On the Per-User Probability of Error in Gaussian Many-Access Channels

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***Abstract*—We consider a Gaussian multiple-access channel where the number of users grows with the blocklength n. For this setup, the maximum number of bits per unit-energy that can be transmitted reliably as a function of the order of growth of the users is analyzed. For the *per-user probability of error*, we show that if the number of users grows sublinearly with the blocklength, then each user can achieve the capacity per unit-energy of the Gaussian single-user channel. Conversely, if the number of users grows at least linearly with the blocklength, then the capacity per unit-energy is zero. Thus, there is a sharp transition between orders of growth where interference-free communication is feasible and orders of growth where reliable communication at a positive rate per unit-energy is infeasible. The same observation was made by Ravi and Koch (*Proc. IEEE Int. Symp. Inf. Theory*, Jul. 2019) when the *per-user probability of error* is replaced by the *joint probability of error*, with the difference that the transition threshold is located at $n/\log n$ rather than at n. We further discuss the rates per unit-energy that can be achieved if one allows for a non-vanishing error probability.**

I. Introduction

Recently, Chen *et al.* [1] introduced the many-access channel (MnAC) as a multiple-access channel (MAC) where the number of users grows with the blocklength. The MnAC model is motivated by systems consisting of a single receiver and many transmitters, the number of which is comparable or even larger than the blocklength. This situation may occur, *e.g.*, in a machine-to-machine communication system with many thousands of devices in a given cell. In [1], Chen *et al.* considered a Gaussian MnAC with k_n users and determined the number of messages M_n each user can transmit reliably with a codebook of average power not exceeding P. Since then, MnACs have been studied in various papers under different settings. For example, Polyanskiy [2] considered a Gaussian MnAC where the number of active users grows linearly in the blocklength and each user's payload is fixed. Zadik *et al.* [3] presented improved bounds on the tradeoff between user density and energy-per-bit of this channel. Generalizations to quasi-static fading MnACs can be found in [4]–[7]. Shahi *et al.* [8] studied the capacity region of strongly asynchronous MnACs. Ravi and Koch [9], [10] characterized the capacity per unit-energy of Gaussian MnACs as a function of the order of growth of the number of users.

Roughly, papers on the MnAC can be divided into two groups: The first group, including [1], [8]–[10], considers a classical information-theoretic setting where the number of messages M_n transmitted by each user grows with n and the probability of a decoding error is defined as

$$P_{e,J}^{(n)} \triangleq \Pr\{(\hat{W}_1, \ldots, \hat{W}_{k_n}) \neq (W_1, \ldots, W_{k_n})\}. \quad (1)$$

Here, W_i denotes the message transmitted by user i and $\hat{W}_i$ denotes the decoder's estimate of this message. The second group, including [2]–[7], assumes that M_n is fixed and defines the probability of a decoding error as

$$P_{e,A}^{(n)} \triangleq \frac{1}{k_n} \sum_{i=1}^{k_n} \Pr\{\hat{W}_i \neq W_i\}. \quad (2)$$

The error probability $P_{e,A}^{(n)}$ is sometimes referred to as *per-user probability of error*. In this paper, we shall refer to it as *average probability of error (APE)*. In contrast, we shall refer to $P_{e,J}^{(n)}$ as *joint probability of error (JPE)*.

This paper aims at a better understanding of the implications of the above assumptions on the *capacity per unit-energy*, defined as the largest number of bits per unit-energy that can be transmitted with vanishing error probability [11]. To this end, we consider the APE and study the behavior of the capacity per unit-energy of Gaussian MnACs as a function of the order of growth of the number of users k_n. We demonstrate that, if the order of growth of k_n is sublinear, then each user can achieve the capacity per unit-energy $\frac{\log e}{N_0}$ of the single-user Gaussian channel (where $N_0/2$ is the noise power). Conversely, if the growth of k_n is linear or superlinear, then the capacity per unit-energy is zero. Thus, there is a sharp transition between orders of growth where interference-free communication is feasible and orders of growth where reliable communication at a positive rate is infeasible. The same behavior has been observed for the JPE, but with the transition threshold located at $n/\log n$ [9], [10]. Consequently, relaxing the error probability from JPE to APE merely shifts the transition threshold from $n/\log n$ to n.

