# IMPROVED BLIND EQUALIZATION VIA ADAPTIVE COMBINATION OF CONSTANT MODULUS ALGORITHMS

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**ABSTRACT**

Adaptive blind equalization plays an essential role in modern communication systems. We propose to improve the performance of constant modulus algorithm (CMA) based equalizers by using an adaptive convex combination of two CMA filters with a large and small step size, respectively, in order to simultaneously obtain fast convergence with low misadjustment during stationary periods. Some experiments show the effectiveness of the new algorithm and suggest that it is a reasonable alternative to blind equalizers that commute between the CMA and the (decision-directed) least mean square filter.

## 1. INTRODUCTION

Adaptive equalization techniques are of great importance in modern high-efficiency communication systems. Among all possible schemes, blind equalizers that do not require the use of training sequences, but make use of some statistical knowledge about the transmitted signal, present a number of important advantages [1]:

- Simplified protocols in point-to-point communications, avoiding the retransmission of training sequences after abrupt changes of the channel.
- Higher bandwidth efficiency in broadcast networks.
- Reduced interoperability problems derived from the use of different training sequences.

Among all algorithms for blind equalization, the constant modulus algorithm (CMA) [2] plays a preeminent role. However, the use of CMA equalizers with constellations whose symbols have non-constant norms is subject, even for infinite signal-to-noise ratios, to a component of gradient noise that is proportional to the step size [3]. Using a small step size to minimize this pernicious effect results in a slow convergence of the algorithm.

To avoid the above problem, it is common to use CMA to get a coarse equalization of the channel, before transferring to a decision-directed (DD) mode where the least mean square (LMS) algorithm can be applied. However, this approach requires the design of appropriate procedures for transfer between both operation modes (see, among many others, [4], [5]).

In this paper we propose an alternative solution based on the adaptive combination of adaptive filters [6], [7], which consists of a convex combination of a fast and a slow CMA filters (high and low adaptation steps, respectively). The resulting scheme is able to extract the best properties of each component, namely fast convergence after abrupt changes in the channel and low residual error in steady-state.

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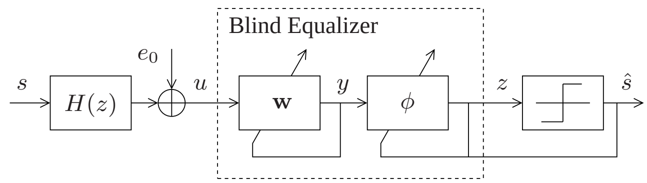


**Fig. 1**. Baseband model of a blind equalization system.

## 2. THE CONSTANT MODULUS ALGORITHM

In Fig. 1 we have depicted a generic baseband model for the blind equalization problem that we will consider throughout the paper. The input to the equalizer, $u(n)$, is a distorted version of the transmitted signal, $s(n)$, corrupted by selective attenuation, inter-symbol interference, and additive noise. Assuming that the channel can be modelled by a linear filter of length $Q$, $u(n)$ is given by

$$u(n) = \sum_{i=0}^{Q-1} h_i(n)s(n-i) + e_0(n), \tag{1}$$

where $\mathbf{h}(n) = [h_0(n), \cdots, h_{Q-1}(n)]^T$ is the impulse response vector of the channel at time $n$, and $e_0(n)$ is i.i.d. Gaussian noise.

The aim of the adaptive equalization block is to recover a signal $z(n)$ that is as close as possible to $s(n)$, so that the decision on which symbol was originally transmitted, based on $z(n)$, results in a minimum number of errors. The adaptive equalizers that we consider in this paper consist of two different stages (see Fig. 1). First, signal $u(n)$ is passed through an adaptive filter $\mathbf{w}(n)$ that aims to recover the original constellation. Second, a phase recovery block is used to obtain the correct constellation rotation.

Probably, the most popular algorithm for the blind optimization of $\mathbf{w}(n)$ is Godard's CMA [2], which consists in stochastic gradient minimization of the following error function

$$J[\mathbf{w}(n)] = E\{|R_{pq} - |y(n)|^p|^q\}, \tag{2}$$

where $|y(n)|$ is the modulus of the output of the filter, $y(n) = \mathbf{w}^T(n)\mathbf{u}(n)$, with $\mathbf{u}(n) = [u(n), u(n-1), \cdots, u(n-M+1)]^T$, and $R_{pq}$ is a positive constant whose value depends on the constellation. In the following, we will just consider the case $q = 2$, using the simplified notation $R_p = R_{p2}$.

