sqrt{s}|\langle x_i|\Pi_t|y_j\rangle| + \sqrt{1-s}\sqrt{\delta}|\langle x_i|\bar{\Pi}_t|y_j\rangle|\Big)^2 \le |\langle x_i|\Pi_t|y_j\rangle|^2 + \delta|\langle x_i|\bar{\Pi}_t|y_j\rangle|^2. \tag{14}$$

By choosing $\delta = s/(1-s)$ in (14), we obtain

$$s\left(|\langle x_i|\Pi_t|y_j\rangle| + |\langle x_i|\bar{\Pi}_t|y_j\rangle|\right)^2 \leq |\langle x_i|\Pi_t|y_j\rangle|^2 + \frac{s}{1-s}|\langle x_i|\bar{\Pi}_t|y_j\rangle|^2. \quad (15)$$

This particular choice of δ enforces $\delta \geq 0$ for $0 \leq s < 1$. We next note that, by the triangle inequality $|u_1|+|u_2| \geq |u_1+u_2|$ and the fact that $\Pi_t + \bar{\Pi}_t = I$,

$$|\langle x_i|\Pi_t|y_j\rangle| + |\langle x_i|\bar{\Pi}_t|y_j\rangle| \geq |\langle x_i|y_j\rangle|. \quad (16)$$

Applying (15) and (16) to (11) then yields

$$\beta_\alpha(\rho,\, \sigma) \geq s\left(\sum_{ij} \min(t'\lambda_i, \mu_j)|\langle x_i|y_j\rangle|^2 - \frac{1}{1-s}t'\alpha\right). \quad (17)$$

We next relate the right-hand side of (17) to $\beta_\alpha(P, Q)$. Indeed, using the definitions of P and Q in (6), we note that

$$\min(t'\lambda_i, \mu_j)|\langle x_i|y_j\rangle|^2 = t'P_{ij}\mathbf{1}_{\{t'\lambda_i<\mu_j\}} + Q_{ij}\mathbf{1}_{\{t'\lambda_i\geq\mu_j\}}, \quad (18)$$

where $\mathbf{1}_\mathcal{E}$ denotes the indicator function for the event $\mathcal{E}$. Specializing the variational formulation (4) for ρ, σ being diagonal operators with P, Q in their diagonal yields

$$\beta_\alpha(P, Q) = \sup_{t\geq 0}\left\{\sum_{ij} Q_{ij}\mathbf{1}_{\{tP_{ij}\geq Q_{ij}\}} + t\Big(\sum_{ij} P_{ij}\mathbf{1}_{\{tP_{ij}<Q_{ij}\}} - \alpha\Big)\right\}. \quad (19)$$

Combining (18) with (17), taking the supremum over $t' \geq 0$, and expressing the result using (19), we obtain the bound (7). ■

B. Pure-State Discrimination

The bound in Theorem 1 is asymptotically tight for a test between pure states that are asymptotically orthogonal. Consider the discrimination between the pure states

$$\mathrm{H}_0\colon \rho = |x_1\rangle\langle x_1|, \qquad \mathrm{H}_1\colon \sigma = |y_1\rangle\langle y_1|, \quad (20)$$

where $|x_1\rangle$ and $|y_1\rangle$ are assumed to satisfy $0 < |\langle x_1|y_1\rangle|^2 < 1$.

1) Classical test: For the above quantum states ρ and σ, the Nussbaum-Szkoła mapping from (6) becomes

$$P_{ij} = \begin{cases}|\langle x_1|y_j\rangle|^2, & i = 1,\\ 0, & \text{otherwise},\end{cases} \qquad Q_{ij} = \begin{cases}|\langle x_i|y_1\rangle|^2, & j = 1,\\ 0, & \text{otherwise}.\end{cases} \quad (21)$$

The distributions P and Q exhibit non-overlapping supports, except in the singular case $(i, j) = (1, 1)$, where

$$P_{11} = Q_{11} = |\langle x_1|y_1\rangle|^2 = \mathrm{Tr}[\rho\sigma] \triangleq a. \quad (22)$$

The optimal classical test for this problem decides on the correct hypothesis with no error, except when $(i, j) = (1, 1)$. In this case, the optimal test can select between H_0 and H_1 at random with some probabilities p and $1-p$, thus incurring an error with probabilities

$$\alpha = (1-p)P_{11} = (1-p)a, \qquad \beta = pQ_{11} = pa. \quad (23)$$

After solving for p in α and substituting into β, we obtain the error trade-off

$$\beta_\alpha(P, Q) = a - \alpha, \quad 0 \leq \alpha \leq a. \quad (24)$$

2) Quantum test: For the pure quantum states ρ, σ given in (20), the quantum error trade-off is characterized in the following lemma.