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Our results imply that, when the number of users grows linearly in n, as assumed, *e.g.*, in [2]–[7], the capacity per unit-energy is zero, irrespective of whether one considers the APE or the JPE. We further show that, for the JPE, this holds true even if we allow for a non-vanishing error probability. We thus conclude that, when the number of users of the Gaussian MnAC grows linearly in n, a positive rate per unit-energy can be achieved only if one considers the APE and one allows for a non-vanishing error probability.

The rest of the paper is organized as follows. In Section II, we introduce the system model. In Section III, we characterize the capacity per unit-energy of the Gaussian MnAC with APE and compare it to the capacity per unit-energy of the Gaussian MnAC with JPE obtained in [9], [10]. Section IV discusses the rates per unit-energy that can be achieved if one allows for a non-vanishing error probability. Section V concludes the paper with a discussion of the obtained results.

II. Problem Formulation and Definitions

A. Model and Definitions

Suppose there are k users that wish to transmit their messages $W_i, i = 1, \dots, k$, which are assumed to be independent and uniformly distributed on $\{1, \dots, M_n^{(i)}\}$, to one common receiver. To achieve this, they send a codeword of n symbols over the channel, where n is referred to as the *blocklength*. We consider a many-access scenario where the number of users k grows with n, hence, we denote it as k_n. We further consider a Gaussian channel model where, for k_n users and blocklength n, the received vector $\mathbf{Y}$ is given by

$$\mathbf{Y} = \sum_{i=1}^{k_n} \mathbf{X}_i(W_i) + \mathbf{Z}.$$

Here, $\mathbf{X}_i(W_i)$ is the length-n transmitted codeword by user i for message W_i and $\mathbf{Z}$ is a vector of n i.i.d. Gaussian components $Z_j \sim \mathcal{N}(0, N_0/2)$ independent of $\mathbf{X}_i$.

We next introduce the notion of an $\left(n, \{M_n^{(\cdot)}\}, \{E_n^{(\cdot)}\}, \epsilon\right)$ code. We use the subscripts "J" and "A" to indicate whether the JPE or the APE is considered.

Definition 1: For $0 \le \epsilon < 1$, an $\left(n, \{M_n^{(\cdot)}\}, \{E_n^{(\cdot)}\}, \epsilon\right)_J$ code for the Gaussian MnAC consists of:

1) k_n encoding functions $f_i : \{1, \dots, M_n^{(i)}\} \to \mathcal{X}^n$, which map user i's message to the codeword $\mathbf{X}_i(W_i)$, satisfying the energy constraint

$$\sum_{j=1}^{n} x_{ij}^2(w_i) \le E_n^{(i)}. \quad (3)$$

Here, x_{ij} is the jth symbol of the transmitted codeword.

2) A decoding function $g : \mathcal{Y}^n \to \{M_n^{(\cdot)}\}$, which maps the received vector $\mathbf{Y}$ to the messages of all users and whose JPE, defined in (1), satisfies $P_{e,J}^{(n)} \le \epsilon$.

An $\left(n, \{M_n^{(\cdot)}\}, \{E_n^{(\cdot)}\}, \epsilon\right)_A$ code for the Gaussian MnAC consists of the same encoding functions f_i, $i = 1, \dots, k_n$ and a decoding function $g : \mathcal{Y}^n \to \{M_n^{(\cdot)}\}$ whose APE, defined in (2), satisfies $P_{e,A}^{(n)} \le \epsilon$.

We shall say that the $(n, \{M_n^{(\cdot)}\}, \{E_n^{(\cdot)}\}, \epsilon)_\xi$ code ($\xi \in \{J, A\}$) is *symmetric* if $M_n^{(i)} = M_n$ and $E_n^{(i)} = E_n$ for all $i = 1, \dots, k_n$. For compactness, we denote a symmetric code by $(n, M_n, E_n, \epsilon)_\xi$, $\xi \in \{J, A\}$. In this paper, we restrict ourselves to symmetric codes.