For constant modulus constellations, it can be seen that by minimizing (2), the CMA filter will try to output constant modulus values satisfying $|y(n)|^p \approx R_p$. However, it is shown in [2] that this cost function can also be used to recover constellations with non-constant modulus, such as pulse amplitude modulation (PAM) or quadrature amplitude modulation (QAM). When $s(n)$ is not a constant modulus signal, the optimum value for $R_p$ is given by

$$R_p = \frac{E\{|s(n)|^{2p}\}}{E\{|s(n)|^p\}}. \tag{3}$$

As discussed in [3], $p = 2$ offers superior performance to that of other values, and so we will use this setting. Now, taking the gradient of (2) with respect to $\mathbf{w}(n)$ results in the following CMA update rule:

$$\mathbf{w}(n+1) = \mathbf{w}(n) + \mu[R_2 - |y(n)|^2]y(n)\mathbf{u}^*(n), \tag{4}$$

where $\mu$ is the step size and the * superscript denotes scalar or vector complex conjugation.

In general, and given the insensitivity of (2) to rotations, it is necessary to rotate the output of the CMA filter, so that the final decision is based on $z(n) = y(n)\exp[-j\phi(n)]$, where $\phi(n)$ can be optimized, for instance, using the recursion [4]

$$\phi(n+1) = \phi(n) - \mu_\phi \Im[z(n)e^*(n)], \tag{5}$$

where $\Im[\cdot]$ denotes the imaginary part, and the error signal is defined as $e(n) = z(n) - \hat{s}(n)$, $\hat{s}(n)$ being the decoded symbol, i.e., if $A$ is the set of symbols in the constellation,

$$\hat{s}(n) = \arg\min_{s' \in A} |z(n) - s'|. \tag{6}$$

The random component that appears when CMA is applied to constellations whose symbols do not have a constant norm results in a residual error term proportional to $\mu$ [3]. Consequently, the step size of CMA imposes a tradeoff between speed of convergence and final misalignment, even for infinite signal-to-noise ratio (SNR). To illustrate this tradeoff, Fig. 2 shows the convergence of two CMA filters with $M = 35$ taps, using step sizes $\mu_1 = 10^{-4}$ and $\mu_2 = 5 \cdot 10^{-6}$. The symbols in $s(n)$ belong to a 4-PAM constellation[1]: $A = \{-3, -1, 1, 3\}$ ($R_2 = 8.2$). The response of the channel is initially given by $\mathbf{h}_1(n) = [0.1, 0.3, 1, -0.1, 0.5, 0.2]^T$, and then it is changed to $\mathbf{h}_2(n) = [0.25, 0.64, 0.8, -0.55]^T$ at $n = 50000$[2]. Figure 3 represents the steady-state symbol error rate (SER) as a function of the SNR for both filters.

We can see that the $\mu_1$-CMA offers very fast initial convergence, as well as after the change in the channel. A smaller step size obtains lower error and SER but makes the convergence slow. Indeed, these results show that speed of convergence and steady-state performance are conflicting requirements for CMA-based equalization.

## 3. ADAPTIVE COMBINATION OF CMA FILTERS

### 3.1. The Basic CCMA Algorithm

To obtain fast blind equalization together with low residual error, several researchers have proposed to revert to a decision-directed

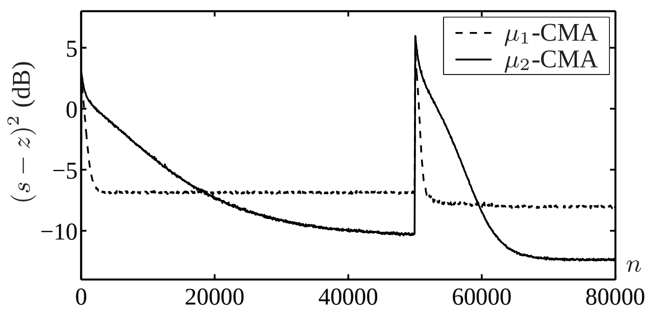


**Fig. 2**. Quadratic error incurred by two CMA equalizers ($\mu_1 = 10^{-4}$ and $\mu_2 = 5 \cdot 10^{-6}$) in a system using 4-PAM modulation. The SNR at the input of the equalizer is 20 dB.