Lemma 2: Let ρ, σ be pure states. Then, the optimal quantum error trade-off is given by

$$\beta_\alpha(\rho, \sigma) = \alpha - 2\alpha a + a - 2\sqrt{(1-\alpha)\alpha(1-a)a}, \quad (25)$$

for $0 \leq \alpha \leq a$, where a is given in (22).

Proof: The fidelity between two states is defined as

$$F(\rho, \sigma) \triangleq \mathrm{Tr}\left[|\sqrt{\rho}\sqrt{\sigma}|\right], \quad (26)$$

where $|A| = \sqrt{A^\dagger A}$. Combining the fidelity lower bound from [2, Eq. (10)], which is tight for pure states, with the fact that $F^2 = a$, the lemma follows. ■

Expanding (25) as $a \to 0$, we obtain

$$\beta_\alpha(\rho, \sigma) = \alpha - 2\sqrt{\alpha a} + a + o(a), \quad 0 \leq \alpha \leq a, \quad (27)$$

where $o(a)$ summarizes terms that vanish faster than a as $a \to 0$. Furthermore, using (24), the lower bound of Theorem 1 can be written as

$$\beta_\alpha(\rho, \sigma) \geq s\beta_{\frac{\alpha}{1-s}}(P, Q) = sa - \frac{s}{1-s}\alpha, \quad 0 \leq s < 1, \quad (28)$$

which, for $s = 1 - \sqrt{\frac{\alpha}{a}}$, becomes

$$\beta_\alpha(\rho, \sigma) \geq \alpha - 2\sqrt{\alpha a} + a. \quad (29)$$

Comparing (27) with (29), we observe that the quantum error trade-off in (27) coincides with the lower bound in (29) up to terms of order $o(a)$. For a test between the n-th tensor product states $\rho^{\otimes n}, \sigma^{\otimes n}$, we have that $a = |\langle x_1|y_1\rangle|^{2n}$. So the assumption $|\langle x_1|y_1\rangle|^2 < 1$ implies that $a \to 0$ as $n \to \infty$. This proves that, for distinct pure states, the bound is asymptotically tight as $n \to \infty$.

As an example, we consider the n-th tensor product of the states $\rho = |0\rangle\langle 0|$ and $\sigma = |+\rangle\langle +|$. Fig. 1 shows the quantum error trade-off β_α for the hypothesis test between $\rho^{\otimes n}$ and $\sigma^{\otimes n}$ when $n = 5$, and compares it to the lower bound of Theorem 1 for different values of s. Observe that the upper envelope of the resulting bounds provides an accurate characterization of the quantum error trade-off when both states are pure and approximately orthogonal (in this example $a = 2^{-5} \approx 0.03$). We therefore conclude that the multiplicative factors s and $1-s$ from Theorem 1 cannot be improved in general.

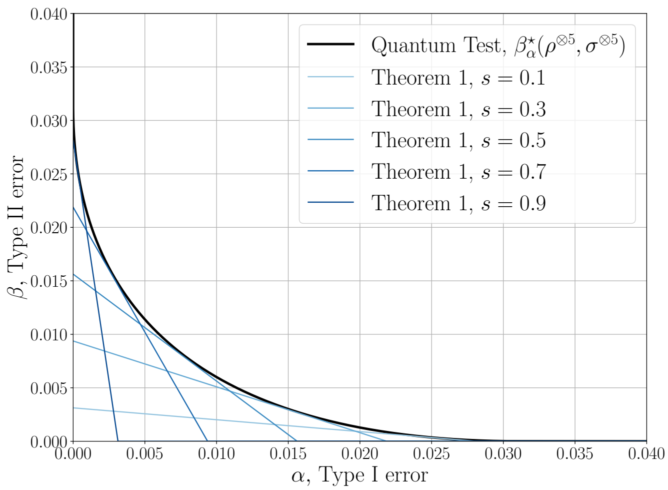

Fig. 1. Quantum error trade-off compared to the lower bounds from Theorem 1 for a hypothesis test between the states $|0\rangle\langle 0|^{\otimes 5}$ and $|+\rangle\langle +|^{\otimes 5}$.