Definition 2: Let $\xi \in \{J, A\}$. For a symmetric code, the rate per unit-energy $\dot{R}^\xi$ is said to be ϵ-achievable if for every $\alpha > 0$ there exists an n_0 such that if $n \ge n_0$, then an $(n, M_n, E_n, \epsilon)_\xi$ code can be found whose rate per unit-energy satisfies $\frac{\log M_n}{E_n} > \dot{R}^\xi - \alpha$. Furthermore, $\dot{R}^\xi$ is said to be achievable if it is ϵ-achievable for all $0 < \epsilon < 1$. The ϵ-capacity per unit-energy $\dot{C}_\epsilon^\xi$ is the supremum of all ϵ-achievable rates per unit-energy. Similarly, the capacity per unit-energy $\dot{C}^\xi$ is the supremum of all achievable rates per unit-energy.

Remark 1: In [11, Def. 2], a rate per unit-energy $\dot{R}$ is said to be ϵ-achievable if for every $\alpha > 0$ there exists an E_0 such that if $E \ge E_0$, then an (n, M, E, ϵ) code can be found whose rate per unit-energy satisfies $\frac{\log M}{E} > \dot{R} - \alpha$. Thus, the energy E is supposed to be large rather than the blocklength n, as required in Definition 2. For the MnAC, where the number of users grows with the blocklength, we believe it is more natural to impose that $n \to \infty$. Definition 2 is also consistent with the definition of energy-per-bit in [2], [3]. Further note that, for the capacity per unit-energy, where a vanishing error probability is required, our definition is actually equivalent to [11, Def. 2]. Indeed, as observed in [9, Lemma 1] for the JPE, and as we argue below for the APE, a vanishing error probability can only be achieved if $E_n \to \infty$ as $n \to \infty$.

B. Order Notations

Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers. We write $a_n = o(b_n)$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$. We further write $a_n = \Omega(b_n)$ if $\liminf_{n\to\infty} \frac{a_n}{b_n} > 0$ and $a_n = \omega(b_n)$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$.

III. Capacity Per Unit-Energy of Gaussian Many-Access Channels

In this section, we discuss the behavior of the capacity per unit-energy as a function of the growth of k_n. Specifically, in Subsection III-A we review the results for the case of JPE that we originally presented in [9], [10]. In Subsection III-B, we then present one of the main results of this paper, a characterization of the capacity per unit-energy as a function of the growth of the number of users for APE (Theorem 2). The proof of Theorem 2 is given in Subsection III-C.

A. Joint Probability of Error

Theorem 1: The capacity per unit-energy $\dot{C}^J$ for JPE has the following behavior:

1) If $k_n = o(n/\log n)$, then $\dot{C}^J = \frac{\log e}{N_0}$.
2) If $k_n = \omega(n/\log n)$, then $\dot{C}^J = 0$.

Proof: Part 1) is [9, Th. 2]. Part 2) is [9, Th. 1]. ■

In words, if the order of growth is below $n/\log n$, then each user can achieve the single-user capacity per unit-energy.

Conversely, for any order of growth above $n/\log n$, no positive rate per unit-energy is achievable. Thus, there is a sharp transition between orders of growth where interference-free communication is feasible and orders of growth where reliable communication at a positive rate per unit-energy is infeasible.

B. Average Probability of Error

Theorem 2: The capacity per unit-energy $\dot{C}^A$ for APE has the following behavior:

1) If $k_n = o(n)$, then $\dot{C}^A = \frac{\log e}{N_0}$.
2) If $k_n = \Omega(n)$, then $\dot{C}^A = 0$.

Proof: See Section III-C. ■

We observe a similar behavior as for JPE. Again, there is a sharp transition between orders of growth where interference-free communication is feasible and orders of growth where reliable communication at a positive rate per unit-energy is infeasible. The main difference is that the transition threshold is shifted from $n/\log n$ to n.