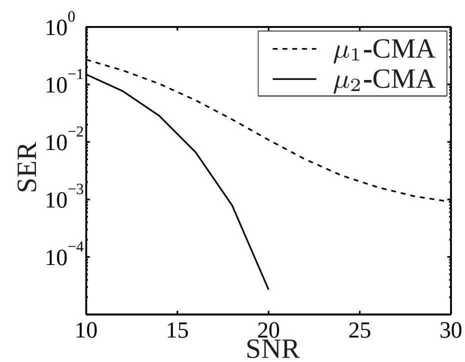


**Fig. 3**. Symbol error rate (SER) achieved by two CMA filters ($\mu_1 = 10^{-4}$ and $\mu_2 = 5 \cdot 10^{-6}$) in a 4-PAM system as a function of the SNR at the input of the equalizer.

(DD) least mean square (LMS) filtering scheme after coarse equalization of the channel is achieved by the CMA filter. However, this approach requires the design of procedures for commuting between the CMA and LMS filters and vice-versa.

In this paper we present an alternative solution that is based on the combination of adaptive filters of [6], [7]. The idea is to adaptively combine the outputs of one fast and one slow CMA filters with step sizes $\mu_1 > \mu_2$. The output of the overall combination of CMA filters (CCMA) is given by (see Fig. 4):

$$y(n) = \lambda(n)y_1(n) + [1 - \lambda(n)]y_2(n), \tag{7}$$

where $y_1(n)$ and $y_2(n)$ are the outputs of both component filters, and $\lambda(n)$ is a mixing coefficient. The idea is that if $\lambda(n)$ is assigned appropriate values at each iteration, then the combination scheme will extract the best properties of each component filter.

In principle, both CMA filters are independently adapted using their own outputs, i.e.,

$$\mathbf{w}_i(n+1) = \mathbf{w}_i(n) + \mu_i[R_2 - |y_i(n)|^2]y_i(n)\mathbf{u}^*(n); \quad i = 1, 2, \tag{8}$$

while the mixing parameter is optimized, using also a stochastic gradient rule, to minimize

$$J = [R_2 - |y(n)|^2]^2. \tag{9}$$

However, instead of directly adapting $\lambda(n)$, we will update a parameter $a(n)$ that is related to $\lambda(n)$ via a sigmoid function

$$\lambda(n) = \mathrm{sigm}[a(n)] = \{1 - \exp[-a(n)]\}^{-1}.$$

[1]Note that it is not necessary to recover the phase when using real constellations.

[2]These channels are taken from [3] and [8], respectively.

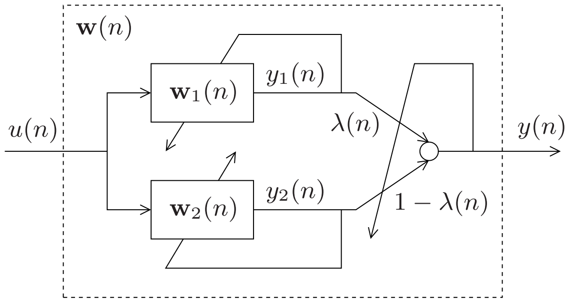


**Fig. 4**. Adaptive convex combination of two CMA filters. Each component is adapted using its own output, while the mixing parameter $\lambda(n)$ uses the overall output of the filter.

The update equation for $a(n)$ is then given by

$$\begin{aligned} a(n+1) &= a(n) - \frac{\mu_a}{4}\frac{\partial J(n)}{\partial a(n)} \qquad (10)\\ &= a(n) + \frac{\mu_a}{2}[R_2 - |y(n)|^2]\lambda(n)[1-\lambda(n)] \\ &\qquad \times \left[\frac{\partial y(n)}{\partial \lambda(n)} y^*(n) + y(n)\frac{\partial y^*(n)}{\partial \lambda(n)}\right]. \end{aligned}$$

Now, substituting in the above expression the derivatives

$$\frac{\partial y(n)}{\partial \lambda(n)} = y_1(n) - y_2(n) \qquad (11)$$

$$\frac{\partial y^*(n)}{\partial \lambda(n)} = [y_1(n) - y_2(n)]^* \qquad (12)$$

we obtain the final adaption rule for $a(n)$

$$a(n+1) = a + \mu_a[R_2 - |y|^2]\Re\{y[y_1 - y_2]^*\}\lambda[1-\lambda], \quad (13)$$

where $\Re[\cdot]$ denotes the real part of a complex number, and where we have omitted the time index $n$ on the right hand side for reasons of compactness.