IV. Asymptotic Analysis

In this section, we show that the lower bound in Theorem 1, with an adequate choice of the parameter s, provides a unified and conceptually simple framework for recovering converse results in the asymptotic small-deviations (second-order), moderate-deviations, and large-deviations regimes. To this end, we extend the hypothesis test from single copies ρ, σ to n-fold tensor products $\rho^{\otimes n}, \sigma^{\otimes n}$. Furthermore, we introduce two relevant classical quantities associated with the Nussbaum-Szkoła-mapping distributions P and Q, namely,

$$V(P\|Q) \triangleq \mathsf{E}_P\left[\left(\log\frac{P}{Q} - D(P\|Q)\right)^2\right] \tag{30}$$

$$T(P\|Q) \triangleq \mathsf{E}_P\left[\left|\log\frac{P}{Q} - D(P\|Q)\right|^3\right] \tag{31}$$

where $D(P\|Q)$ denotes the relative entropy between P and Q. We further define the hypothesis testing relative entropy

$$D_h^\varepsilon(\rho^{\otimes n}\|\sigma^{\otimes n}) \triangleq -\log \beta_\varepsilon\left(\rho^{\otimes n}, \sigma^{\otimes n}\right). \tag{32}$$

For the classical Nussbaum-Szkoła distributions P and Q, $D_h^\varepsilon(P^n\|Q^n)$ is defined analogously.

Corollary 1 (Small Deviations): Let $V(P \,\|\, Q) > 0$. Then,

$$D_h^\varepsilon(\rho^{\otimes n} \,\|\, \sigma^{\otimes n}) \le nD(P \,\|\, Q) + \sqrt{nV(P \,\|\, Q)}\Phi^{-1}(\varepsilon) + O(\log n), \tag{33}$$

where $\Phi(x)$ denotes the cumulative distribution function of the standard normal distribution and $O(\log n)$ summarizes terms of order $\log n$.

Proof: Taking logarithms on both sides of (7), and replacing s by s_n to indicate that this parameter may depend on n, we obtain from Theorem 1 that

$$D_h^\varepsilon(\rho^{\otimes n} \,\|\, \sigma^{\otimes n}) \le D_h^{\frac{\varepsilon}{1-s_n}}(P^n \,\|\, Q^n) - \log s_n. \tag{34}$$

From the small-deviations asymptotic behavior of the classical hypothesis testing problem [15, Prop. 2.3], we have

$$D_h^\varepsilon(P^n \,\|\, Q^n) = nD(P \,\|\, Q) + \sqrt{nV(P \,\|\, Q)}\Phi^{-1}(\varepsilon) + \frac{1}{2}\log n + O(1). \tag{35}$$

Using (35) in (34), and setting $s_n = 1/\sqrt{n}$, we obtain that

$$D_h^\varepsilon(\rho^{\otimes n}\|\sigma^{\otimes n}) \le nD(P\|Q) + \sqrt{nV(P \,\|\, Q)}\Phi^{-1}\left(\frac{\varepsilon}{1-n^{-1/2}}\right) + \log n + O(1). \tag{36}$$

Performing a first-order Taylor expansion of $\Phi^{-1}(\frac{\varepsilon}{1-n^{-1/2}})$ around ε, and grouping terms of order $\log n$ and 1 in one $O(\log n)$-term, proves the corollary. ■

Corollary 2 (Moderate Deviations): Let $\{a_n\}$ be a nonnegative sequence satisfying $a_n \to 0$ and $na_n^2 \to \infty$ as $n \to \infty$. Further let $\varepsilon_n = e^{-na_n^2}$. Then,

$$D_h^{\varepsilon_n}(\rho^{\otimes n}\|\sigma^{\otimes n}) \le nD(P\|Q) - \sqrt{2V(P\|Q)}na_n + o(na_n) \tag{37}$$

where $o(na_n)$ summarizes terms that grow more slowly than na_n as $n \to \infty$.

Proof: We start by upper-bounding $D_h^{\varepsilon_n}(\rho^{\otimes n}\|\sigma^{\otimes n})$ using Theorem 1 with $s \equiv s_n$ and applying Lemma 1 to the term $\beta_{\frac{1}{1-s_n}\varepsilon_n}(P^n, Q^n)$. We then set $t = 2^{-R_n}$ and use that $Q^n\left(\log\frac{P^n}{Q^n} > R_n\right) \ge 0$ to obtain the upper bound

$$D_h^{\varepsilon_n}(\rho^{\otimes n}\|\sigma^{\otimes n}) \le R_n - \log\left(P^n\left(\log\frac{P^n}{Q^n} \le R_n\right) - \frac{1}{1-s_n}\varepsilon_n\right) - \log s_n \tag{38}$$

for arbitrary sequences $\{R_n\}$ and $\{s_n\}$. Specifically, we set

$$R_n = nD(P\|Q) - na_n\lambda_n, \quad s_n = \frac{1}{na_n}, \tag{39}$$

for a sequence $\{\lambda_n\}$ that satisfies

$$\lambda_n^2 = 2V(P\|Q) + o(1) \tag{40}$$

and

$$P^n\left(\log\frac{P^n}{Q^n} \le R_n\right) \ge \frac{1}{1-2s_n}\varepsilon_n. \tag{41}$$