C. Proof of Theorem 2

Part 1): We first argue that $P_{e,A}^{(n)} \to 0$ only if $E_n \to \infty$, and that in this case $\dot{C}^A \leq \frac{\log e}{N_0}$. Indeed, let $P_i \triangleq \Pr\{\hat{W}_i \neq W_i\}$ denote the probability that message W_i is decoded erroneously. We then have that $P_{e,A}^{(n)} \geq \min_i P_i$. Furthermore, P_i is lower-bounded by the error probability of the Gaussian single-user channel, since a single-user channel can be obtained from the MnAC if a genie informs the receiver about the codewords transmitted by users $j \neq i$. By applying the lower bound [12, eq. (30)] on the error probability of the Gaussian single-user channel, we thus obtain

$$P_{e,A}^{(n)} \geq Q\left(\sqrt{\frac{2E_n}{N_0}}\right), \quad M_n \geq 2. \tag{4}$$

Hence $P_{e,A}^{(n)} \to 0$ only if $E_n \to \infty$. As mentioned in Remark 1, when E_n tends to infinity as $n \to \infty$, the capacity per unit-energy $\dot{C}^A$ coincides with the capacity per unit-energy defined in [11], which for the Gaussian single-user channel is given by $\frac{\log e}{N_0}$ [11, Ex. 3]. Furthermore, if $P_{e,A}^{(n)} \to 0$ as $n \to \infty$, then there exists at least one user i for which $P_i \to 0$ as $n \to \infty$. By the above genie argument, this user's rate per unit-energy is upper-bounded by the capacity per unit-energy of the Gaussian single-user channel. Since for the class of symmetric codes considered in this paper each user transmits at the same rate per unit-energy, we conclude that $\dot{C}^A \leq \frac{\log e}{N_0}$.

We next show that any rate per unit-energy $\dot{R}^A < \frac{\log e}{N_0}$ is achievable. For a given $0 < \epsilon < 1$, let $0 < \epsilon' < \epsilon$, and define

$$A_n \triangleq \frac{1}{k_n}\sum_{i=1}^{k_n} \mathbf{1}(\hat{W}_i \neq W_i)$$

where $\mathbf{1}(\cdot)$ denotes the indicator function. Further define $\mathcal{A}_n \triangleq \{0, 1/k_n, \ldots, 1\}$ and $\mathcal{A}_n^{\epsilon'} \triangleq \{a \in \mathcal{A}_n : a \geq \epsilon'\}$. Noting that $P_{e,A}^{(n)} = \mathsf{E}[A_n]$, we then obtain that

$$\begin{aligned} P_{e,A}^{(n)} &= \sum_{a\in\mathcal{A}_n} a\Pr\{A_n = a\} \\ &= \sum_{a\in\mathcal{A}_n\setminus\mathcal{A}_n^{\epsilon'}} a\Pr\{A_n = a\} + \sum_{a\in\mathcal{A}_n^{\epsilon'}} a\Pr\{A_n = a\} \\ &\leq \epsilon' + \sum_{a\in\mathcal{A}_n^{\epsilon'}} \Pr\{A_n = a\} \end{aligned} \tag{5}$$

where we used that $a \leq \epsilon'$ for $a \in \mathcal{A}_n \setminus \mathcal{A}_n^{\epsilon'}$ and $a \leq 1$ for $a \in \mathcal{A}_n^{\epsilon'}$. Next we show that if $\dot{R}^A < \frac{\log e}{N_0}$, then

$$\lim_{n\to\infty} \sum_{a\in\mathcal{A}_n^{\epsilon'}} \Pr\{A_n = a\} = 0. \tag{6}$$

It then follows from (5) that $P_{e,A}^{(n)} \leq \epsilon$ for sufficiently large n and all $0 < \epsilon < 1$. Thus, any rate per unit-energy $\dot{R}^A < \frac{\log e}{N_0}$ is achievable which proves Part 1) of Theorem 2.