As explained in [6], [7], the advantages of using the sigmoid activation for $\lambda(n)$ are twofold. First, it is an easy way to guarantee that $\lambda(n)$ remains within the desired interval $[0, 1]$. Second, the factor $\lambda(n)[1 - \lambda(n)]$ in (13) reduces the adaptation speed, and consequently also the gradient noise, near $\lambda = 0$ and $\lambda = 1$, when the combination is expected to perform like one of the component filters. Nevertheless, the update for $a(n)$ could stop whenever $\lambda(n)$ is too close to one of these limits. To circumvent this difficulty, we restrict the values of $a(n)$ to lie inside the interval $[-4, 4]$.

The proposed scheme has a very simple interpretation: when fast or abrupt changes appear, the $\mu_1$-CMA filter achieves a lower error according to the CMA cost function and, consequently, the minimization of (9) results in $\lambda(n) \rightarrow 1$. On the contrary, in steady-state situations, it is the slow filter that works better, making $\lambda(n) \rightarrow 0$.

### 3.2. Speeding up Convergence of the Slow Component

A limitation of the basic CCMA scheme is that, after an abrupt change in the channel, the steady-state error of the $\mu_2$-CMA filter can not be achieved until this component has completely converged. To improve the performance of the overall scheme we will use a modification of the method presented in [9], transferring at each iteration a part of the weights from the fast filter to $\mathbf{w}_2$:

$$\mathbf{w}_2(n+1) \leftarrow \alpha\mathbf{w}_2(n+1) + (1-\alpha)\mathbf{w}_1(n+1), \qquad (14)$$

where $\alpha$ is a parameter close to 1.

The use of this weight transfer procedure over successive iterations will serve to speed up the convergence of the slow CMA filter and, consequently, the convergence of the overall equalizer. However, an uninterrupted application of (14) would increase the steady-state error of the slow filter. To avoid this, weight transfer should only be applied when the fast filter is clearly achieving a better equalization of the channel, i.e., when $\lambda(n) > \beta$, where $\beta$ is a positive constant, which must be fixed close to the maximum allowed value for $\lambda(n)$.

Although this “speeding-up” mechanism requires two extra parameters, we have checked that the CCMA filter is not very sensitive to the selection of $\alpha$ and $\beta$. In any case, this weight transfer mechanism should be seen as an optional procedure that can be used to improve the performance of the basic combination in very particular situations. Our extensive simulation work shows that $\alpha = 0.9$ and $\beta = 0.98$ are values that obtain good results in most situations, and so these are the settings that we will keep in the following.

## 4. EXPERIMENTS

In this section we will describe the performance of the CCMA filter in a system using 16-QAM modulation: $s(n) \in \{\pm s_R \pm j s_I\}$, with $s_R$ and $s_I \in \{1, 3\}$. For this constellation, we have $R_2 = 13.2$. We will use in our experiments the same channel considered at the end of Section 2, but now the change of the impulse response of the channel occurs at $n = 150000$.

The settings for the CCMA filter are $\mu_1 = 2 \cdot 10^{-5}$ and $\mu_2 = 10^{-6}$ for the component filters, and $\mu_a = 0.1$ to adapt the mixing parameter. The weights of both CMA filters were initialized with zeros, and the initial value for the mixing parameter was $a(0) = 0$. Finally, before delivering the signal to the decision block, $y(n)$ is rotated using (5) with $\mu_\phi = 10^{-5}$.

Figure 5 represents, for two different values of the input SNR, the average over 100 independent runs of the quadratic difference between the transmitted signal and the signal delivered to the decision block, both for equalizers using the fast and slow CMA filters only, and for the CCMA-based equalizer. As discussed in Section 2, the $\mu_1$-CMA filter has very fast convergence for both values of the SNR, while the slow filter is able to achieve reduced misalignment in steady-state, at the cost of slower convergence.

We can see that the combined CCMA scheme inherits the best properties of each of the component filters, presenting fast convergence together with the low residual error of the $\mu_2$-CMA. Furthermore, it is important to remark that the weight transfer procedure allows the combined equalizer to achieve the steady-state misalignment of the slow filter very soon, in comparison to the convergence time of this component.