Using (41) and $\varepsilon_n = e^{-na_n^2}$, and noting that $na_n = \sqrt{n}\sqrt{na_n^2} \to \infty$ as $n \to \infty$ by the assumption $na_n^2 \to \infty$, the last two terms in (38) become

$$-\log\left(P^n\left(\log\frac{P^n}{Q^n} \le R_n\right) - \frac{1}{1-s_n}\varepsilon_n\right) - \log s_n = o(na_n). \tag{42}$$

Substituting (39), (40), and (42) in (38), and noting that $\sqrt{2V(P\|Q) + o(1)} = \sqrt{2V(P\|Q)} + o(1)$, we obtain the desired bound (37).

It remains to show that there exists a sequence $\{\lambda_n\}$ that satisfies (40) and (41). Specifically, we choose

$$\lambda_n^2 = 2V(P\|Q)\left(1 - \frac{A_n}{na_n^2} - \frac{B_n}{na_n^2} + \frac{\log(1-2s_n)}{na_n^2}\right) \tag{43}$$

where

$$A_n \triangleq \frac{c_1\sqrt{8}T(P\|Q)}{V^{3/2}(P\|Q)} n a_n^3 - \ln\left(1 - \frac{c_2\sqrt{2}T(P\|Q)}{V(P\|Q)} a_n\right) \quad (44)$$

$$B_n \triangleq \frac{1}{2}\ln(2na_n^2) - \ln\left(1 - \frac{1}{na_n^2}\right) \quad (45)$$

with constants $c_1, c_2 > 0$. Note that the sequences $\{A_n\}$ and $\{B_n\}$ are nonnegative and of order $o(na_n^2)$ by the assumption $a_n \to 0$. Moreover, for our choice of $s_n = 1/(na_n)$, we have that $\ln(1 - 2s_n) = o(1)$, since $na_n \to \infty$ as $n \to \infty$. Consequently, this choice of $\{\lambda_n\}$ satisfies (40).

To show that $\{\lambda_n\}$ also satisfies (41), we apply the following result due to Rozovsky:

Lemma 3 ([9, Th. B2]): Let $X_1, \ldots, X_n$ be independent random variables with finite third moments

$$\mu_k = \mathsf{E}[X_k], \ \sigma_k^2 = \mathsf{Var}(X_k), \ t_k = \mathsf{E}\left[|X_k - \mu_k|^3\right]. \quad (46)$$

Let $S = \sigma_1^2 + \ldots + \sigma_n^2$ and $T = t_1 + \ldots + t_n$, and let $c_1 > 0$ and $c_2 > 0$ be universal constants. Then, for every $x \geq 1$,

$$\Pr\left(\sum_{k=1}^{n}(X_k - \mu_k) > x\sqrt{S}\right) \geq Q(x)e^{-\frac{c_1 T}{S^{3/2}}x^3}\left(1 - \frac{c_2 T}{S^{3/2}}x\right). \quad (47)$$

Specifically, we consider

$$X_k = -\log\frac{P}{Q}, \qquad \mu_k = -D(P\|Q),$$
$$\sigma_k^2 = V(P\|Q), \qquad t_k = T(P\|Q), \quad (48)$$

so that the probability on the left-hand side of (41) becomes

$$P^n\left(\log\frac{P^n}{Q^n} \leq R_n\right) = \Pr\left(\sum_{k=1}^{n}(X_k - \mu_k) \geq \lambda_n n a_n\right). \quad (49)$$

We apply Lemma 3 with $S = nV(P\|Q)$ and $T = nT(P\|Q)$ to (49) and use that, by the definition of λ_n^2 in (43), $V(P\|Q) \leq \lambda_n^2 \leq 2V(P\|Q)$. Taking natural logarithms on both sides of (49), and further using the lower bound $Q(x) \geq 1/\sqrt{2\pi x^2}e^{-x^2/2}(1 - x^{-2})$, $x > 0$, we obtain that

$$\ln P^n\left(\log\frac{P^n}{Q^n} \leq R_n\right) \geq -na_n^2 - \ln(1 - 2s_n). \quad (50)$$