To prove (6), we need the following lemma.

Lemma 1: For any arbitrary $0 < \rho \leq 1$, we have

$$\Pr\{A_n = a\} \leq \binom{k_n}{ak_n} M_n^{ak_n\rho} e^{-nE_0(a,\rho)}, \quad a \in \mathcal{A}_n \setminus \{0\}$$

where

$$E_0(a,\rho) \triangleq \frac{\rho}{2}\ln\left(1 + \frac{a2k_nE_n}{n(\rho+1)N_0}\right).$$

Proof: See [13, Th. 2]. ■

Using Lemma 1, we can upper-bound the second term on the right-hand side (RHS) of (5) as

$$\begin{aligned} &\sum_{a\in\mathcal{A}_n^{\epsilon'}} \Pr\{A_n = a\} \\ &\leq \left(\max_{a\in\mathcal{A}_n^{\epsilon'}} \exp\left[-nE_0(a,\rho) + \ln M_n^{a\rho k_n}\right]\right) \sum_{a\in\mathcal{A}_n^{\epsilon'}} \binom{k_n}{ak_n} \\ &\leq \max_{a\in\mathcal{A}_n^{\epsilon'}} \exp\left[-E_n f_n(a,\rho)\right] \end{aligned} \tag{7}$$

where

$$f_n(a,\rho) \triangleq \frac{nE_0(a,\rho)}{E_n} - \frac{a\rho k_n \ln M_n}{E_n} - \frac{k_n \ln 2}{E_n}.$$

We next choose $E_n = (\ln(n/k_n)k_n/n)^{-1}$. This implies that $E_n \to \infty$ and $E_nk_n/n \to 0$ as $n \to \infty$ since, by the theorem's assumption, $k_n = o(n)$. We then show that, for this choice of E_n and $\dot{R}^A = \frac{\log e}{(1+\rho)N_0} - \delta$ (for some arbitrary $0 < \delta < \frac{\log e}{(1+\rho)N_0}$), we have

$$\liminf_{n\to\infty} \min_{a\in\mathcal{A}_n^{\epsilon'}} f_n(a,\rho) > 0. \tag{8}$$

Thus, for $\dot{R}^A = \frac{\log e}{(1+\rho)N_0} - \delta$, the RHS of (7) vanishes as $n \to \infty$. Since $0 < \rho < 1$ and $\delta > 0$ are arbitrary, (6) follows.

To obtain (8), we first show that, for any fixed value of ρ and our choices of E_n and $\dot{R}^A$,

$$\liminf_{n\to\infty} \frac{df_n(a,\rho)}{da} > 0, \quad \epsilon' \leq a \leq 1. \tag{9}$$

Hence

$$\liminf_{n\to\infty} \min_{a\in\mathcal{A}_n^{\epsilon'}} f_n(a,\rho) \geq \liminf_{n\to\infty} f_n(\epsilon',\rho). \tag{10}$$

Indeed, basic algebraic manipulations yield for $\epsilon' \leq a \leq 1$

$$\frac{df_n(a,\rho)}{da} \geq \rho k_n \left[\frac{1}{1+\frac{2k_nE_n}{n(\rho+1)N_0}} \frac{1}{(1+\rho)N_0} - \frac{\dot{R}^A}{\log e}\right]. \tag{11}$$

Recall that, for the given choice of E_n, we have $\frac{k_nE_n}{n} \to 0$ as $n\to\infty$. It follows that the bracketed term in (11) tends to $\frac{\delta}{\log e}$ as $n\to\infty$. This proves (9).