Figure 6 shows the effects of the channel and the CCMA equalizer on the transmitted signal $s(n)$. The constellations represent the last 10000 symbols that were received in a single run with $\mathrm{SNR} = 20$ dB. It is interesting to see that both CMA components have converged to the same minimum of (2) (although with a reduced gradient noise in the case of $\mu_2$-CMA), probably because of the application of the weight transfer procedure. Finally,

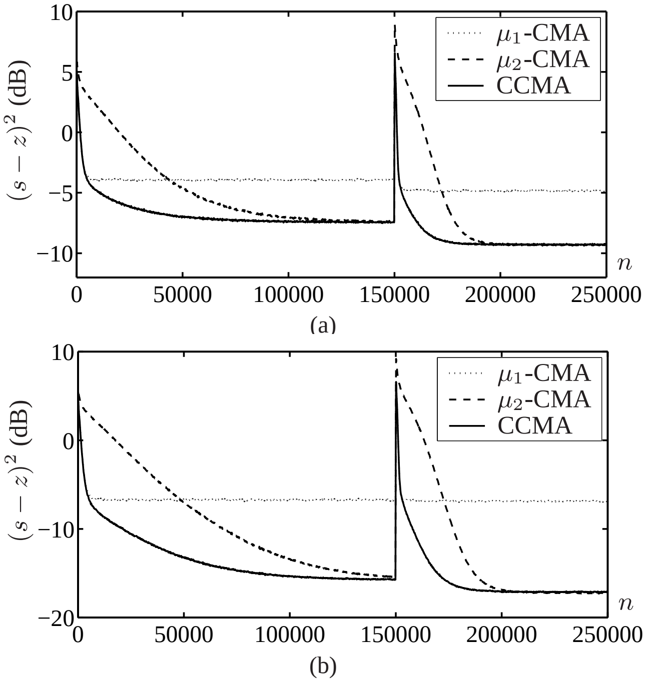


**Fig. 5**. Quadratic error between the transmitted and equalized signals (after phase rotation) achieved by two CMA blind equalizers ($\mu_1 = 2 \cdot 10^{-5}$ and $\mu_2 = 10^{-6}$), as well as by their adaptive combination (CCMA, $\mu_a = 0.1$). The system uses 16-QAM modulation and the SNR at the input of the equalizer was tuned to (a) 20 dB, and (b) 30 dB.

in steady-state, $z(n)$ is a rotated version of $y_2(n)$ that can be used to recover the transmitted information with a very low SER.

The above results allow us to conclude that the CCMA scheme can be used as a reasonable alternative to blind equalizers that commute between CMA and LMS filters, and require the design of appropriate procedures for transfer between algorithms.

## 5. CONCLUSIONS

In this paper we have presented a new algorithm for adaptive blind equalization of communication channels that relies on a convex combination of two CMA filters with different step sizes. Each component filter is adapted independently using its own output, while the combination is adapted in accordance with an overall performance criterion.

Simulation examples show that the proposed scheme retains the best properties of each component filter, namely fast convergence and low residual misadjustment, and constitutes a very simple and effective alternative to other algorithms that pursue the same goal, but require the (not always easy) commutation between CMA and (DD) LMS modes.

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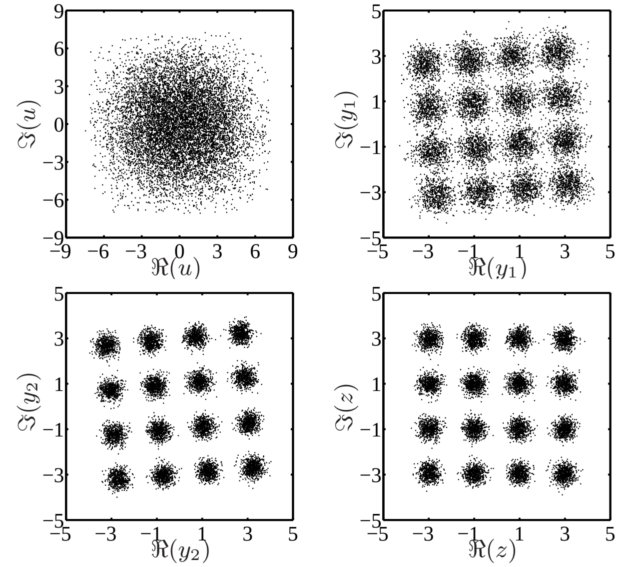

**Fig. 6**. Constellations in a 16-QAM system recovered by a CCMA blind equalizer operating in steady-state. The figure shows the constellations that appear at the input of the equalizer ($u$), at the output of the fast and slow CMA components ($y_1$ and $y_2$, respectively), and at the output of the overall system after phase recovery ($z$). The SNR at the input of the equalizer is 20 dB.


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