Therefore,

$$P^n\left(\log\frac{P^n}{Q^n} \leq R_n\right) \geq e^{-na_n^2 - \ln(1-2s_n)} = \frac{1}{1 - 2s_n}\varepsilon_n, \quad (51)$$

which is (41). ∎

Corollary 3 (Large Deviations): Let $\varepsilon_n \leq e^{-nr}$. Then,

$$\limsup_{n\to\infty} \frac{1}{n} D_h^{\varepsilon_n}(\rho^{\otimes n}\|\sigma^{\otimes n}) \leq \sup_{0\leq s\leq 1}\left\{\frac{1}{s-1}\log \mathrm{Tr}\left[\rho^s\sigma^{1-s}\right] + \frac{s}{s-1}r\right\}. \quad (52)$$

Proof: Setting $s = \frac{1}{2}$ in Theorem 1 recovers the lower bound in [16, Prop. 2] (for details, see [17]). The quantum Hoeffding bound (52) is then obtained by following the steps in the proof of [16, Th. 3]. ∎

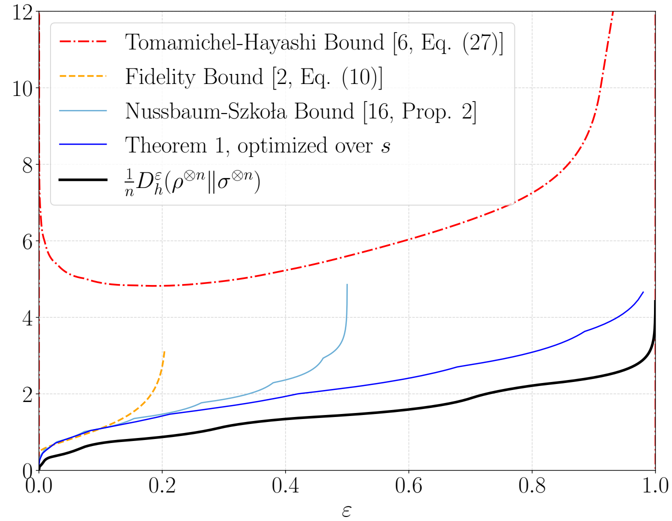

Fig. 2. Upper bounds on the normalized hypothesis testing relative entropy $\frac{1}{n}D_h^\varepsilon(\rho^{\otimes n}\|\sigma^{\otimes n})$ for ρ, σ in (54) and $n = 5$.

V. Comparison with Previous Converse Bounds

In this section, we compare the converse bound of Theorem 1 with previous converse bounds obtained in the literature. In particular, we consider the information-spectrum bound in [6, Eq. (27)], the fidelity bound in [2, Eq. (10)], and the Nussbaum-Szkoła bound on the minimum average error probability [16, Prop. 2] which, extended to the asymmetric setting, coincides with Theorem 1 with $s = 1/2$.

Figure 2 compares the resulting upper bounds with the true normalized hypothesis testing relative entropy

$$\frac{1}{n}D_h^\varepsilon(\rho^{\otimes n}\|\sigma^{\otimes n}) = -\frac{1}{n}\log\beta_\varepsilon(\rho^{\otimes n}, \sigma^{\otimes n}) \quad (53)$$

for $n = 5$ and the mixed states

$$\rho = \begin{bmatrix}0.8 & 0\\ 0 & 0.2\end{bmatrix}, \qquad \sigma = \begin{bmatrix}0.35 & \frac{3\sqrt{3}}{20}\\ \frac{3\sqrt{3}}{20} & 0.65\end{bmatrix}. \quad (54)$$

Observe the limited range of the bounds in [16, Prop. 2] and [2, Eq. (10)], which yield finite non-trivial bounds only for $0 \leq \varepsilon < F(\rho, \sigma)^{2n}$ and $0 \leq \varepsilon < 1/2$, respectively. In contrast, the bound of Theorem 1, optimized over $s \in [0, 1]$, provides a tight characterization of $\frac{1}{n}D_h^\varepsilon(\rho^{\otimes n}\|\sigma^{\otimes n})$ over the entire range $0 \leq \varepsilon < 1$. Furthermore, the bound [6, Eq. (27)] yields finite values over $0 < \varepsilon < 1$, but it is significantly looser than the bound obtained from Theorem 1 for the considered example.

We further note that, while the bound in [6, Eq. (27)] recovers the small- and moderate-deviation results of Corollaries 1 and 2 (see [6] and [7]), it fails to recover the error exponent in the large-deviations analysis beyond Stein's regime. We conclude that the bound presented in Theorem 1 not only constitutes an accurate non-asymptotic approximation of the quantum hypothesis testing performance, it also provides a unified framework for recovering converse results in the asymptotic setting.

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