We next show that the RHS of (10) is positive for every $0<\rho<1$. Let

$$i_n(\epsilon',\rho) \triangleq \frac{nE_0(\epsilon',\rho)}{E_n}$$
$$j_n(\epsilon',\rho) \triangleq \frac{\epsilon'\rho k_n\dot{R}^A}{\log e}$$
$$h_n \triangleq \frac{k_n \ln 2}{E_n}.$$

For our choices of E_n and $\dot{R}^A$, we have that $h_n/j_n(\epsilon',\rho)\to 0$ as $n\to\infty$. Consequently,

$$\liminf_{n\to\infty} f_n(\epsilon',\rho) \geq \liminf_{n\to\infty} j_n(\epsilon',\rho) \liminf_{n\to\infty} \frac{f_n(\epsilon',\rho)}{j_n(\epsilon',\rho)}$$
$$= \liminf_{n\to\infty} j_n(\epsilon',\rho)\left\{\liminf_{n\to\infty}\frac{i_n(\epsilon',\rho)}{j_n(\epsilon',\rho)} - 1\right\}.$$

Note that $j_n(\epsilon',\rho) \geq \epsilon'\rho\dot{R}^A/\log e$, which is bounded away from zero for our choice of $\dot{R}^A$ and $\delta < \frac{\log e}{(1+\rho)N_0}$. The RHS of (10) is thus positive if $\liminf_{n\to\infty} i_n(\epsilon',\rho)/j_n(\epsilon',\rho) > 1$, which is what we show next. Indeed, we have for our choice of E_n and $k_n = o(n)$ that

$$\lim_{n\to\infty}\frac{i_n(\epsilon',\rho)}{j_n(\epsilon',\rho)} = \frac{\log e}{(1+\rho)N_0\dot{R}^A}.$$

For our choice of $\dot{R}^A$, this is strictly larger than 1. We thus conclude that the RHS of (10) is positive, from which (8), and hence also (6), follows. This proves Part 1) of Theorem 2.

Part 2): Fano's inequality yields that

$$\log M_n \leq 1 + P_i \log M_n + I(W_i;\hat{W}_i)$$

for $i=1,\ldots,k_n$. Averaging over all i's then gives

$$\log M_n \leq 1 + \frac{1}{k_n}\sum_{i=1}^{k_n} P_i \log M_n + \frac{1}{k_n} I(\mathbf{W};\hat{\mathbf{W}})$$
$$\leq 1 + P_{e,A}^{(n)} \log M_n + \frac{1}{k_n} I(\mathbf{X};\mathbf{Y})$$
$$\leq 1 + P_{e,A}^{(n)} \log M_n + \frac{n}{2k_n}\log\left(1+\frac{2k_nE_n}{nN_0}\right) \tag{12}$$

where $\mathbf{X} \triangleq (\mathbf{X}_1,\mathbf{X}_2,\ldots,\mathbf{X}_{k_n})$. Here, the first inequality follows because the messages $W_i, i=1,\ldots,k_n$ are independent and because conditioning reduces entropy, the second inequality follows from the definition of $P_{e,A}^{(n)}$ and the data processing inequality, and the third inequality follows by upper-bounding $I(\mathbf{X};\mathbf{Y})$ by $\frac{n}{2}\log\bigl(1+\frac{2k_nE_n}{nN_0}\bigr)$.

Dividing both sides of (12) by E_n, and solving the inequality for $\dot{R}^A$, we obtain the upper bound

$$\dot{R}^A \leq \frac{\frac{1}{E_n} + \frac{n}{2k_nE_n}\log(1+\frac{2k_nE_n}{nN_0})}{1-P_{e,A}^{(n)}}. \tag{13}$$

As argued at the beginning of the proof of Part 1), we have $P_{e,A}^{(n)} \to 0$ only if $E_n\to\infty$. If $k_n = \Omega(n)$, then this implies that $k_nE_n/n\to\infty$ as $n\to\infty$. It thus follows from (13) that, if $k_n=\Omega(n)$, then $\dot{C}^A = 0$, which is Part 2) of Theorem 1.

IV. Non-Vanishing Error Probability

In this section, we briefly discuss how the largest achievable rate per unit-energy changes if we allow for a non-vanishing error probability. With the help of the following example, we first argue that when the number of users is bounded in n, then a simple orthogonal-access scheme achieves an ϵ-achievable rate per unit-energy that can be strictly larger than the single-user capacity per unit-energy $\frac{\log e}{N_0}$.

Example 1: Consider a k-user Gaussian MAC with normalized noise variance $N_0/2 = 1$ and where the number of users is independent of n. Suppose that each user transmits one out of two messages ($M_n = 2$) with energy $E_n = 1$ by following an orthogonal-access scheme where each user gets one channel use and remains silent in the remaining channel uses. In this channel use, each user transmits either $+1$ or -1 to convey its message. Since the access scheme is orthogonal, the receiver can perform independent decoding for each user, which yields $P_i = Q(1)$. Consequently, we can achieve the rate per unit-energy $\frac{\log M_n}{E_n} = 1$ at APE $P_{e,A}^{(n)} = Q(1)$ and at JPE $P_{e,J}^{(n)} = 1-(1-Q(1))^k$ [9, eq. (6)]. Thus, for some $0<\epsilon<1$, we have that $\dot{C}_\epsilon^\xi > \frac{\log e}{N_0}$, $\xi\in\{J,A\}$.

Remark 2: A crucial ingredient in the above scheme is that the energy E_n is bounded in n. Indeed, it follows from [12, Th. 3] that if $E_n\to\infty$ as $n\to\infty$, then the ϵ-capacity per unit-energy of the Gaussian single-user channel is equal to $\frac{\log e}{N_0}$, irrespective of $0<\epsilon<1$. The genie argument provided at the beginning of Section III-C then yields that the same is true for the Gaussian MnAC.

In the rest of this section, we discuss the ϵ-capacity per unit-energy when the number of users k_n tends to infinity as n tends to infinity. Specifically, in Subsection IV-A we discuss the ϵ-capacity per unit-energy for JPE as a function of the order of growth of the number of users. In Subsection IV-B, we briefly discuss the ϵ-capacity per unit-energy for APE when k_n grows linearly in n.

A. Non-Vanishing JPE

Theorem 3: The ϵ-capacity per unit-energy $\dot{C}_\epsilon^J$ for JPE has the following behavior:

1) If $k_n = \omega(1)$ and $k_n = o(n/\log n)$, then $\dot{C}_\epsilon^J = \frac{\log e}{N_0}$ for every $0<\epsilon<1$.
2) If $k_n = \omega(n/\log n)$, then $\dot{C}_\epsilon^J = 0$ for every $0<\epsilon<1$.

Proof: We first prove Part 1). It follows from [9, eq. (20)] that, for $M_n \geq 2$,[1]

$$P_{e,J}^{(n)} \geq 1 - \frac{64E_n/N_0 + \log 2}{\log k_n}. \quad (14)$$

This implies that $P_{e,J}^{(n)}$ tends to one unless $E_n = \Omega(\log k_n)$. Since by the theorem's assumption $k_n = \omega(1)$, it follows that $E_n \to \infty$ is necessary to achieve a JPE strictly smaller than one. As argued in Remark 2, if $E_n \to \infty$ as $n \to \infty$, then the ϵ-capacity per unit-energy of the Gaussian MnAC cannot exceed the single-user capacity per unit-energy $\frac{\log e}{N_0}$. Furthermore, by Theorem 1, if $k_n = o(n/\log n)$ then any rate per unit-energy satisfying $\dot{R}^J < \frac{\log e}{N_0}$ is achievable, hence it is also ϵ-achievable. We thus conclude that, if $k_n = \omega(1)$ and $k_n = o(n/\log n)$, then $\dot{C}_\epsilon^J = \frac{\log e}{N_0}$ for every $0 < \epsilon < 1$.

To prove Part 2), we use that, by Fano's inequality, we can upper-bound $\dot{R}^J$ as [9, eq. (2)]

$$\dot{R}^J \leq \frac{\frac{1}{k_n E_n} + \frac{n}{2k_n E_n}\log(1 + \frac{2k_n E_n}{nN_0})}{1 - P_{e,J}^{(n)}}. \quad (15)$$

By (14), $P_{e,J}^{(n)}$ tends to one unless $E_n = \Omega(\log k_n)$. For $k_n = \omega(n/\log n)$, this implies that $k_n E_n/n \to \infty$ as $n \to \infty$, so the RHS of (15) vanishes as n tends to infinity. We thus conclude that, if $k_n = \omega(n/\log n)$, then $\dot{C}_\epsilon^J = 0$ for every $0 < \epsilon < 1$. ■

B. Non-Vanishing APE

For the APE, we restrict ourselves to the case where $k_n = \mu n$ for some $\mu > 0$, since it is a common assumption in the analysis of MnACs; see, *e.g.*, [2]–[7]. By inspecting the proof of Part 1) of Theorem 2, one can show that, for every $\mu > 0$ and $0 < \epsilon' < \epsilon < 1$, there exists an E independent of n and a $0 < \rho \leq 1$ such that the RHS of (7) vanishes with n for some positive $\dot{R}^A$. By (5), it then follows that $P_{e,A}^{(n)} \leq \epsilon$ for sufficiently large n, hence, there exists a positive rate per unit-energy $\dot{R}^A$ that is ϵ-achievable.

While (5) and (7) yield an upper bound on $P_{e,A}^{(n)}$ that is sufficient to demonstrate the qualitative behavior of $\dot{C}_\epsilon^A$, this bound is looser than the bounds obtained in [2], [3]. Specifically, [2], [3] derived bounds on the minimum energy-per-bit $\mathcal{E}^*(M, \mu, \epsilon)$ required to send M messages at an APE not exceeding ϵ when the number of users is given by $k_n = \mu n$. Since the rate per unit-energy is the inverse of the energy-per-bit, these bounds also apply to $\dot{C}_\epsilon^A$. The achievability and converse bounds presented in [3] further suggest that there exists a critical user density μ below which interference-free communication is feasible. This conjectured effect can be confirmed when each user sends only one bit ($M = 2$), since in this case $\mathcal{E}^*(M, \mu, \epsilon)$ can be evaluated in closed form for $\mu \leq 1$. For simplicity, assume that $N_0/2 = 1$. Then,

$$\mathcal{E}^*(2, \mu, \epsilon) = \left(\max\{0, Q^{-1}(\epsilon)\}\right)^2, \quad 0 \leq \mu \leq 1. \quad (16)$$

Indeed, that $\mathcal{E}^*(2, \mu, \epsilon) \geq (\max\{0, Q^{-1}(\epsilon)\})^2$ follows from (4). Furthermore, when $\mu \leq 1$, applying the orthogonal-access scheme presented in Example 1 with energy $(\max\{0, Q^{-1}(\epsilon)\})^2$ achieves $P_{e,A}^{(n)} = \epsilon$. Observe that the RHS of (16) does not depend on μ and agrees with the minimum energy-per-bit required to send one bit over the Gaussian single-user channel with error probability ϵ. Thus, when $\mu \leq 1$, we can send one bit free of interference.

[1] A similar bound was presented in [14, p. 84] for the case where $M_n = 2$.

V. Conclusion

A common assumption in the analysis of MnACs is that the number of users grows linearly with the blocklength. Theorems 1 and 2 imply that in this case the capacity per unit-energy is zero, irrespective of whether one considers the APE or the JPE. Theorem 3 further demonstrates that, for the JPE, this holds true even if we allow for a non-vanishing error probability. The situation changes for the APE. Here a positive rate per unit-energy can be achieved if one allows for a non-vanishing error probability. Another crucial assumption is that the energy E_n and payload $\log M_n$ are bounded in n. Indeed, for $k_n = \mu n$, the RHS of (13) vanishes as E_n tends to infinity, so when $E_n \to \infty$ no positive rate per unit-energy is ϵ-achievable. Moreover, for $k_n = \mu n$ and a bounded E_n, (12) implies that the payload $\log M_n$ is bounded, too. We conclude that the arguably most common assumptions in the literature on MnACs—linear growth of the number of users, a non-vanishing APE, and a fixed payload—are the only set of assumptions under which a positive rate per unit-energy is achievable, unless we consider nonlinear growths of k_n.

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